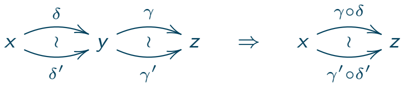
Congruences on small categories

A congruence on a small category $\mathcal{C}$ is an equivalence relation $\sim$ over $\mathrm{Mo}(\mathcal{C})$ such that:

- $\gamma \sim \gamma' \quad \Rightarrow \quad \partial^- \gamma = \partial^- \gamma'$ and $\partial^+ \gamma = \partial^+ \gamma'$
- $\gamma \sim \gamma'$, $\delta \sim \delta'$ and $\partial^- \gamma = \partial^+ \delta \quad \Rightarrow \quad \gamma \circ \delta \sim \gamma' \circ \delta'$

In diagrams we have



Hence the $\sim$-equivalence class of $\gamma \circ \delta$ only depends on the $\sim$-equivalence classes of γ and δ and we have a quotient category $\mathcal{C}/\sim$ in which the composition is given by

$$[\gamma] \circ [\delta] \quad = \quad [\gamma \circ \delta]$$

The quotient map $q : \gamma \in \mathrm{Mo}(\mathcal{C}) \mapsto [\gamma] \in \mathrm{Mo}(\mathcal{C})/\sim$ induces a functor $q : \mathcal{C} \to \mathcal{C}/\sim$

Natural congruences on a functor $P : \mathcal{C} \to \mathcal{C}at$

A natural congruence on a functor $P : \mathcal{C} \to \mathcal{C}at$ is a collection of congruences $\sim_X$ on PX, for X ranging through the objects of $\mathcal{C}$, such that for all morphisms $f : X \to Y$ of $\mathcal{C}$, for all $\alpha, \beta \in PX$,

$$\alpha \sim_X \beta \quad \Rightarrow \quad P(f)(\alpha) \sim_Y P(f)(\beta)$$

Then we can define the functor $\overrightarrow{\pi_1} : \mathcal{C} \to \mathcal{C}at$ as follows:

- for all $X \in \mathcal{C}$, $\pi_1(X) = P(X)/ \sim_X$
- for all $f : X \to Y$ in $\mathcal{C}$

$$X \xrightarrow{f} Y$$

$$\begin{array}{ccc} PX & \xrightarrow{Pf} & PY \\ {\scriptstyle q_X}\downarrow & & \downarrow{\scriptstyle q_Y} \\ \overrightarrow{\pi_1}X & \xrightarrow[\overrightarrow{\pi_1}f]{} & \overrightarrow{\pi_1}Y \end{array}$$

The collection of quotient functors q_X, for X ranging through the objects of $\mathcal{C}$, provides a natural transformation from P to $\overrightarrow{\pi_1}$.

Object part

Let X be a locally ordered space.

- The objects of PX are the points of X.
- The homset $PX(a, b)$ is

$$\bigcup_{r \geqslant 0} \{\gamma \in \mathcal{L}po([0,r], X) \mid \gamma(0) = a \text{ and } \gamma(r) = b\}$$

- For $\delta : [0,r] \to X$ and $\gamma : [0,r'] \to X$ with $\delta(r) = \gamma(0)$, define the concatenation

$$\gamma \cdot \delta : \; [0, r + r'] \longrightarrow X$$

$$t \longmapsto \begin{cases} \delta(t) & \text{if } t \leqslant r \\ \gamma(t - r) & \text{if } t \geqslant r \end{cases}$$

Morphism part

The (Moore) path category construction gives rise to a functor P from $\mathcal{Lpo}$ to $\mathcal{Cat}$ since for all $f \in \mathcal{Lpo}(X, Y)$ and all paths γ on X, the composite $f \circ \gamma$ is a path on Y.

$$P : \mathcal{Lpo} \longrightarrow \mathcal{Cat}$$

$$\begin{array}{ccc} X & & PX \\ \downarrow f & \longmapsto & Pf \downarrow \\ Y & & PY \end{array}$$

with

$$Pf : PX \longrightarrow PY$$

$$\begin{array}{ccc} p & & f(p) \\ \downarrow \gamma & \longmapsto & f\circ\gamma \downarrow \\ q & & f(q) \end{array}$$

Equivalent directed paths on a local pospace X

An elementary homotopy is a *finite* concatenation of directed and anti-directed homotopies.

If $\theta : [0,r] \to [0,r]$ is a reparametrization and $\gamma \in \mathcal{L}po([0,r],X)$, then γ and $\gamma \circ \theta$ are dihomotopic.

Two directed paths $\gamma : [0,r'] \to X$ and $\delta : [0,r''] \to X$ on a local pospace are said to be equivalent (denoted by $\sim_X$) when there exists two reparametrizations $\theta : [0,r] \to [0,r']$ and $\psi : [0,r] \to [0,r'']$ such that there is an elementary homotopy between $\gamma \circ \theta$ and $\delta \circ \psi$.

The relation $\sim_X$ is symmetric because if $h(s,t)$ is an elementary homotopy, then so is the mapping $(s,t) \mapsto h(-s,t)$.

The relation $\sim_X$ is transitive because a concatenation of elementary homotopies is an elementary homotopy.

Given $x, y \in X$ and $r \in \mathbb{R}_+$, the relation $\sim_X$ is an equivalence relation on the set

$$\bigcup_{r \in \mathbb{R}_+} \{\gamma \in \mathcal{L}po([0,r],X) \mid \gamma(0) = x;\ \gamma(r) = y\}$$

Juxtaposition of homotopies

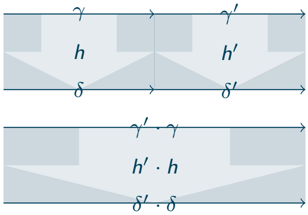
horizontal composition

Let $h : [0,r] \times [0,q] \to X$ and $h' : [0,r'] \times [0,q] \to X$ be homotopies from γ to δ and from γ' to δ' with $\partial^{+}\gamma = \partial^{-}\gamma'$.

The mapping $h' * h : [0, r + r'] \times [0,q] \to X$ defined by

$$h' * h(t,s) \quad = \quad \begin{cases} h(t,s) & \text{if } 0 \leqslant t \leqslant r \\ h'(t-r,s) & \text{if } r \leqslant t \leqslant r + r' \end{cases}$$

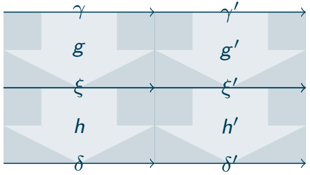
is a homotopy from γ to δ.



If h and h' are ((weakly) directed) homotopies, then so is their juxtaposition $h' \cdot h$.

Godement exchange law

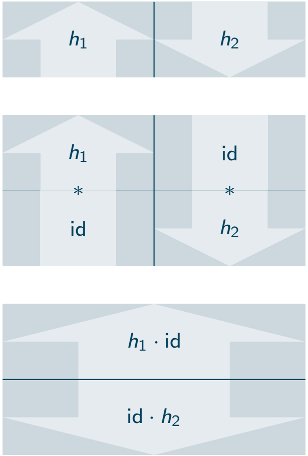
Suppose we have



then it comes

$$(g' * h') \cdot (g * h) = (g' \cdot g) * (h' \cdot h)$$

Applying Godement exchange law



Equivalences are congruences

If:

- h is an elementary homotopy between $\gamma \circ \theta$ and $\delta \circ \psi$
- h' is an elementary homotopy between $\gamma' \circ \theta'$ and $\delta' \circ \psi'$
- the endpoint of γ is the starting point of γ'

then $h \cdot h'$ is an elementary homotopy from $(\gamma \cdot \gamma') \circ (\theta \cdot \theta')$ to $(\delta \cdot \delta') \circ (\psi \cdot \psi')$.

The relation $\sim_X$ is a congruence on $P(X)$

Naturality

If h is a homotopy from γ to γ' on the topological space X and $f : X \to Y$ is a continuous map, then $f \circ h$ is a homotopy from $f \circ \gamma$ to $f \circ \gamma'$ on the topological space Y.

If h is a (weakly) directed homotopy from γ to γ' on the local pospace space X and $f : X \to Y$ is a local pospace morphism, then $f \circ h$ is a (weakly) directed homotopy from $f \circ \gamma$ to $f \circ \gamma'$ on the local pospace space Y.

If $\gamma, \gamma' : [0, r] \to X$ are ((weakly) di)homotopic, then so are $f \circ \gamma, f \circ \gamma' : [0, r] \to Y$.

Conclusion

- The relations $\sim_X$ form a natural congruence on the directed path functor $P : \mathcal{Lpo} \to \mathcal{Cat}$.
- The fundamental category functor $\overrightarrow{\pi_1} : \mathcal{Lpo} \to \mathcal{Cat}$ is defined accordingly.
- The fundamental groupoid functor $\Pi_1 : \mathcal{Top} \to \mathcal{Grd}$ is obtained by substituting "paths" and "homotopies" to "directed paths" and "elementary homotopies".

- The fundamental category of the locally ordered real line is the corresponding partial order.
- For all local pospaces X and Y we have

$$\overrightarrow{\pi_1}(X \times Y) \quad \cong \quad \overrightarrow{\pi_1}X \times \overrightarrow{\pi_1}Y$$

- Given a pospace X, $\overrightarrow{\pi_1}X$ is loop-free i.e.

$$\overrightarrow{\pi_1}X(x,y) \neq \emptyset \ \text{ and } \ \overrightarrow{\pi_1}X(y,x) \neq \emptyset \qquad \Rightarrow \qquad x = y \ \text{ and } \ \overrightarrow{\pi_1}X(x,x) = \{\mathrm{id}_x\}$$

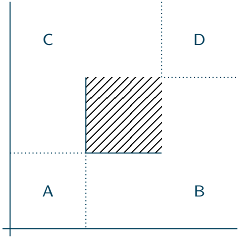
- The fundamental category of a local pospace has no nontrivial null homotopic directed paths i.e. any directed loop that is related to a constant path by an elementary homotopy is actually a constant.
- In particular the fundamental category of a local pospace has no isomorphism but its identities.

The fundamental category of the locally ordered circle

- Given x, y, $\widehat{xy}$ is the anticlockwise arc from x to y.
  It is a singleton if $x = y$.
- $\overrightarrow{\pi_1}\mathbb{S}^1(x, y) = \{x\} \times \mathbb{N} \times \{y\}$
- the identities are the tuples $(x, 0, x)$
- the composition is given by
  - $(y, p, z) \circ (x, n, y) = (x, n + p, z)$ if $\widehat{xy} \cup \widehat{yz} \neq \mathbb{S}^1$
  - $(y, p, z) \circ (x, n, y) = (x, n + p + 1, z)$ if $\widehat{xy} \cup \widehat{yz} = \mathbb{S}^1$

Plane without a square

$x = \mathbb{R}_+^2 \setminus]0,1[^2$



If $x \leqslant^2 y$, then $\overrightarrow{\pi_1}X(x,y)$ only depends on the elements of the partition x and y belong to.

$\rightarrow$	A	B	C	D
A	σ	β	α	$\beta' \circ \beta$ $\alpha' \circ \alpha$
B		σ		β'
C			σ	α'
D				σ

Skeleta and equivalences of categories

- A skeleton of $\mathcal{C}$ is a full subcategory of $\mathcal{C}$ whose class of objects meets every isomorphism class of $\mathcal{C}$ exactly once.
- The skeleton of $\mathcal{C}$ is unique up to isomorphism, it is denoted by sk$\mathcal{C}$.
- Two categories are equivalent (i.e. there exists an equivalence of categories between them) iff their skeleta are isomorphic.
- The skeleton of the category of finite sets is the full subcategory whose objects are $\{0, \ldots, n-1\}$ for $n \in \mathbb{N}$.
- The skeleton of the fundamental groupoid of a path-connected space is the fundamental group of this space.
- Problem: The fundamental category of a local pospace has no isomorphisms but its identities, hence it is its own skeleton.

The categories $\mathcal{LfCat}$ and $\mathcal{OwCat}$

- A category $\mathcal{C}$ is said to be one-way when all its endomorphisms are identities i.e. $\mathcal{C}(x,x) = \{\mathrm{id}_x\}$ for all x
  Every Grothendieck topos has a one-way site. C. MacLarty. Theor. Appl. of Cat. 16(5) pp 123-126 (2006).
- A one-way category $\mathcal{C}$ is said to be loop-free when for all x, y

$$\mathcal{C}(x,y) \neq \emptyset \text{ and } \mathcal{C}(y,x) \neq \emptyset \text{ implies } x = y$$

  Complexes of groups and orbihedra *in* Group theory from a geometrical viewpoint.
  A. Haefliger. World Scientific (1991).
- A loop-free category is its own skeleton
- A category is one-way iff its skeleton is loop-free

Generalized congruences

M. A. Bednarczyk, A. M. Borzyszkowski, W. Pawlowski. Theor. Appl. Cat. 5(11). 1999

- Given a binary relation $\mathcal{R}$ on the set of morphisms of a category $\mathcal{C}$, there is a unique category $\mathcal{C}/\mathcal{R}$ and a unique functor $q : \mathcal{C} \to \mathcal{C}/\mathcal{R}$ such that for all functors $f : \mathcal{C} \to \mathcal{D}$, if $\alpha\mathcal{R}\beta \Rightarrow f(\alpha) = f(\beta)$, then there is a unique functor $g : \mathcal{C}/\mathcal{R} \to \mathcal{D}$ such that $f = g \circ q$

$$\begin{array}{ccc} \mathcal{C}/\mathcal{R} & & \\ \uparrow q & \searrow g & \\ \mathcal{C} & \xrightarrow{f} & \mathcal{D} \end{array}$$

- Examples
  - any congruence is a generalized congruence.
  - $\mathcal{C}$ freely generated by $x \xrightarrow{\alpha} y$ with $\mathrm{id}_x \mathcal{R} \mathrm{id}_y$ (resp. with $\alpha \mathcal{R} \mathrm{id}_x$).
  - $(\mathbb{N}, +, 0)$ with $0\mathcal{R}n$ for some $n \in \mathbb{N}$.

Goal

Let $\mathcal{C}$ be a one-way category:

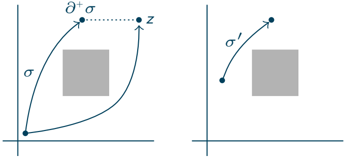
- Define a class Σ of morphisms of $\mathcal{C}$ so we can keep one representative in each class of Σ-related objects without loss of information
- To do so, we are in search for a class that behaves much like the one of isomorphisms
- From now on $\mathcal{C}$ denotes a one-way category

Potential weak isomorphisms

Let $\mathcal{C}$ be a one-way category

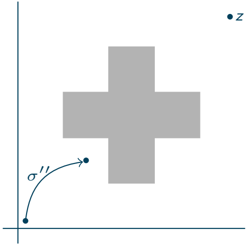
- For all morphisms σ and all objects z define
  - the σ, z-precomposition as $\gamma \in \mathcal{C}(\partial^+\sigma, z) \to \gamma\circ\sigma \in \mathcal{C}(\partial^-\sigma, z)$
  - the z, σ-postcomposition as $\delta \in \mathcal{C}(z, \partial^-\sigma) \mapsto \sigma\circ\delta \in \mathcal{C}(z, \partial^+\sigma)$
- One may have $\mathcal{C}(\partial^+\sigma, z) = \emptyset$ or $\mathcal{C}(z, \partial^-\sigma) = \emptyset$
- Note that σ is an isomorphism iff for all z both precomposition and postcomposition are bijective.
- The latter condition is weakened: σ is said to preserve the future cones (resp. past cones) when for all z if $\mathcal{C}(\partial^+\sigma, z) \neq \emptyset$ (resp. $\mathcal{C}(z, \partial^-\sigma) \neq \emptyset$) then the precomposition (resp. postcomposition) is bijective.
- Then σ is a potential weak isomorphism when it preserves both future cones and past cones. Potential weak isomorphisms compose.
- If $\mathcal{C}(x, y)$ contains a potential weak isomorphism, then it is a singleton
  Requires the assumption that $\mathcal{C}$ is one-way

An example of potential weak isomorphism



Due to the lower dipath, the σ, z-precomposition is not bijective; yet σ' is a potential weak isomorphism.

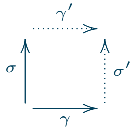
An unwanted example of potential weak isomorphism



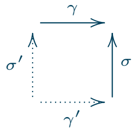
Note that σ'' is a potential weak isomorphism though there exists a morphism from $\partial^{-}\sigma''$ to z but none from $\partial^{+}\sigma''$ to z.

Stability under pushout and pullback

- A collection of morphisms Σ is said to be stable under pushout when for all $\sigma \in \Sigma$, for all γ with $\partial^{-}\gamma = \partial^{-}\sigma$, the pushout of σ along γ exists and belongs to Σ



- A collection of morphisms Σ is said to be stable under pullback when for all $\sigma \in \Sigma$, for all γ with $\partial^{+}\gamma = \partial^{+}\sigma$, the pullback of σ along γ exists and belongs to Σ



Greatest inner collection stable under pushout and pullback

- Any collection Σ of morphisms of a category $\mathcal{C}$ admits a greatest subcollection that is stable under pushout and pullback
- Construction:
  - Start with $\Sigma_0 = \Sigma$
  - For $n \in \mathbb{N}$ define Σ_{n+1} as the collection of morphisms $\sigma \in \Sigma_n$ s.t. the pushout and the pullback of σ along with all morphisms exist (when sources or targets match) and belong to Σ_n

$$\Sigma_0 \supseteq \cdots \Sigma_1 \supseteq \cdots \supseteq \Sigma_n \supseteq \Sigma_{n+1} \supseteq \cdots$$

  - The expected subcollection is the decreasing intersection

$$\Sigma_\infty := \bigcap_{n\in\mathbb{N}}^{\downarrow} \Sigma_n$$

- The collection Σ_∞ is stable under the action of $\mathrm{Aut}(\mathcal{C})$

Systems of weak isomorphisms

- The class of isomorphisms of any category is stable under pushout and pullback
- A system of weak isomorphisms is a collection of potential weak isomorphisms that is stable under pushout and pullback
- The class of all isomorphisms of any category is a system of weak isomorphisms
- If Σ is a system of weak isomorphisms, then so is its closure under composition
- Hence we suppose the systems of weak isomorphisms are closed under composition

Examples of systems of weak isomorphisms

- Given a partition $\mathcal{P}$ of $\mathbb{R}$ into intervals, the following collection is a system of weak isomorphisms

$$\{(x, y) \mid x \leqslant y;\ \exists I \in \mathcal{P},\ [x, y] \subseteq I\}$$

- In the preceding example, $\mathbb{R}$ can be replaced by any totally ordered set
- Let $\Sigma_i \subseteq \mathcal{C}_i$ be a family of collections of morphisms, then

$$\boxed{\prod_i \Sigma_i \text{ is a swi of } \prod_i \mathcal{C}_i \text{ iff each } \Sigma_i \text{ is a swi of } \mathcal{C}_i}$$

- The inverse image (resp. the direct image) of a system of weak isomorphisms by an equivalence of categories is a system of weak isomorphisms.

Pureness

- A collection Σ of morphisms is said to be pure when

$$\gamma \circ \delta \in \boldsymbol{\Sigma} \;\Rightarrow\; \gamma, \delta \in \boldsymbol{\Sigma}$$

- Given a one-way category $\mathcal{C}$ we have:

All the systems of weak isomorphisms of $\mathcal{C}$ are pure

The locale of systems of weak isomorphisms

- A locale is a complete lattice whose binary meet distributes over arbitrary join i.e.

$$x \wedge \Big(\bigvee_i y_i\Big) = \bigvee_i (x \wedge y_i)$$

- The collection ΩX open subsets of a topological space X form a locale and we have the functor $L : \mathcal{Top} \to \mathcal{Loc}$ (that admits a left adjoint) defined by
  - $L(X) = \Omega X$
  - $L(f)(W) = f^{-1}(W)$ for all $f : X \to Y$ and $W \in \Omega Y$
- The collection of systems of weak isomorphisms of a category has a greatest element
- Given a one-way category $\mathcal{C}$ we have:
  - The collection of systems of weak isomorphisms of $\mathcal{C}$ forms a locale
  - The greatest swi is invariant under the action of $\mathrm{Aut}(\mathcal{C})$

Components of a one-way category $\mathcal{C}$

- From now on $\mathcal{C}$ is a one-way category and Σ is a system of weak isomorphisms on it
- Recall that if $\mathcal{C}(x, y)$ meets Σ, then $\mathcal{C}(x, y)$ is a singleton, a fact that we represent on diagrams by: $x \xrightarrow{\Sigma} y$
- Given two objects x and y of $\mathcal{C}$ t.f.a.e.:
  - there exists a Σ-zigzag between x and y
  - there exists z such that $x \xleftarrow{\Sigma} z \xrightarrow{\Sigma} y$
  - there exists z such that $x \xrightarrow{\Sigma} z \xleftarrow{\Sigma} y$
- When any of the following property is satisfied x and y are said to be Σ-connected
- Σ-connectedness is an equivalence relation on the objects of $\mathcal{C}$
- The equivalence classes are called a Σ-components

Structure of the Σ-components

Σ system of weak isomorphisms of $\mathcal{C}$ one-way category

A prelattice is a preordered set in which $x \wedge y$ and $x \vee y$ exist for all x and y.
However they are defined only up to isomorphism

Let K be a Σ-component of $\mathcal{C}$ and $\mathcal{K}$ be the full subcategory of $\mathcal{C}$ whose objects are the elements of K. The following properties are satisfied:

1. The category $\mathcal{K}$ is isomorphic with the preorder $(K, \preccurlyeq)$ where $x \preccurlyeq y$ stands for $\mathcal{C}[x, y] \neq \emptyset$. In particular, every diagram in $\mathcal{K}$ commutes.
2. The preordered set $(K, \preccurlyeq)$ is a prelattice.
3. If d and u are respectively a greatest lower bound and a least upper bound of the pair $\{x, y\}$, then Diagram 1 is both a pullback and a pushout in $\mathcal{C}$, and all the arrows appearing on the diagram belong to Σ.
4. $\mathcal{C} = \mathcal{K}$ iff $\mathcal{C}$ is a prelattice, and Σ is the greatest system of weak isomorphisms of $\mathcal{C}$ i.e. all the morphisms in this case.

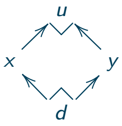


Diagram 1

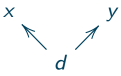


Diagram 2

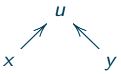
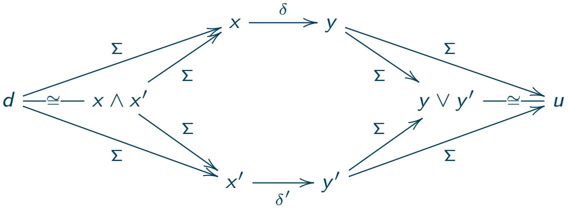


Diagram 3

Equivalent morphisms with respect to Σ

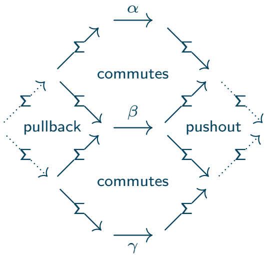
- Let $\delta \in \mathcal{C}(x, y)$ and $\delta' \in \mathcal{C}(x', y')$. Then write $\delta \sim \delta'$ when
  - $x \sim x'$ and $y \sim y'$, and
  - the inner hexagon of the next diagram commutes



  - Note that if $d \cong x \wedge x'$ and $u \cong y \vee y'$ then the outter hexagon also commutes, hence the relation $\sim$ is well defined.
- If $\gamma \sim \delta$ then $\partial^- \gamma \sim \partial^- \delta$ and $\partial^+ \gamma \sim \partial^+ \delta$

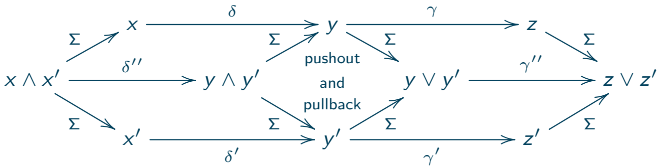
The relation $\sim$ is an equivalence

- The relation $\sim$ is:
  - reflexive since Σ contains all identities
  - symmetric by definition
  - transitive



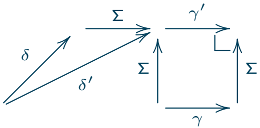
The relation $\sim$ fits with composition

- Suppose $\partial^- \gamma = \partial^+ \delta$, $\partial^- \gamma' = \partial^+ \delta'$ and $\gamma \sim \gamma'$ and $\delta \sim \delta'$.
- Then we have $\gamma \circ \delta \sim \gamma' \circ \delta'$



The category of components $\mathcal{C}/\Sigma$

- The quotient category $\mathcal{C}/\Sigma$ (obtained by turning each morphism of Σ into an identity) can be defined as follows:
  - The objects are the Σ-components
  - The morphisms are the $\sim$-equivalence classes
- If $\partial^- \gamma \sim \partial^+ \delta$ then
  - there exists γ' and δ' such that $\gamma' \sim \gamma$, $\delta' \sim \delta$, and $\partial^- \gamma' = \partial^+ \delta'$



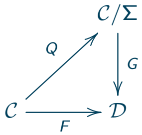
  - so we define $[\gamma] \circ [\delta] = [\gamma' \circ \delta']$
- We have the quotient functor $Q : \mathcal{C} \to \mathcal{C}/\Sigma$
- The category of components is $\mathcal{C}/\Sigma$ with Σ being the greatest swi of $\mathcal{C}$

Characterizing the identities of $\mathcal{C}/\Sigma$

For any morphism δ of $\mathcal{C}$ t.f.a.e.

- $\delta \in \Sigma$
- $[\delta] \subseteq \Sigma$
- $[\delta]$ is an identity of $\mathcal{C}/\Sigma$

The quotient functor $Q : \mathcal{C} \to \mathcal{C}/\Sigma$ satisfies the following universal property:
for all functors $F : \mathcal{C} \to \mathcal{D}$ s.t. $F(\Sigma) \subseteq \{\text{identities of } \mathcal{D}\}$
there exists a unique $G : \mathcal{C}/\Sigma \to \mathcal{D}$ s.t. $F = G \circ Q$



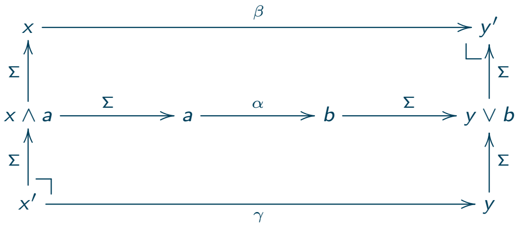
The fundamental properties of $\mathcal{C}/\Sigma$

with Σ being a system of weak isomorphisms of a one-way category $\mathcal{C}$

- The quotient functor $Q : \mathcal{C} \to \mathcal{C}/\Sigma$ is surjective on morphisms
- The quotient category $\mathcal{C}/\Sigma$ is loop-free
- If $\mathcal{C}(x, y) \neq \emptyset$ then the following map is a bijection.

$$\delta \in \mathcal{C}(x, y) \;\mapsto\; Q(\delta) \in \mathcal{C}/\Sigma\big(Q(x), Q(y)\big)$$

- If $\mathcal{C}/\Sigma\big(Q(x), Q(y)\big) \neq \emptyset$ then there exist x' and y' such that $\Sigma(x', x)$, $\Sigma(y, y')$, $\mathcal{C}(x', y)$, and $\mathcal{C}(x, y')$ are nonempty.

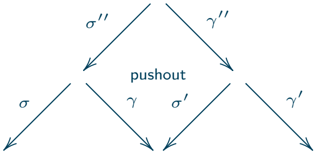


- The quotient functor Q preserves and reflects potential weak isomorphisms
- If $\mathcal{C}$ is finite then so is the quotient $\mathcal{C}/\Sigma$
- $\mathcal{C}$ is a preorder iff $\mathcal{C}/\Sigma$ is a poset

Describing the localization of $\mathcal{C}$ by Σ

with Σ being a system of weak isomorphisms of a one-way category $\mathcal{C}$

- The objects of $\mathcal{C}[\Sigma^{-1}]$ are the objects of $\mathcal{C}$
- The morphisms are the equivalence classes of ordered pairs of coinitial morphisms (γ, σ) with $\sigma \in \Sigma$,
  - Two pairs (γ, σ) and (γ', σ') being equivalent when $\partial^- \sigma = \partial^- \sigma'$, $\partial^- \gamma = \partial^- \gamma'$, and $Q(\gamma) = Q(\gamma')$
  - In the diagram below we have $Q(\gamma' \circ \gamma'') = Q(\gamma') \circ Q(\gamma'') = Q(\gamma') \circ Q(\gamma)$ hence the composite $(\gamma' \circ \gamma'', \sigma \circ \sigma'')$ neither depend on the choice of the pushout nor on the representatives (γ, σ) and (γ', σ').



The canonical comparison $P : \mathcal{C}[\Sigma^{-1}] \to \mathcal{C}/\Sigma$

with Σ being a system of weak isomorphisms of a one-way category $\mathcal{C}$

- Define I by $I(\gamma) := (\gamma, \mathrm{id}_{\partial^- \gamma})$ and the identity on objects
- Given a functor $F : \mathcal{C} \to \mathcal{D}$ s.t. $F(\Sigma) \subseteq \{\text{isomorphisms of } \mathcal{D}\}$ define
  - $G(x) := F(x)$ for all objects x of $\mathcal{C}[\Sigma^{-1}]$ and
  - $G(\gamma, \sigma) := F(\gamma) \circ (F(\sigma))^{-1}$ for any representative (γ, σ) of a morphism of $\mathcal{C}[\Sigma^{-1}]$
- The functor $I : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ then satisfies the universal property: for all functors $F : \mathcal{C} \to \mathcal{D}$ there exists a unique $G : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ s.t. $F = G \circ I$
- In particular there is a unique functor P s.t. $Q = P \circ I$ with $Q : \mathcal{C} \to \mathcal{C}/\Sigma$ and we have

The functor P is an equivalence of categories

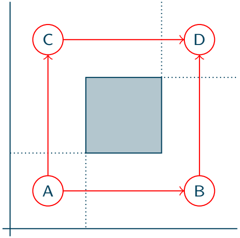
- The skeleton of $\mathcal{C}[\Sigma^{-1}]$ is $\mathcal{C}/\Sigma$ and $\mathcal{C}[\Sigma^{-1}]$ is one-way.

Embeding $\mathcal{C}/\Sigma$ into $\mathcal{C}$

- Let ϕ : Σ-components of $\mathcal{C} \to \mathrm{Ob}(\mathcal{C})$ such that
  - for all Σ-components K, K', if there exists $x \in K$ and $x' \in K'$ such that $\mathcal{C}(x, x') \neq \emptyset$, then $\mathcal{C}(\phi(K), \phi(K')) \neq \emptyset$
  - in this case $\mathcal{C}/\Sigma$ is isomorphic with the full subcategory of $\mathcal{C}$ whose set of objects is $\mathrm{im}(\phi)$.
  - the mapping ϕ is called an admissible choice (of canonical objects)
- Write $\phi \preccurlyeq \phi'$ when $\mathcal{C}(\phi(K), \phi'(K)) \neq \emptyset$ for all Σ-components K
  - The collection of admissible choice then forms a (pre)lattice
  - If $\mathcal{C}/\Sigma$ is finite then there exists an admissible choice
  - If $\mathcal{C}/\Sigma$ is infinite the existence of an admissible choice is a open question

Plane without a square

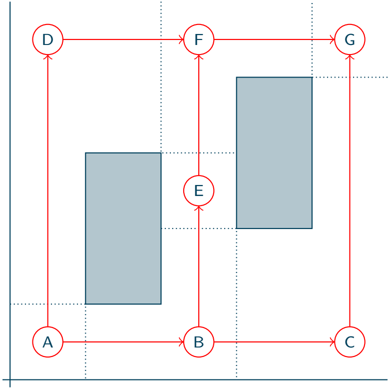
$x = \mathbb{R}^2_+ \setminus]0,1[^2$



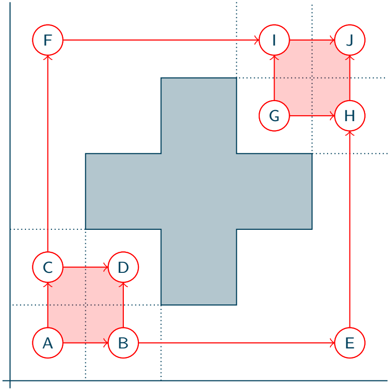
Let x, y such that $x \leqslant^2 y$, then $\overrightarrow{\pi_1}X(x,y)$ only depends on which elements of the partition x and y belong to

$\to$	A	B	C	D
A	σ	β	γ	$\beta' \circ \beta$ $\alpha' \circ \alpha$
B		σ		β'
C			σ	γ'
D				σ

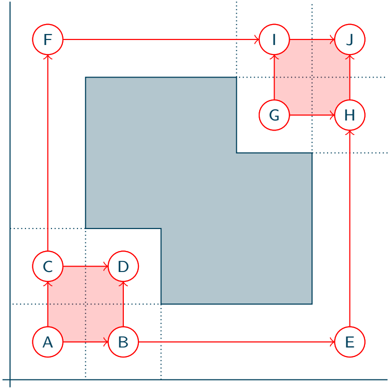
Two rectangles



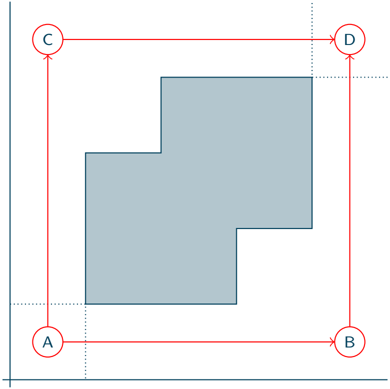
Swiss Flag



Achronal overlaping square

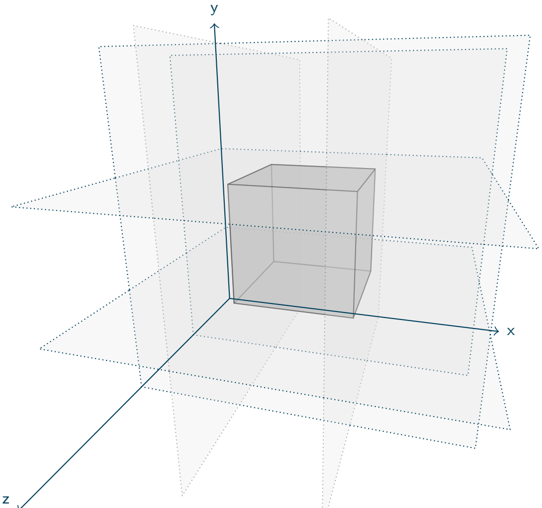


Diagonal overlaping squares



The floating cube

boundaries of the components



Commutative monoid

of nonempty finite connected loop-free categories

- The Cartesian product of categories $\mathcal{A} \times \mathcal{B}$ is non-empty iff so are $\mathcal{A}$ and $\mathcal{B}$.
  If $\mathcal{A}$ and $\mathcal{B}$ are indeed nonempty then we also have
  - $\mathcal{A} \times \mathcal{B}$ finite iff so are $\mathcal{A}$ and $\mathcal{B}$
  - $\mathcal{A} \times \mathcal{B}$ connected iff so are $\mathcal{A}$ and $\mathcal{B}$
  - $\mathcal{A} \times \mathcal{B}$ loop-free iff so are $\mathcal{A}$ and $\mathcal{B}$
- $\mathcal{A} \cong \mathcal{A}'$ and $\mathcal{B} \cong \mathcal{B}'$ implies $\mathcal{A} \times \mathcal{A}' \cong \mathcal{B} \times \mathcal{B}'$
- $(\mathcal{A} \times \mathcal{B}) \times \mathcal{C} \cong \mathcal{A} \times (\mathcal{B} \times \mathcal{C})$
- $1 \times \mathcal{A} \cong \mathcal{A} \cong \mathcal{A} \times 1$
- $\mathcal{A} \times \mathcal{B} \cong \mathcal{B} \times \mathcal{A}$
- The collection of isomorphism classes of nonempty finite connected loop-free categories is thus a commutative monoid $\mathcal{M}$

The commutative monoid $\mathcal{M}$ is free.

Criteria for primality

- The monoid $\mathcal{M}$ is graded by the following morphisms
  - $\#Ob : \mathcal{C} \in \mathcal{M} \mapsto \mathrm{card}(\mathrm{Ob}(\mathcal{C})) \in (\mathbb{N} \setminus \{0\}, \times, 1)$
  - $\#Mo : \mathcal{C} \in \mathcal{M} \mapsto \mathrm{card}(\mathrm{Mo}(\mathcal{C})) \in (\mathbb{N} \setminus \{0\}, \times, 1)$
  - $\#Mo(\mathcal{C}) \geqslant 2 \times \#Ob(\mathcal{C}) - 1$, for all $\mathcal{C} \in \mathcal{M}$
- In particular if $\#Ob(\mathcal{C})$ or $\#Mo(\mathcal{C})$ is prime, then so is $\mathcal{C}$. The converse is false.
- Any element of $\mathcal{M}$ freely generated by a graph, is prime

Comparing decompositions

- The mapping $\mathcal{C} \in \mathcal{M} \mapsto \overrightarrow{\pi_0}(\mathcal{C}) \in \mathcal{M}$ is a morphism of monoids
- We would like to know which prime element of $\mathcal{M}$ are preserved by it
- We known that $\overrightarrow{\pi_0}(\mathcal{C})$ is null iff $\mathcal{C}$ is a lattices (e.g. $\overrightarrow{\pi_0}(0 < 1) = \{0\}$ though $\{0 < 1\}$ is prime in $\mathcal{M}$)
- For all d-spaces X and Y, $\overrightarrow{\pi_1}(X \times Y) \cong \overrightarrow{\pi_1}X \times \overrightarrow{\pi_1}Y$
- Hence $\mathcal{N}' := \{X \in \mathcal{H}_f\upharpoonleft G \downharpoonright \mid \overrightarrow{\pi_1}X \text{ is nonempty, connected, and loop-free}\}$
  is a pure submonoid of $\mathcal{H}_f\upharpoonleft G \downharpoonright$
- Then $\mathcal{N} := \{X \in \mathcal{N}' \mid \overrightarrow{\pi_0}(\overrightarrow{\pi_1}X) \text{ is finite}\}$ is a pure submonoid of $\mathcal{N}'$
- Therefore it is free commutative and we would like to know which prime elements are preserved by
  $X \in \mathcal{N} \mapsto \overrightarrow{\pi_0}(\overrightarrow{\pi_1}X) \in \mathcal{M}$
- Conjecture

> If $P \in \mathcal{N}$ is prime and $\overrightarrow{\pi_1}(P)$ is not a lattice, then $\overrightarrow{\pi_0}(\overrightarrow{\pi_1}(P))$ is prime