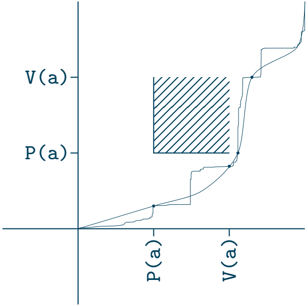
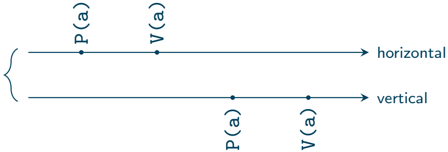
```
sem 1 a
proc:  p = P(a);V(a)
init:  2p
```





$$\text{blowup}\left\{\begin{array}{ccccc} \|P\| \subseteq & \|G_1\| & \times \cdots \times & \|G_n\| \\ & \big\downarrow \beta_1 & \times \cdots \times & \big\downarrow \beta_n \\ |P| \subseteq & |G_1| & \times \cdots \times & |G_n| \end{array}\right.$$

$$\underbrace{|G_1| \times \cdots \times |G_n|}_{\text{sets}}$$

$$\overbrace{\mathcal{E}_1 \times \cdots \times \mathcal{E}_n}^{\text{euclidean ordered bases}} \rightsquigarrow \overbrace{(\mathcal{A}_1, f_1) \times \cdots \times (\mathcal{A}_n, f_n)}^{\text{parallelized atlas}}$$

$$\begin{array}{ccc} \mathcal{E}_1 & \times \cdots \times & \mathcal{E}_n \\ \big\downarrow \beta_1 & \times \cdots \times & \big\downarrow \beta_n \\ \mathcal{X}_1 & \times \cdots \times & \mathcal{X}_n \end{array}$$

$$\underbrace{\mathcal{X}_1 \times \cdots \times \mathcal{X}_n}_{\text{ordered bases}}$$

$$\underbrace{G_1 \;,\; \ldots \;,\; G_n}_{\text{graphs}}$$

$$\underbrace{P_1 \;|\; \cdots \;|\; P_n}_{\text{program } P}$$

Cartesian product in a category $\mathcal{C}$

The object c is the Cartesian product (in $\mathcal{C}$) of a and b when there exist two morphisms $\pi_a : c \to a$ and $\pi_b : c \to b$ such that for all objects x of $\mathcal{C}$ the following map is a bijection

$$\mathcal{C}[x, c] \longrightarrow \mathcal{C}[x, a] \times \mathcal{C}[x, b]$$

$$h \longmapsto (\ \pi_a \circ h\ ,\ \pi_b \circ h\ )$$

When such an object c exists we write $c = a \times b$

Cartesian product in the category of graphs ($\mathcal{Grph}$)

$$\left(\begin{array}{c} A \\ \mathsf{t}\Big\downarrow\Big\downarrow\mathsf{s} \\ V \end{array}\right) \times \left(\begin{array}{c} A' \\ \mathsf{t}'\Big\downarrow\Big\downarrow\mathsf{s}' \\ V' \end{array}\right) \cong \left(\begin{array}{c} A \times A' \\ \mathsf{t}\times\mathsf{t}'\Big\downarrow\Big\downarrow\mathsf{s}\times\mathsf{s}' \\ V \times V' \end{array}\right)$$

The Cartesian product in $\mathcal{Grph}$ is deduced form the Cartesian product in $\mathcal{Set}$

Examples of Cartesian products

- The product of (X, Ω_X) and (Y, Ω_Y) in $\mathcal{Top}$ is given by $X \times Y$ together with unions of subsets of the form $U \times V$ with $U \in \Omega_X$ and $V \in \Omega_Y$. It is the least topology making the projections continuous.
- The product of $(X, \sqsubseteq_X)$ and $(Y, \sqsubseteq_Y)$ in $\mathcal{Pos}$ is given by $X \times Y$ and the partial order $\sqsubseteq$ defined by $(x, y) \sqsubseteq (x', y')$ when $x \sqsubseteq_X x'$ and $y \sqsubseteq_Y y'$. It is the greatest partial order such that the projection are poset morphisms.
- The product of $(X, \sqsubseteq_X)$ and $(Y, \sqsubseteq_Y)$ in $\mathcal{PoSp}$ is given by $X \times Y$ and the product order $\sqsubseteq_X \times \sqsubseteq_Y$.
- The product of $(X, [\mathcal{U}]_\sim)$ and $(Y, [\mathcal{V}]_\sim)$ in $\mathcal{Lpo}$ is given by $X \times Y$ together with the collection of ordered charts $U \times V$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$.
- The product of (X, d_X) and (Y, d_Y) in $\mathcal{Met}_{emb}$ does not exist.
- The product of (X, d_X) and (Y, d_Y) in $\mathcal{Met}_{ctr}$ is given by $X \times Y$ together with $d\big((x, y), (x', y')\big) = \max\{d_X(x, x'), d_Y(y, y')\}$.
- The product of (X, d_X) and (Y, d_Y) in $\mathcal{Met}_{top}$ can also be given by $X \times Y$ together with the Euclidean product

$$d\big((x, y), (x', y')\big) = \sqrt{d_X^2(x, x') + d_Y^2(y, y')}$$

- Categories of models of algebraic theories.

Infinite Cartesian product

The product of a family $(A_i)_{i \in \mathcal{I}}$ of objects of a category $\mathcal{C}$, when it exists, is an object

$$\prod_i A_i$$

together with projections

$$\pi_{A_j} : \prod_i A_i \longrightarrow A_j$$

such that the next mapping is a bijection.

$$\begin{array}{ccc} \mathcal{C}(X, \prod_i A_i) & \longrightarrow & \prod_i \mathcal{C}(X, A_i) \\ h & \longmapsto & (\ \pi_{A_i} \circ h\ ) \end{array}$$

Infinite products of directed circle does not exist in $\mathcal{L}po$.

Canonical partition

$$G : A \underset{\partial^-}{\overset{\partial^+}{\rightrightarrows}} V \qquad\qquad \lceil G \rfloor \;=\; V \;\sqcup\; A \times]0,1[$$

$$\lceil G_1 \rfloor \times \cdots \times \lceil G_n \rfloor \;=\; (\; V_1 \sqcup A_1 \times]0,1[\;) \;\times\; \cdots \;\times\; (\; V_n \sqcup A_n \times]0,1[\;)$$

$$\lceil G_1 \rfloor \times \cdots \times \lceil G_n \rfloor \;=\; \bigsqcup_{\substack{\text{points } p \text{ of} \\ G_1,\ldots,G_n}} \{p\} \times]0,1[^{\,\dim(p_1,\ldots,p_n)}$$

where $p = (p_1, \ldots, p_n)$, $p_i \in V_i \sqcup A_i$, and $\dim p = \#\{i \in \{1, \ldots, n\} \mid p_i \in A_i\}$

$B_p = \{p\} \times]0,1[^{\,\dim(p_1,\ldots,p_n)}$ is called a canonical block

The collection of canonical blocks forms the canonical partition of $\lceil G_1 \rfloor \times \cdots \times \lceil G_n \rfloor$.

The geometric model of a conservative program

The forbidden region of a conservative program $\Pi = (G_1, \ldots, G_n)$ is the disjoint union of canonical blocks

$$\bigsqcup_{\substack{\text{forbidden points } p \\ \text{of } (G_1, \ldots, G_n)}} B_p$$

The geometric model of Π is the locally ordered metric space

$$\upharpoonleft G_1 \downharpoonright \times \cdots \times \upharpoonleft G_n \downharpoonright \setminus \{\text{forbidden region}\}$$

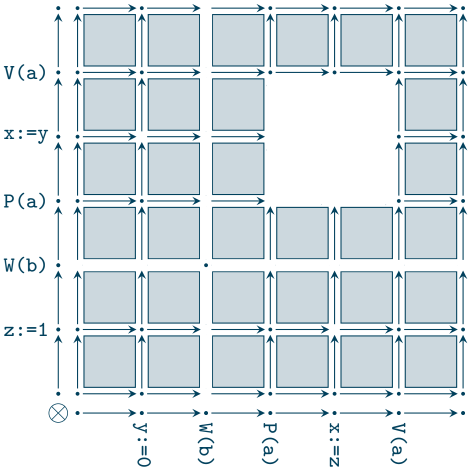
the distance being given by

$$d(p, p') \quad = \quad \max\left\{ d_{\upharpoonleft G_i \downharpoonright}(p_i, p'_i) \mid i \in \{1, \ldots, n\} \right\}$$

in accordance with the fact that the execution time of the simultaneous execution of many processes is the longest execution time among that of the processes considered individually.

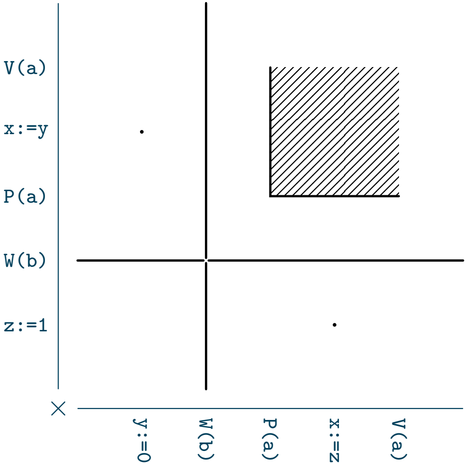
From discrete to continuous

`sem: 1 a        sync: 1 b`



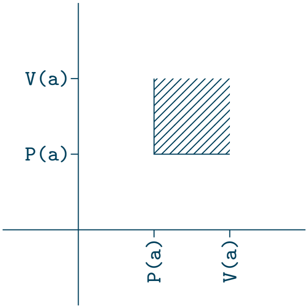
From discrete to continuous

sem: 1 a   sync: 1 b



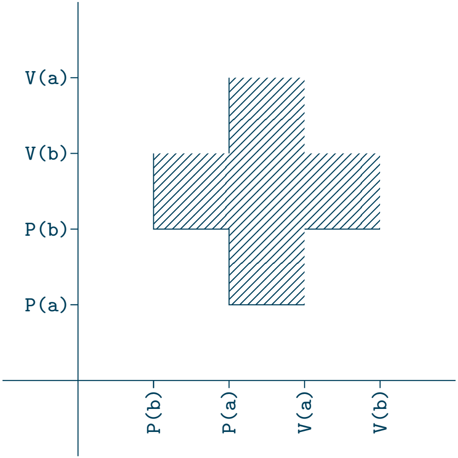
Square

```
sem 1 a
proc:  p = P(a);V(a)
init:  2p
```



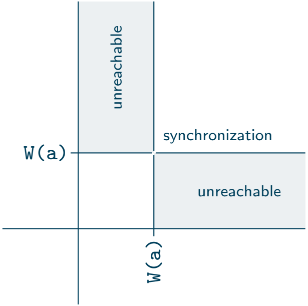
Swiss Cross

```
sem 1 a b
proc:
p = P(a);P(b);V(b);V(a)
q = P(b);P(a);V(a);V(b)
init:  p q
```



Binary synchronization

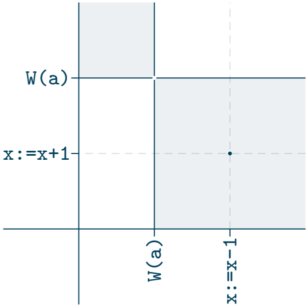
```
sync 1 a
proc:  p = W(a)
init:  2p
```



Producer/Consumer

nonlooping

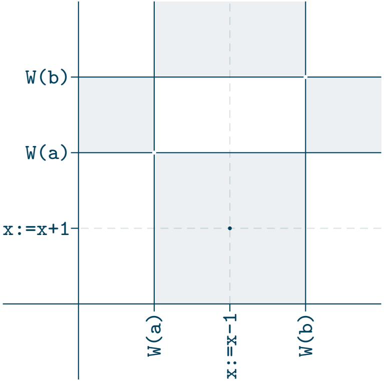
```
sync 1 a
proc:
  p = x:=x+1 ; W(a)
  c = W(a)   ; x:=x-1
init:  p c
```



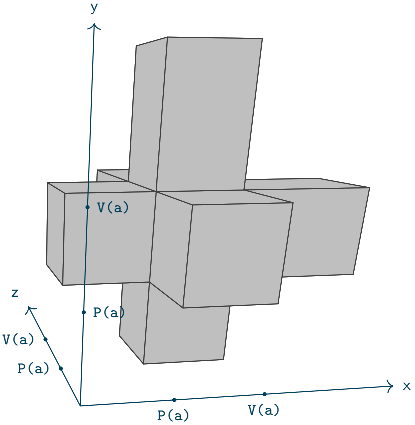
Producer/Consumer

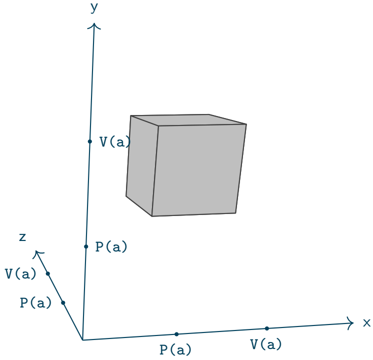
looping

```
sync 1 a b
proc:
  p = x:=x+1 ; W(a)    ; W(b) ; J(p)
  c = W(a)   ; x:=x-1 ; W(b) ; J(c)
init:  p c
```

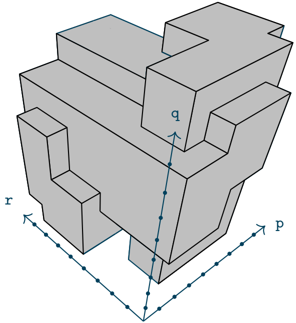


3D Swiss Cross (tetrahemihexacron) and floating cube





The Lipski algorithm




```
sem 1:  u v w x y z
proc:
  p = P(x);P(y);P(z);V(x);P(w);V(z);V(y);V(w)
  q = P(u);P(v);P(x);V(u);P(z);V(v);V(x);V(z)
  r = P(y);P(w);V(y);P(u);V(w);P(v);V(u);V(v)
init:  p q r
```

Justifying de definition of discrete directed paths

Let B_p and $B_{p'}$ be canonical blocks.

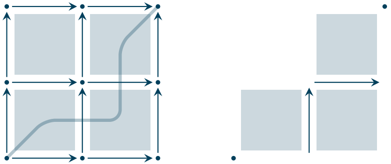
If there exists a directed path starting in B_p, ending in $B_{p'}$, and whose image is contained in $B_p \cup B_{p'}$ then one of the following facts is satisfied:

- for all $i \in \{1, \ldots, n\}$, $p_i = p'_i$ or p_i is the source of the arrow p'_i, or
- for all $i \in \{1, \ldots, n\}$, $p_i = p'_i$ or p'_i is the target of the arrow p_i.

Discretization and lifting

- Given a directed path γ on the local pospace $\upharpoonleft G_1 \downharpoonleft \times \cdots \times \upharpoonleft G_n \downharpoonleft$ we have a finite partition $I_0 < \cdots < I_N$ of $\mathrm{dom}(\gamma)$ such that for all $k \in \{0, \ldots, N\}$, there exists a (necessarily unique) point p^k such that $\gamma(I_k) \subseteq B_{p^k}$.
- The sequence $p^0, \ldots, p^N$ is a directed path on $(G_1, \ldots, G_n)$, it is called the discretization of γ and denoted by $D(\gamma)$.
- Given a directed path δ on $(G_1, \ldots, G_n)$ there exists a directed path γ on $\upharpoonleft G_1 \downharpoonleft \times \cdots \times \upharpoonleft G_n \downharpoonleft$ whose discretization is δ, such a directed path γ is said to be a lifting of δ.

Example of discretization



Admissible directed paths and execution traces

on $\upharpoonleft G_1 \downharpoonright \times \cdots \times \upharpoonleft G_n \downharpoonright$

The sequence of multi-instructions of a directed path γ on $\upharpoonleft G_1 \downharpoonright \times \cdots \times \upharpoonleft G_n \downharpoonright$ is that of its discretization of $D(\gamma)$.

A directed path on $\upharpoonleft G_1 \downharpoonright \times \cdots \times \upharpoonleft G_n \downharpoonright$ is admissible (resp. an execution trace) iff so is its discretization.

The action of a directed path γ on $\upharpoonleft G_1 \downharpoonright \times \cdots \times \upharpoonleft G_n \downharpoonright$ on the right of a state σ is that of its discretization of $D(\gamma)$.

Example

```
var x = 0
var y = 0
var z = 0
sync 1 b
sem 1 a
```

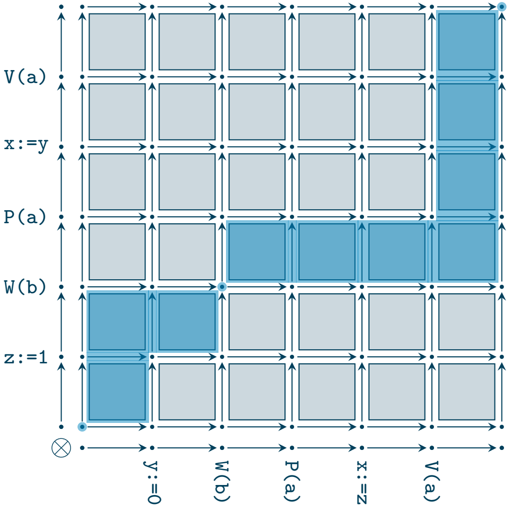
```
proc p = y:=0 ; W(b) ; P(a) ; x:=z ; V(a)

proc q = z:=1 ; W(b) ; P(a) ; x:=y ; V(a)
```

```
init p q
```

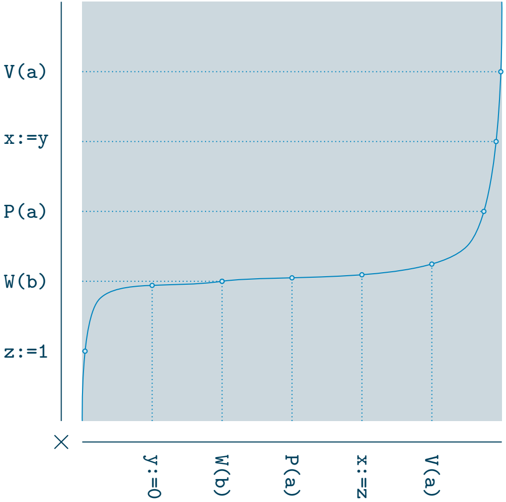
Discretization of an execution trace

`sem: 1 a` `sync: 1 b`



Discretization of an execution trace

`sem: 1 a`   `sync: 1 b`



Potential function on $\upharpoonleft G_1 \downharpoonright \times \cdots \times \upharpoonleft G_n \downharpoonright$

If the program under consideration is conservative, then we have the potential function

$$F : \upharpoonleft G_1 \downharpoonright \times \cdots \times \upharpoonleft G_n \downharpoonright \times \mathcal{S} \to \{\text{multisets over } \{1, \ldots, n\}\}$$

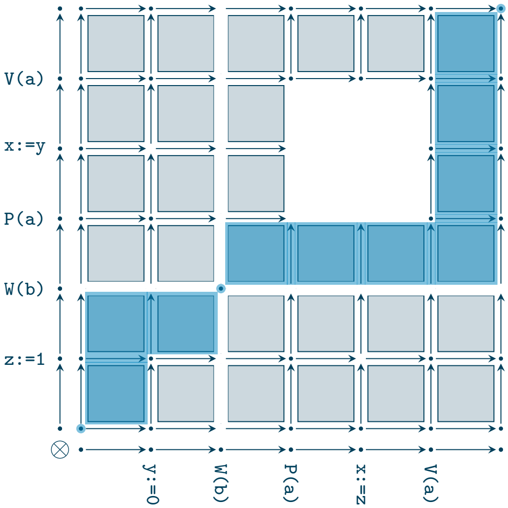
The function F is constant on each canonical block B_p, its value is given by $\tilde{F}(p)$ where $\tilde{F}$ denotes the "discrete" potential function.

Geometric models are sound and complete

- Any directed path on a continuous model is admissible.
- Conversely, for each admissible path on a continuous model which meets a forbidden point, there exists a directed path which avoids them and such that both directed paths induce the same sequence of multi-instructions.

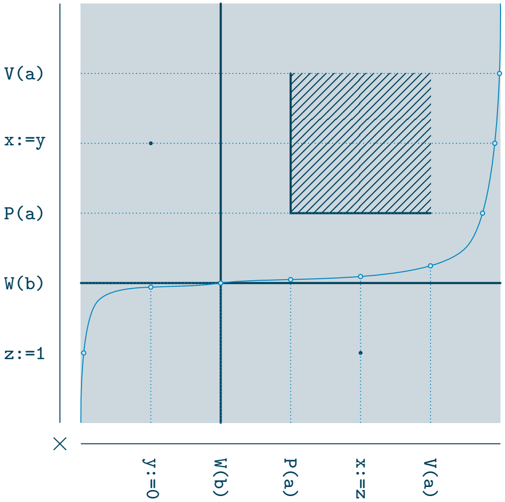
Directed paths on the geometric model are admissible

sem: 1 a   sync: 1 b



Directed paths on the geometric model are admissible

`sem: 1 a` `sync: 1 b`



Trade off

More mathematics for more properties?

- Both discrete and geometric models are sound and complete.
- The continuous models satisfy extra properties that are “naturally” expressed in terms of metrics.

Uniform distance between directed paths

Given a compact Hausdorff space K and a metric space (X, d_X), the set of continuous maps from K to X can be equipped with the uniform distance

$$d(f,g) \quad = \quad \max\{d_X(f(k),g(k)) \mid k \in K\} \ .$$

We consider the case where $K = [0,r]$ is the domain of definition of a directed path and (X, d_X) is the geometric model of a conservative program.

The main theorem

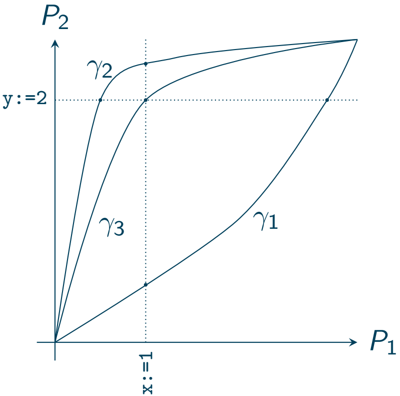
Let B_p and $B_{p'}$ be canonical blocks of the geometric model X of a conservative program.

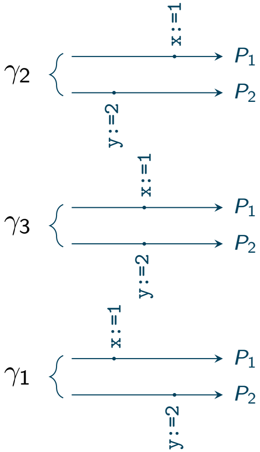
Let $dX^{[0,r]}(B_p, B_{p'})$ be the set of directed paths on X whose sources and targets lie in B_p and $B_{p'}$ respectively.

Let γ be an element of $dX^{[0,r]}(B_p, B_{p'})$.

There exists an open ball Ω of $dX^{[0,r]}(B_p, B_{p'})$, centred in γ, such that all the elements of Ω induce the same action on valuations. Moreover, if γ is an execution trace, then so are all the elements of Ω.

Illustration





Homotopy of paths

Let γ and δ be two paths on X defined over the segment $[0, r]$

A homotopy from γ to δ is a continuous map h from $[0, r] \times [0, q]$ to X such that

- The mappings $h(0, -) : [0, q] \to X$ and $h(r, -) : [0, q] \to X$ are constant
- The mappings $h(-, 0) : [0, r] \to X$ and $h(-, q) : [0, r] \to X$ are γ and δ

As a consequence we have $\gamma(0) = \delta(0)$ and $\gamma(r) = \delta(r)$.

Uniform distance and Curryfication

Suppose that X is a metric space.

For all compact Hausdorff space K, the homset $\mathcal{Top}(K,X)$ with the (topology induced by the) uniform distance is denoted by X^K

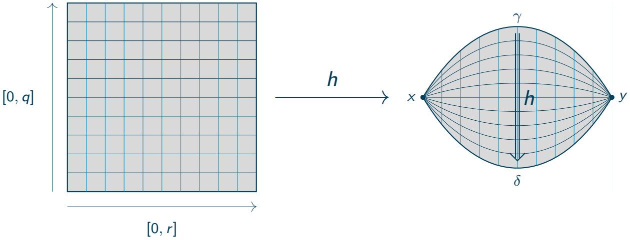
The Curryfication $(\hat{\_})$ induces a homeomorphism from $X^{[0,r]\times[0,q]}$ to $\left(X^{[0,r]}\right)^{[0,q]}$

$$(h : [0,r] \times [0,q] \to X) \to (\hat{h} : [0,q] \to X^{[0,r]})$$

The two faces of homotopies

h is a continuous map from $[0,r] \times [0,q]$ to X i.e. $h \in \mathcal{Top}\big[[0,r] \times [0,q], X\big]$

but is also a path from γ to δ in the space $X^{[0,r]}$ i.e. $h \in \mathcal{Top}\big[[0,q], X^{[0,r]}\big]$



We introduce the following notation

Concatenation of homotopies

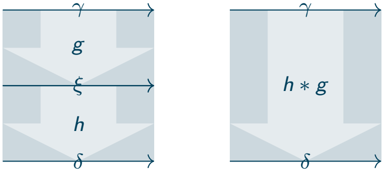
vertical composition

Let $g : [0, r] \times [0, q'] \to X$ and $h : [0, r] \times [0, q] \to X$ be homotopies from γ to ξ and from ξ to δ.

The mapping $h * g : [0, r] \times [0, q + q'] \to X$ defined by

$$h * g(t, s) = \begin{cases} g(t, s) & \text{if } 0 \leqslant s \leqslant q \\ h(t, s - q) & \text{if } q \leqslant s \leqslant q + q' \end{cases}$$

is a homotopy from γ to δ.



Directed homotopy on a locally ordered space

Let $\gamma, \delta \in \mathcal{Lpo}([0,r], X)$ such that $\partial^- \gamma = \partial^- \delta$ and $\partial^+ \gamma = \partial^+ \delta$.

- A directed homotopy from γ to δ is a local pospace morphism $h : [0,r] \times [0,q] \to X$ whose underlying map $U(h)$ is a homotopy from $U(\gamma)$ to $U(\delta)$.
- An anti-directed homotopy from γ to δ is a homotopy of paths $h : [0,r] \times [0,q] \to X$ such that $(t,s) \mapsto h(t, q-s)$ is a directed homotopy from δ to γ.
- An elementary homotopy between γ to δ is a homotopy of paths $h : [0,r] \times [0,q] \to X$ obtained as a finite concatenation of directed homotopies and anti-directed homotopies.
- A weakly directed homotopy from γ to δ is a homotopy of paths $h : [0,r] \times [0,q] \to X$ whose intermediate paths $h(\_, s)$, for $s \in [0,q]$, are directed.
- Any elementary homotopy is a weakly directed homotopy. The converse is false.
- Each of the preceding class of homotopies is stable under concatenation.

Homotopy and dihomotopy relations

Two paths γ and γ' are said to be homotopic when there exists a homotopy between them.
We have the equivalence relation $\sim_h$ between paths on a topological space.

They are said to be dihomotopic when there exists an elementary homotopy between them.
We have the equivalence relation $\sim_d$ between directed paths on a locally ordered space.

They are said to be weakly dihomotopic when there exists a weakly directed homotopy between them. We have the equivalence relation $\sim_w$ between directed paths on a locally ordered space.

Reparametrization

An increasing and surjective map $\theta : [0, r] \to [0, r]$ is called a reparametrization.
The mapping

$$h : (t, s) \in [0, r] \times [0, 1] \mapsto \theta(t) + s \cdot (\max(t, \theta(t)) - \theta(t)) \in [0, r]$$

is a directed homotopy from θ to $\max(\mathrm{id}_{[0,r]}, \theta)$.

If $\gamma : [0, r] \to X$ is a directed path on the local pospace X, then $\gamma \circ h$ is a directed homotopy from $\gamma \circ \theta$ to $\gamma \circ \max(\mathrm{id}_{[0,r]}, \theta)$

Therefore γ and $\gamma \circ \theta$ are dihomotopic.

Images of directed paths on a pospace

Theorem

The image of a nonconstant directed path on a pospace is isomorphic to $[0, 1]$.

Corollary

Two directed paths on a posapce having the same image are dihomotopic.

proof:
Suppose that $\text{im}(\gamma) = \text{im}(\gamma')$.
$\phi : [0, r] \to \text{im}(\gamma)$ a pospace isomorphism.
$\phi^{-1} \circ \gamma$ and $\phi^{-1} \circ \gamma'$ are reparametrization.
We have h an elementary homotopy from $\phi^{-1} \circ \gamma$ to $\phi^{-1} \circ \gamma'$.
Hence $\phi \circ h$ is an elementary homotopy from γ and γ'.

Main theorem

Two weakly dihomotopic paths on the geometric model of a conservative program induce the same action on valuations. Moreover, if one of them is an execution trace, then so is the other.

Proof

By a standard result from general topology, the Curryfication of h

$$\hat{h} : s \in [0, q] \mapsto (t \in [0, r] \mapsto h(t, s) \in X)$$

is a continuous path on $dX^{[0,r]}(p, p')$.

The image of $\hat{h}$ is thus compact, so we cover it with open balls given by the main theorem of geometric models.

By the Lebesgue number theorem there exists a real number $\varepsilon > 0$ such that $|s - s'| \leqslant \varepsilon$ implies that $\hat{h}(s)$ and $\hat{h}(s')$ belong to the same open ball from the covering.

The conclusion follows considering the sequence

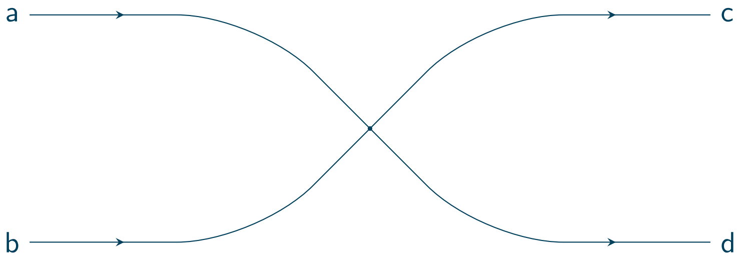
$$\hat{h}(0), \hat{h}(\varepsilon), \hat{h}(2\varepsilon), \hat{h}(3\varepsilon), \cdots, \hat{h}(n\varepsilon), \hat{h}(q)$$

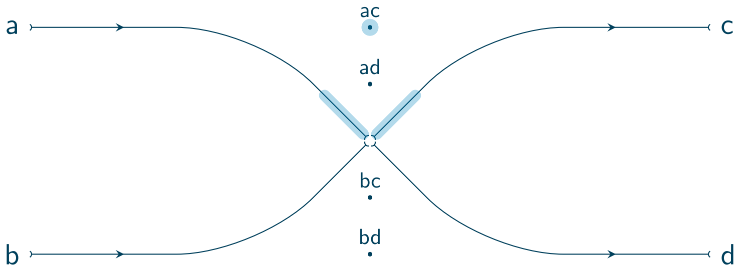
where n is the greatest natural number such that $n\varepsilon \leqslant q$.

Programs with mutex only

Directed Homotopy in Non-Positively Curved Spaces, *É. Goubault and S. Mimram*, LMCS 2020

Let X be the geometric model of a conservative program whose semaphores have arity 1 (mutex), then two directed paths on X are dihomotopic if and only if they are homotopic.


a
b
c
d


a
b
ac
ad
bc
bd
c
d

$$G = \left(G^{(1)} \underset{src}{\overset{tgt}{\rightrightarrows}} G^{(0)} \right) \quad : \quad \text{graph}$$

$$\|G\| = \left(G^{(1)} \times]0,1[\right) \cup \{(a,b) \in G^{(1)} \times G^{(1)} \mid \partial^+(a) = \partial^-(b)\} \quad : \quad \text{set}$$

For small $\varepsilon > 0$, the ε-neighborhoods of (a, t) and (a, b) are

$$\begin{cases} \{a\} \times]t-\varepsilon, t+\varepsilon[& (\text{for } \varepsilon \leq \min\{t, 1-t\}) \\ \{a\} \times]1-\varepsilon, 1[\ \cup \ \{(a,b)\} \cup \{b\} \times]0, \varepsilon[& (\text{for } \varepsilon \leq \frac{1}{2}) \end{cases}$$

The *standard ordered base* $\mathcal{E}_G$ of G is the collection of ε-neighborhoods (each of them being equipped with the obvious total order).

The *blowup* of G is the map

$$\begin{array}{rcccl} \beta_G & : & \|G\| & \to & |G| \\ & & (a,b) & \mapsto & \partial^+(a)(=\partial^-(b)) \\ & & (a,t) & \mapsto & (a,t) \end{array}$$

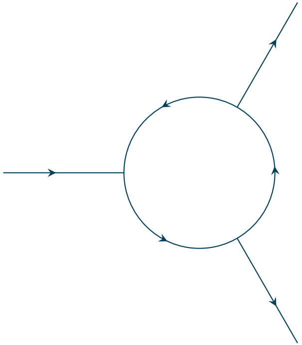
The blowup β_G is locally order-preserving from $\mathcal{E}_G$ to $\mathcal{X}_G$.

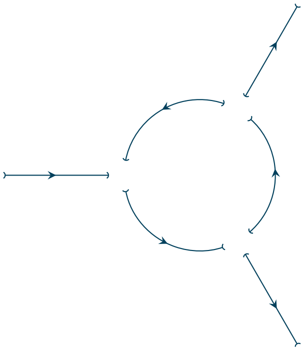
An ordered base $\mathcal{E}$ is said to be *euclidean* of dimension $n \in \mathbb{N}$ when every point p of $\mathcal{E}$ is contained in some $E \in \mathcal{E}$ with $E \cong \mathbb{R}^n$ (as ordered spaces).

A locally order-preserving map $f : \mathcal{E} \to \mathcal{X}$ is a *local $\vee$-embedding* when for every point p of $\mathcal{E}$ and $X \in \mathcal{X}$ containing $f(p)$, there exists $E \in \mathcal{E}$ containing p such that $E \cong \mathbb{R}^n$ and $f : E \to X$ is an ordered space embedding preserving $\vee$.

Theorem (Universal property of graph blowups)

For every euclidean ordered base $\mathcal{E}$, and every local $\vee$-embedding $f : \mathcal{E} \to \mathcal{X}_{G_1} \times \cdots \times \mathcal{X}_{G_n}$ of dimension n, there is a unique continuous map $g : \mathcal{E} \to \mathcal{E}_{G_1} \times \cdots \times \mathcal{E}_{G_n}$ such that $f = \bar{\beta} \circ g$ with $\bar{\beta} = \beta_{G_1} \times \cdots \times \beta_{G_n}$; moreover g is a local $\vee$-embedding of dimension n.

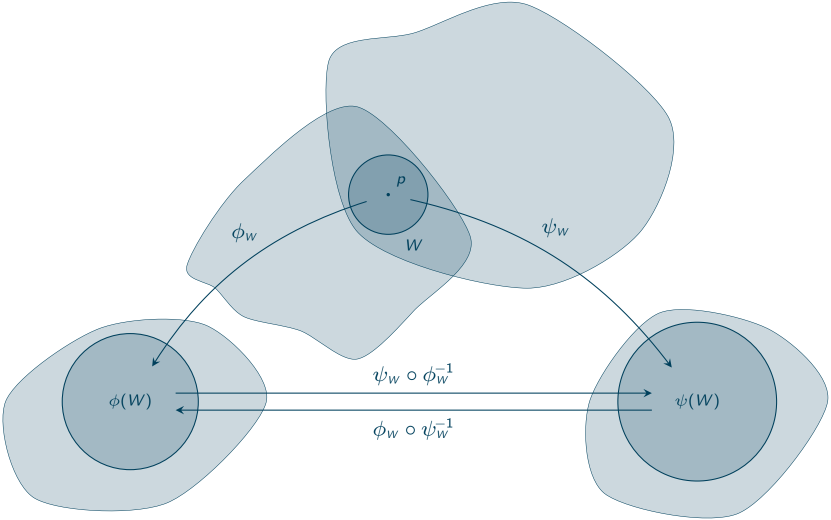




A *chart* of dimension $n \in \mathbb{N}$ is a bijection ϕ whose codomain is an open subset of $\mathbb{R}^n$.

$U \subseteq \mathrm{dom}(\phi)$ is said to be *open* when so is $\phi(U)$ in $\mathbb{R}^n$; we deduce $\phi_U : U \to \phi(U)$.

The n-charts ϕ and ψ are *compatible at $p \in \operatorname{dom}(\phi) \cap \operatorname{dom}(\psi)$* when there exists W open in $\operatorname{dom}(\phi)$ and in $\operatorname{dom}(\psi)$ such that $\phi_W \circ \psi_W^{-1}$ and $\psi_W \circ \phi_W^{-1}$ are smooth.
We say that W is a *witness of compatibility* of ϕ and ψ at p.



The n-charts ϕ and ψ are *compatible* when they are compatible at every $p \in \mathrm{dom}(\phi) \cap \mathrm{dom}(\psi)$.

$$\Updownarrow$$

$W = \mathrm{dom}(\phi) \cap \mathrm{dom}(\psi)$ is open in $\mathrm{dom}(\phi)$ and in $\mathrm{dom}(\psi)$ and the maps $\phi_W \circ \psi_W^{-1}$ and $\psi_W \circ \phi_W^{-1}$ are smooth.

An *atlas* of dimension $n \in \mathbb{N}$ is a collection $\mathcal{A}$ of pairwise compatible n-charts.

Given atlases $\mathcal{A}$, $\mathcal{B}$, map $f : \mathcal{A} \to \mathcal{B}$ is said to be *smooth* when for all $\phi \in \mathcal{A}$, $p \in \mathrm{dom}(\phi)$, $\psi \in \mathcal{B}$ with $f(p) \in \mathrm{dom}(\psi)$, $\psi \circ f \circ \phi^{-1}$ is smooth (as a map between open subsets of euclidean spaces).

The *standard charts* of G are the following bijections

$$\begin{aligned} \phi_a &: \{a\} \times]0,1[\to]0,1[\;, \quad \text{and} \\ \phi_{ab} &: \{a\} \times]\tfrac{1}{2},1[\cup \{(a,b)\} \cup \{b\} \times]0,\tfrac{1}{2}[\to]-\tfrac{1}{2},\tfrac{1}{2}[\\ &\text{with} \quad (a,t) \mapsto t-1 \;, \quad (a,b) \mapsto 0 \;, \quad (b,t) \mapsto t \end{aligned}$$

for all arrows a and all 2-tuples of arrows (a, b) such that $\partial^+(a) = \partial^-(b)$.

The *standard atlas* $\mathcal{A}_G$ of G is the collection of its standard charts.

The *transition maps* are translations:

$$\begin{aligned} \phi_{ab} \circ \phi_a^{-1} &: t \in]\tfrac{1}{2},1[\mapsto t-1 \in]-\tfrac{1}{2},0[\\ \phi_{ab} \circ \phi_b^{-1} &: t \in]0,\tfrac{1}{2}[\mapsto t \in]0,\tfrac{1}{2}[\end{aligned}$$

The set of *tangent vectors* of $\mathcal{A}$ is the quotient

$$\{(p, \phi, u) \mid \phi \in \mathcal{A};\ p \in \mathrm{dom}(\phi);\ u \in \mathbb{R}^n\}/\sim$$

with $(p, \phi, u) \sim (q, \psi, v)$ when $p = q$ and $d(\psi_W \circ \phi_W^{-1})_{\phi(p)}(u) = v$ (with W a witness of compatibility of ϕ and ψ at p).
Denote by $[\![p, \phi, u]\!]$ the $\sim$-equivalence class of (p, ϕ, u).

We have $(p, \phi, u) \sim (p, \phi, v) \Rightarrow u = v$, and the collection $T\mathcal{A} = \{T\phi \mid \phi \in \mathcal{A}\}$ with $T\phi[\![p, \phi, u]\!] = (\phi(p), u)$ is an atlas.

The *tangent bundle* of $\mathcal{A}$ is the smooth map $\pi_{\mathcal{A}} : T\mathcal{A} \to \mathcal{A}$ sending a tangent vector to its attachment point;
i.e. $\pi_{\mathcal{A}}([\![p, \phi, u]\!]) = p$.

The *tangent space* at p is $T_p\mathcal{A} = \pi_{\mathcal{A}}^{-1}(\{p\})$; it is a vector space with

$$[\![p, \phi, u]\!] + \lambda[\![p, \phi, v]\!] \quad = \quad [\![p, \phi, u + \lambda v]\!]\ .$$

A *vector field* on $\mathcal{A}$ is a smooth map $f : \mathcal{A} \to T\mathcal{A}$ such that $\pi_{\mathcal{A}} \circ f = \mathrm{id}_{\mathcal{A}}$,
i.e. $f(p) \in T_p\mathcal{A}$ for every point p of $\mathcal{A}$.

If ϕ and ψ are standard charts of G, then $d(\psi \circ \phi^{-1})_{\phi(p)} = \mathrm{id}_{\mathbb{R}}$,
so $[\![p, \phi, u]\!]$ does not depend on $\phi \in \mathcal{A}_G$.

$$T\mathcal{A}_G \quad \cong \quad \mathcal{A}_G \times \mathbb{R} \qquad \text{and} \qquad T_p\mathcal{A}_G \quad \cong \quad \{p\} \times \mathbb{R}$$

The *standard vector field* on the standard atlas is

$$\begin{array}{rcl} \mathcal{A}_G & \to & T\mathcal{A}_G \\ p & \mapsto & (p, 1) \end{array}$$

For every smooth map $f : \mathcal{A} \to \mathcal{B}$ we have $Tf : T\mathcal{A} \to T\mathcal{B}$ defined by

$$Tf[\![p, \phi, u]\!] = [\![fp, \psi, d(\psi \circ f \circ \phi^{-1})_{\phi(p)}(u)]\!]$$

with $\phi \in \mathcal{A}$, $\psi \in \mathcal{B}$ charts around p and $f(p)$.

A *curve* is a smooth map defined on an open interval of $\mathbb{R}$; a *smooth path* is the restriction of a curve to a compact subinterval.

For every smooth path γ on $\mathcal{A}_G$, every $\phi \in \mathcal{A}_G$ we have

$$T\gamma(t, u) = T\gamma[\![t, \mathrm{id}_I, u]\!] = [\![\gamma(t), \phi, d(\phi \circ \gamma \circ \mathrm{id}_I^{-1})_t(u)]\!] = (\gamma(t), \gamma'(t) \cdot u) \,.$$

The tangent vector to γ at t is of the form $(\gamma(t), \gamma'(t))$; γ is locally order-preserving iff $\gamma'(t) \geqslant 0$ for every t.

Proposition (standard vector field vs standard ordered base)

For every $\phi \in \mathcal{A}_G$, for all p, $q \in \mathrm{dom}(\phi)$, we have $p \leqslant q$ (with $(\mathrm{dom}(\phi), \leqslant) \in \mathcal{A}_G$) iff there exists a smooth path γ on $\mathcal{A}_G$ from p to q with $\mathrm{im}(\gamma) \subseteq \mathrm{dom}(\phi)$ and $\gamma' \geqslant 0$, i.e. $\phi \circ \gamma$ is a smooth map between open intervals of $\mathbb{R}$ with nonnegative derivative, $\min(\phi \circ \gamma) = \phi(p)$, and $\max(\phi \circ \gamma) = \phi(q)$.

The above result is a special instance of Lawson's correspondence:

Ordered manifolds, invariant cone fields, and semigroups. Lawson, J. D., Forum Mathematicum, 1989.

From every norm $|\_|$ on $\mathbb{R}^n$ one defines the length of a smooth path $\gamma = (\gamma_1, \ldots, \gamma_n)$ on $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ by

$$\mathcal{L}(\gamma) \quad = \quad \int_{t \in I} |\gamma'(t)| dt$$

with $\gamma'(t) = (\gamma_1'(t), \ldots, \gamma_n'(t))$ the coordinates of the tangent vector to γ at t in the standard base $((\gamma_1(t), 1), \ldots, (\gamma_n(t), 1))$ of the tangent space at $\gamma(t)$.

We also define the distance between p, $q \in |G_1| \times \cdots \times |G_n|$ as $d(p, q) = |d_{G_1}(p_1, q_1), \ldots, d_{G_n}(p_n, q_n)|$ from which we deduce the length $L(\gamma)$ of any path γ on $|G_1| \times \cdots \times |G_n|$.

If δ is a smooth path on $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ then $\mathcal{L}(\delta) = L((\beta_{G_1} \times \cdots \times \beta_{G_n}) \circ \delta)$.

$\|x_1, \ldots, x_n\|_2$	$=$	$\sqrt{\sum_{i=1}^n x_i^2}$	Riemannian
$\|x_1, \ldots, x_n\|_1$	$=$	$\sum_{i=1}^n \|x_i\|$	cumulative execution time
$\|x_1, \ldots, x_n\|_\infty$	$=$	$\max\{x_1, \ldots, x_n\}$	parallel execution time

A subset X of $|G_1| \times \cdots \times |G_n|$ is said to be *tile compatible* when for all $p, q \in |G_1| \times \cdots \times |G_n|$ such that $(\pi_{G_1}, \ldots, \pi_{G_n})(p) = (\pi_{G_1}, \ldots, \pi_{G_n})(q)$, we have $p \in X$ iff $q \in X$.

The *standard cone* of $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$ at $p = (p_1, \ldots, p_n)$ is the cone $C_p = \left\{ \sum_{i=1}^{n} (p_i, \lambda_i) \mid \lambda_i \geqslant 0 \right\} \subseteq T_p\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$.

A *conal path* on a subset Y of $\|G_1\| \times \cdots \times \|G_n\|$ is a smooth path δ on $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$ such that $\delta(t) \in Y$ and $T\delta(t) \in C_{\delta(t)}$ for every $t \in \mathrm{dom}(\delta)$.

Theorem (Approximation)

For every directed path $\gamma = (\gamma_1, \ldots, \gamma_n)$ on a tile compatible subset X of $|G_1| \times \cdots \times |G_n|$, and every $\varepsilon > 0$, there exists a conal path $\delta = (\delta_1, \ldots, \delta_n)$ on $(\beta_{G_1} \times \cdots \times \beta_{G_n})^{-1}(X)$ such that:

- *γ and $(\beta_{G_1} \times \cdots \times \beta_{G_n}) \circ \delta$ start (resp. finish) at the same point,*
- $\max \left\{ d_i(\gamma_i(t), \beta_i(\delta_i(t))) \mid t \in \mathrm{dom}(\gamma);\ i \in \{1, \ldots, n\} \right\} < \varepsilon$*, and*
- $\mathcal{L}_\infty(\delta) < L_\infty(\gamma)$.