DIRECTED ALGEBRAIC TOPOLOGY

AND

CONCURRENCY

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MPRI : Concurrency (2.3.1)
– Lecture 5 –

2024 – 2025

THE FUNDAMENTAL CATEGORY

Abstract setting

Congruences on small categories

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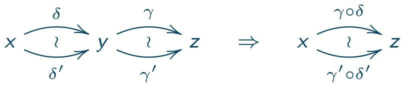

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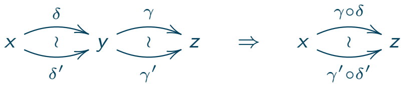

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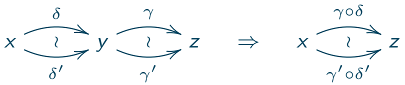

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The quotient map $q : \gamma \in \mathrm{Mo}(\mathcal{C}) \mapsto [\gamma] \in \mathrm{Mo}(\mathcal{C})/\sim$ induces a functor $q : \mathcal{C} \to \mathcal{C}/\sim$

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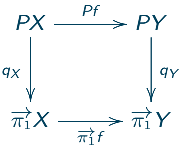

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$$X \xrightarrow{f} Y$$

$$\begin{array}{ccc} PX & \xrightarrow{Pf} & PY \\ {\scriptstyle q_X}\downarrow & & \downarrow{\scriptstyle q_Y} \\ \overrightarrow{\pi_1}X & \xrightarrow[\overrightarrow{\pi_1}f]{} & \overrightarrow{\pi_1}Y \end{array}$$

The collection of quotient functors q_X, for X ranging through the objects of $\mathcal{C}$, provides a natural transformation from P to $\overrightarrow{\pi_1}$.

The directed path functor

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- For $\delta:[0,r]\to X$ and $\gamma:[0,r']\to X$ with $\delta(r)=\gamma(0)$, define the concatenation

$$\gamma\cdot\delta:\ [0,r+r']\longrightarrow X$$

$$t\longmapsto\begin{cases}\delta(t) & \text{if } t\leqslant r\\ \gamma(t-r) & \text{if } t\geqslant r\end{cases}$$

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with

$$Pf : PX \longrightarrow PY$$

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Natural congruences from directed homotopies

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Given $x, y \in X$ and $r \in \mathbb{R}_+$, the relation $\sim_X$ is an equivalence relation on the set

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Juxtaposition of homotopies

horizontal composition

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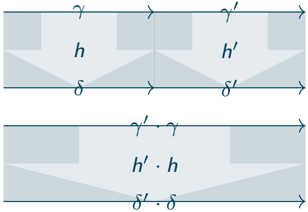
The mapping $h' * h : [0, r+r'] \times [0,q] \to X$ defined by

$$h' * h(t,s) \quad = \quad \begin{cases} h(t,s) & \text{if } 0 \leqslant t \leqslant r \\ h'(t-r,s) & \text{if } r \leqslant t \leqslant r+r' \end{cases}$$

is a homotopy from γ to δ.



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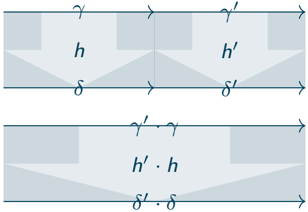

If h and h' are ((weakly) directed) homotopies, then so is their juxtaposition $h' \cdot h$.

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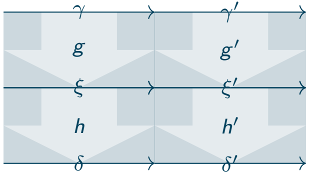

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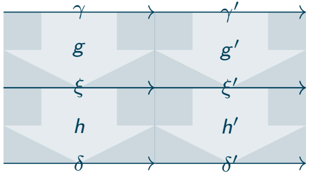

then it comes

$$(g' * h') \cdot (g * h) = (g' \cdot g) * (h' \cdot h)$$

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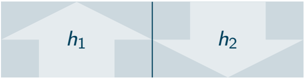

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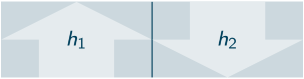

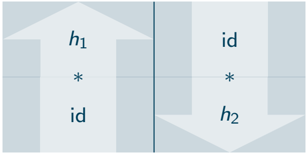

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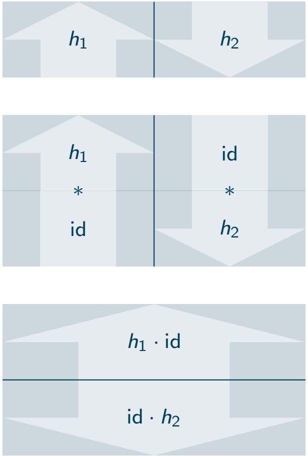

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The relation $\sim_X$ is a congruence on $P(X)$

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If $\gamma, \gamma' : [0, r] \to X$ are ((weakly) di)homotopic, then so are $f \circ \gamma, f \circ \gamma' : [0, r] \to Y$.

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- The fundamental category functor $\overrightarrow{\pi_1} : \mathcal{Lpo} \to \mathcal{Cat}$ is defined accordingly.
- The fundamental groupoid functor $\Pi_1 : \mathcal{Top} \to \mathcal{Grd}$ is obtained by substituting "paths" and "homotopies" to "directed paths" and "elementary homotopies".

Basic properties and computations

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- Given a pospace X, $\overrightarrow{\pi_1}X$ is loop-free i.e.

$$\overrightarrow{\pi_1}X(x,y) \neq \emptyset \text{ and } \overrightarrow{\pi_1}X(y,x) \neq \emptyset \qquad \Rightarrow \qquad x = y \text{ and } \overrightarrow{\pi_1}X(x,x) = \{\mathrm{id}_x\}$$

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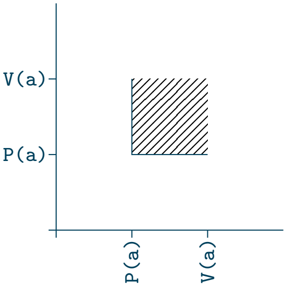

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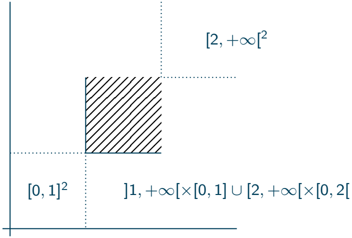

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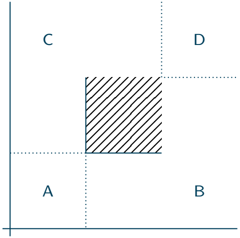

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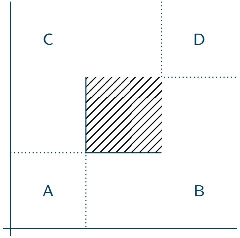

If $x \leqslant^2 y$, then $\overrightarrow{\pi_1} X(x,y)$ only depends on the elements of the partition x and y belong to.

$\to$	A	B	C	D
A	σ	β	α	$\beta' \circ \beta$ $\alpha' \circ \alpha$
B		σ		β'
C			σ	α'
D				σ

CATEGORY OF COMPONENTS

Motivation

Skeleta and equivalences of categories

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- The skeleton of the fundamental groupoid of a path-connected space is the fundamental group of this space.
- Problem: The fundamental category of a local pospace has no isomorphisms but its identities, hence it is its own skeleton.

Loop-free and one-way categories

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- A loop-free category is its own skeleton
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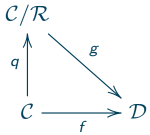

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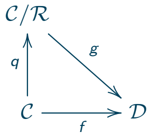

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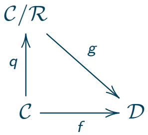

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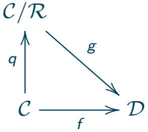

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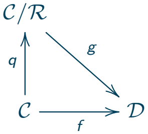

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Systems of weak isomorphisms

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- Then σ is a potential weak isomorphism when it preserves both future cones and past cones. Potential weak isomorphisms compose.
- If $\mathcal{C}(x, y)$ contains a potential weak isomorphism, then it is a singleton
  Requires the assumption that $\mathcal{C}$ is one-way

An example of potential weak isomorphism

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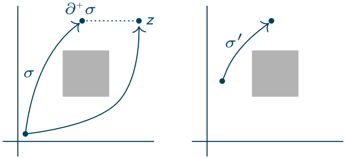

Due to the lower dipath, the σ, z-precomposition is not bijective; yet σ' is a potential weak isomorphism.

An unwanted example of potential weak isomorphism

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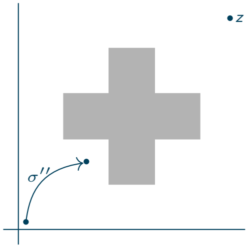

Note that σ'' is a potential weak isomorphism though there exists a morphism from $\partial^{-}\sigma''$ to z but none from $\partial^{+}\sigma''$ to z.

Stability under pushout and pullback

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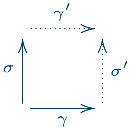

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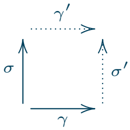

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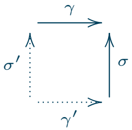

Greatest inner collection stable under pushout and pullback

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  - For $n \in \mathbb{N}$ define Σ_{n+1} as the collection of morphisms $\sigma \in \Sigma_n$ s.t. the pushout and the pullback of σ along any morphism exists (when sources or targets match accordingly) and belongs to Σ_n

$$\Sigma_0 \supseteq \cdots \Sigma_1 \supseteq \cdots \supseteq \Sigma_n \supseteq \Sigma_{n+1} \supseteq \cdots$$

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- Any collection Σ of morphisms of a category $\mathcal{C}$ admits a greatest subcollection that is stable under pushout and pullback
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- The inverse image (resp. the direct image) of a system of weak isomorphisms by an equivalence of categories is a system of weak isomorphisms.

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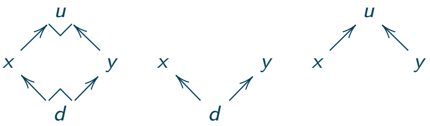

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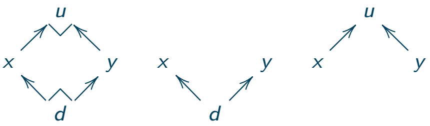

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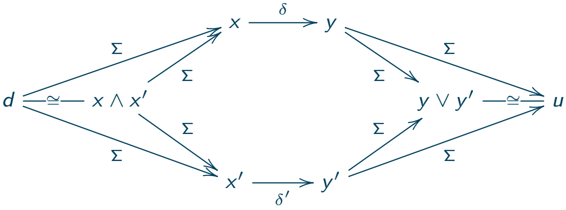

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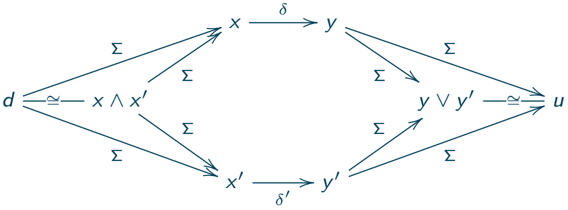

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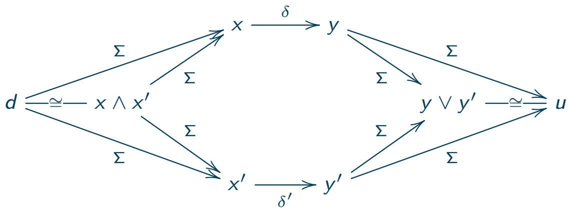

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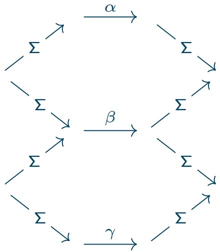

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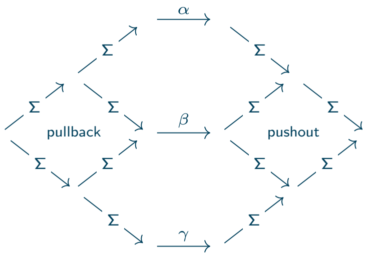

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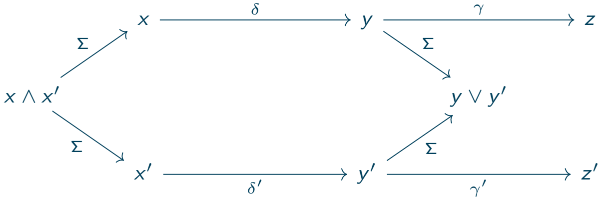

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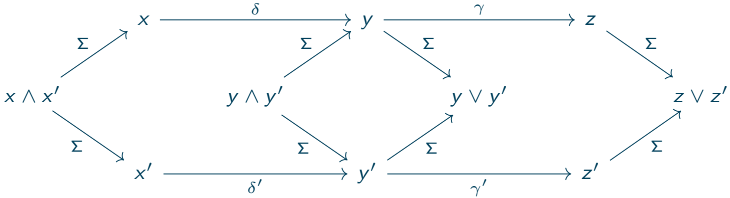

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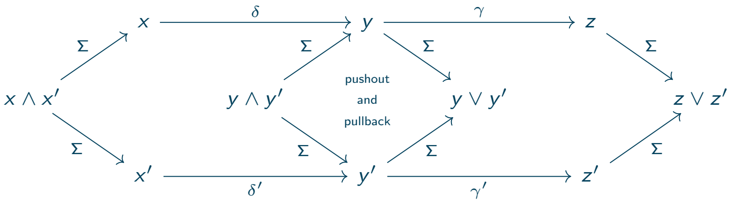

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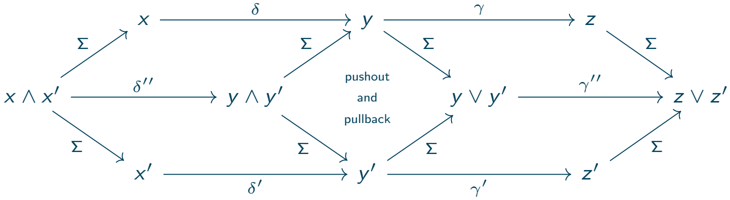

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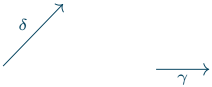

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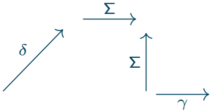

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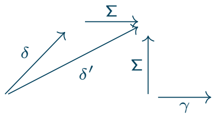

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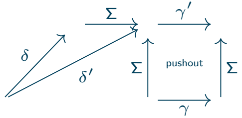

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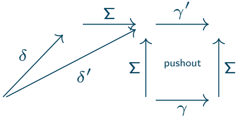

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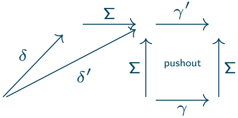

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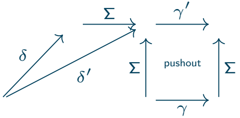

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Properties

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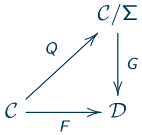

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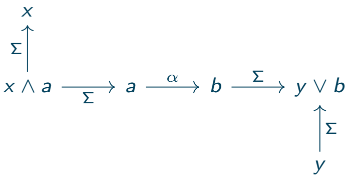

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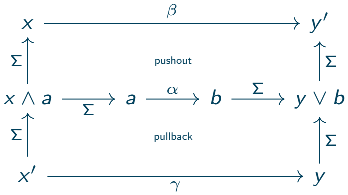

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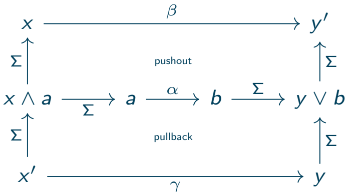

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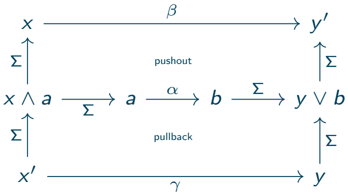

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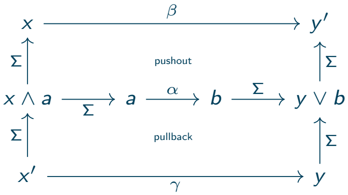

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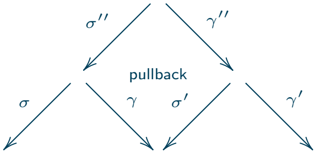
  - In the diagram below we have $Q(\gamma' \circ \gamma'') = Q(\gamma') \circ Q(\gamma'') = Q(\gamma') \circ Q(\gamma)$ hence the composite $(\gamma' \circ \gamma'', \sigma \circ \sigma'')$ neither depend on the choice of the pullback nor on the representatives (γ, σ) and (γ', σ').



The canonical comparison $P : \mathcal{C}[\Sigma^{-1}] \to \mathcal{C}/\Sigma$

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  - the mapping ϕ is called an admissible choice (of canonical objects)
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  - If $\mathcal{C}/\Sigma$ is finite then there exists an admissible choice
  - If $\mathcal{C}/\Sigma$ is infinite the existence of an admissible choice is a open question

Examples

Plane without a square

$x = \mathbb{R}_+^2 \setminus]0,1[^2$

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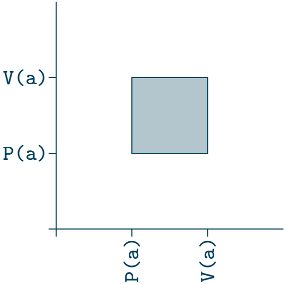

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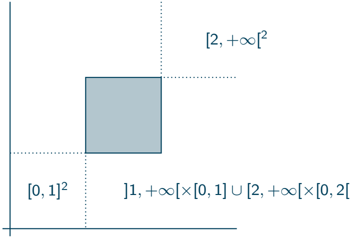

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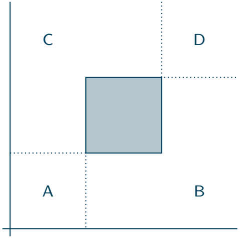

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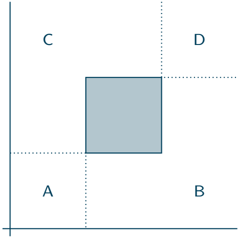

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$\to$	A	B	C	D
A	σ	β	γ	$\beta' \circ \beta$ $\alpha' \circ \alpha$
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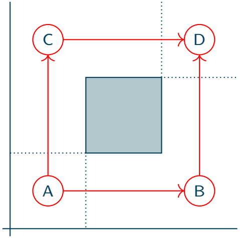

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Two rectangles

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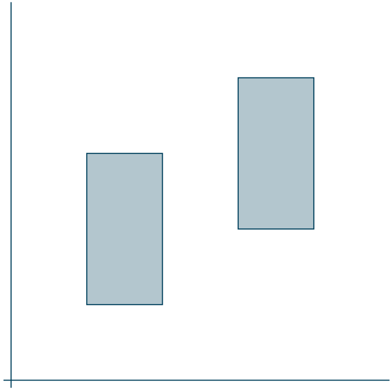

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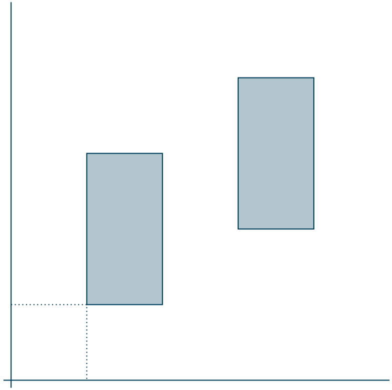

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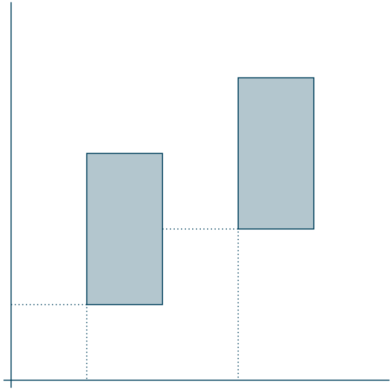

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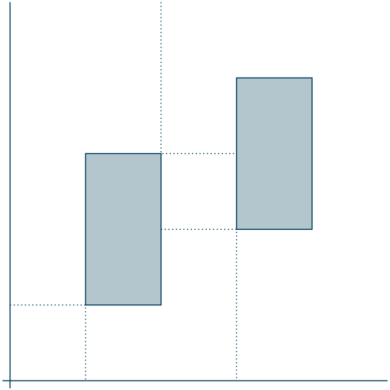

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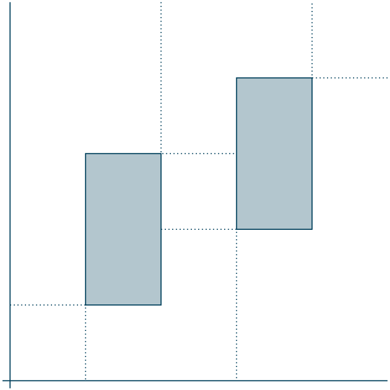

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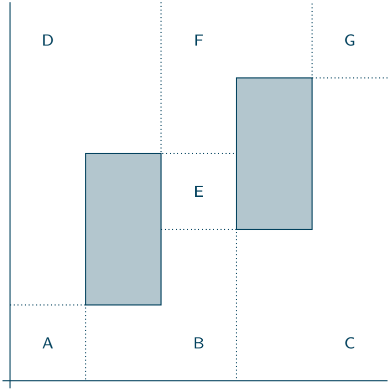

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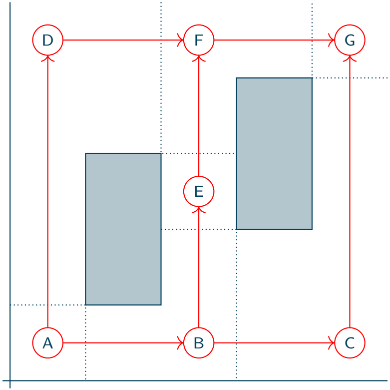

Swiss Flag

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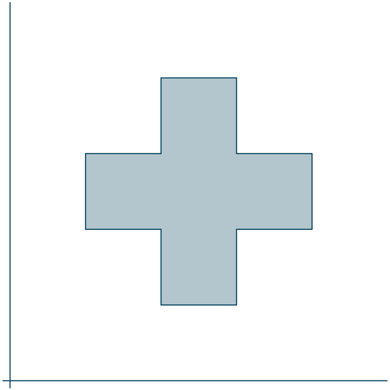

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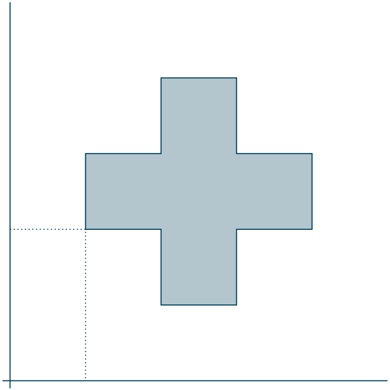

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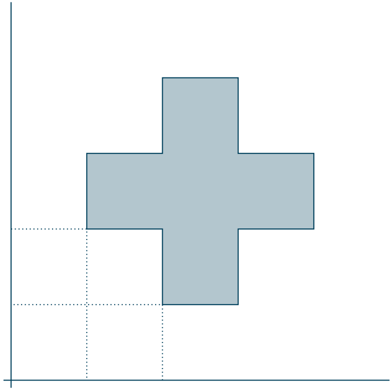

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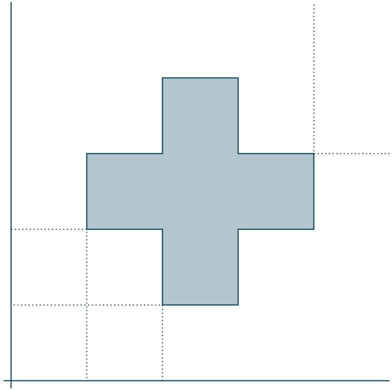

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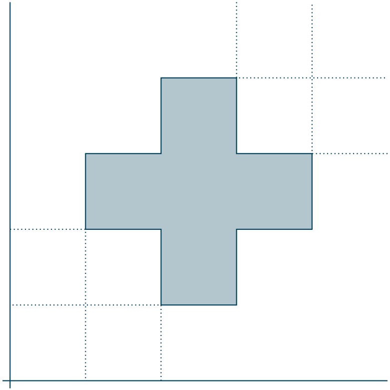

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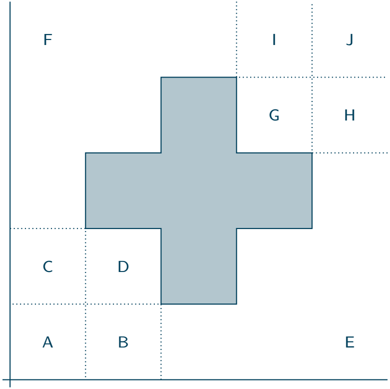

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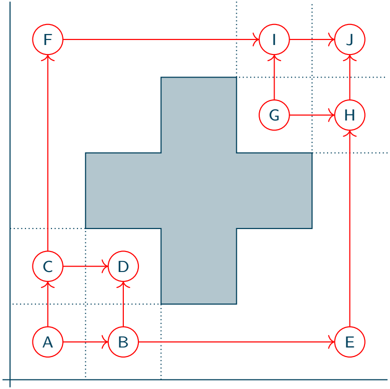

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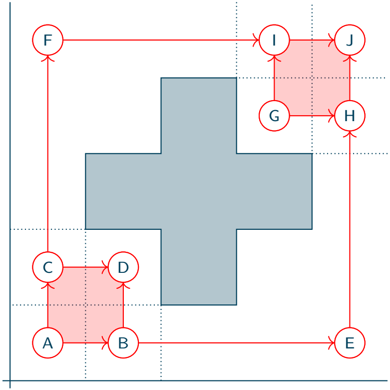

Achronal overlaping square

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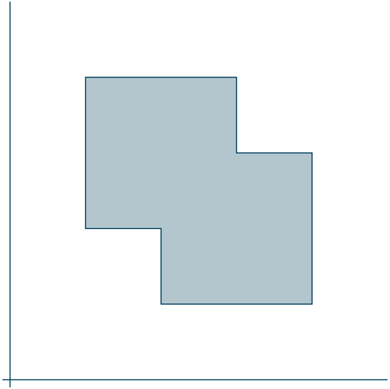

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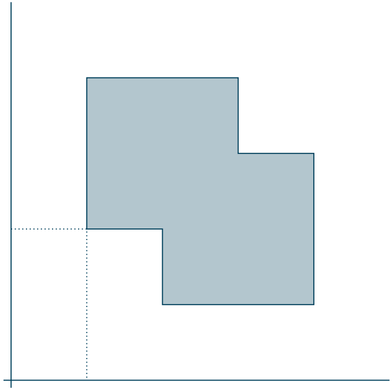

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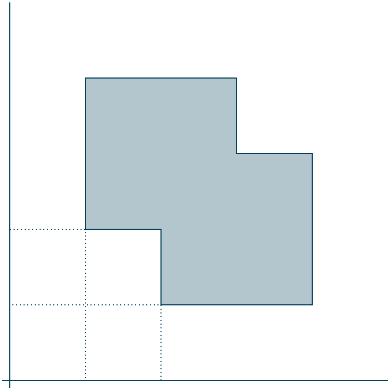

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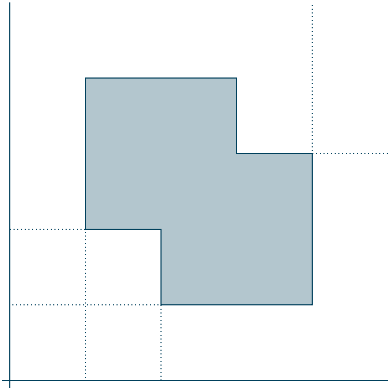

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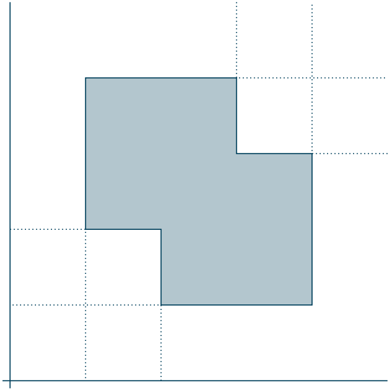

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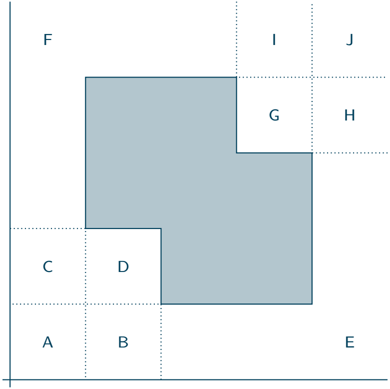

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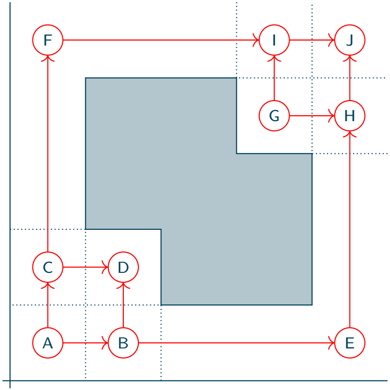

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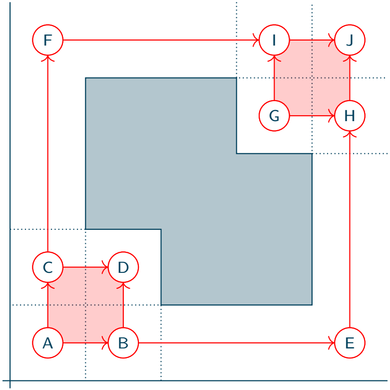

Diagonal overlaping squares

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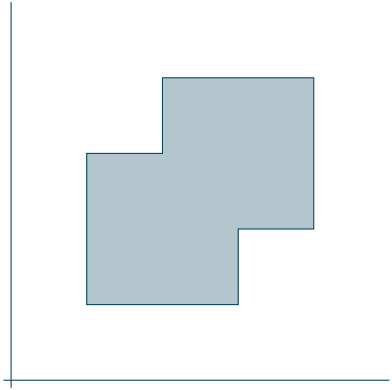

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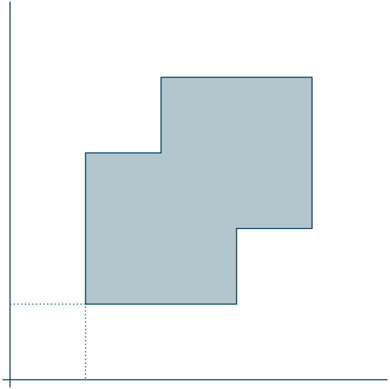

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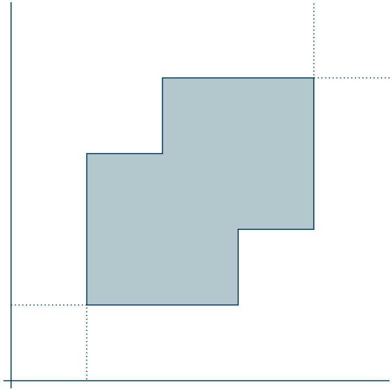

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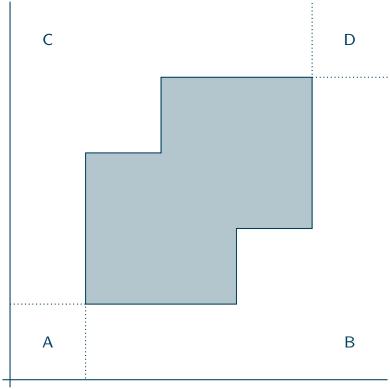

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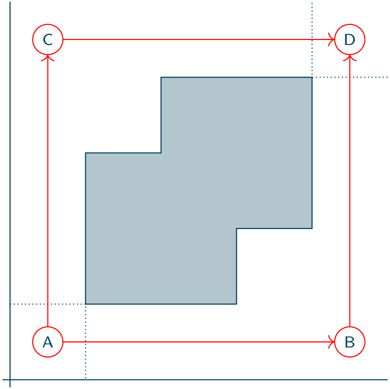

The floating cube

Non potential weak isomorphisms

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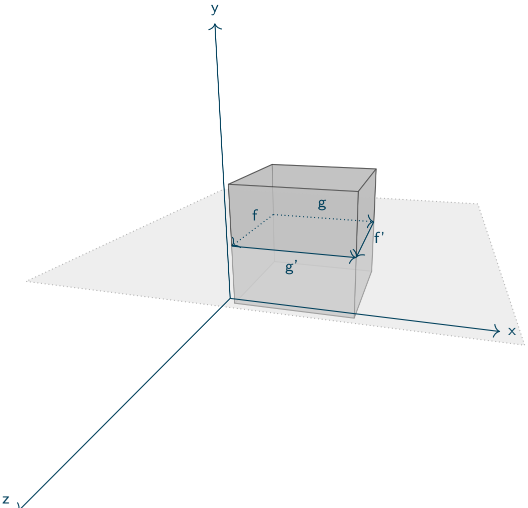

The floating cube

A "vee" that does not extend to a pushout

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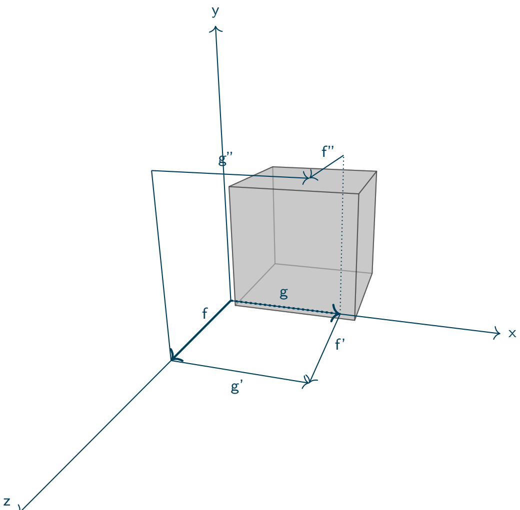

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Some pushouts squares

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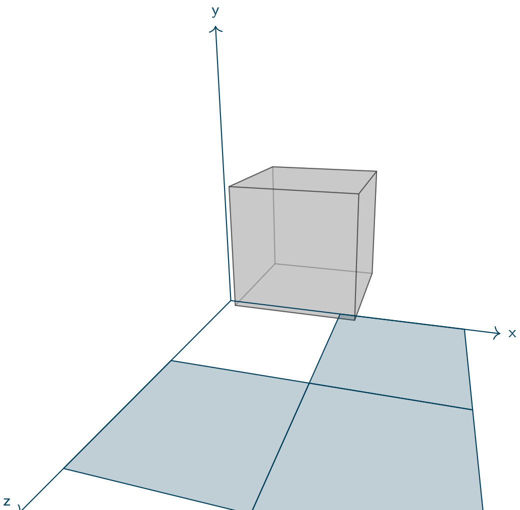

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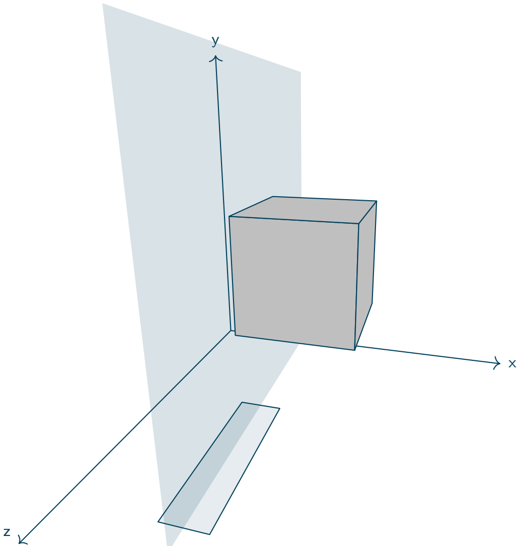

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  - Therefore $f', g' \notin \Sigma$ (anyway they do not preserve the future cones)

The floating cube

boundaries of the components

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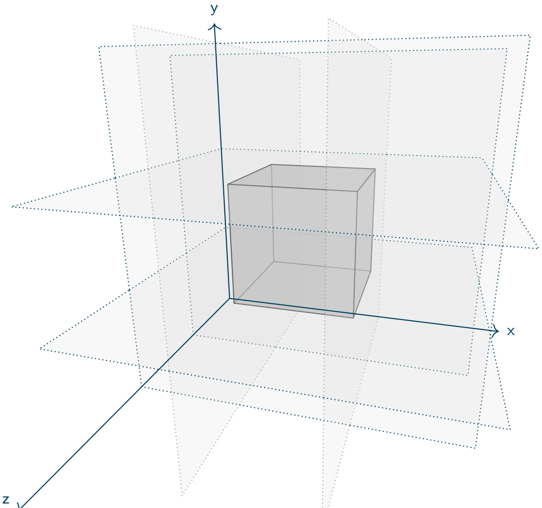

Finite connected loop-free categories

Commutative monoid

of nonempty finite connected loop-free categories

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The commutative monoid $\mathcal{M}$ is free.

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- Any element of $\mathcal{M}$ freely generated by a graph, is prime

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- Conjecture

> If $P \in \mathcal{N}$ is prime and $\overrightarrow{\pi_1}(P)$ is not a lattice, then $\overrightarrow{\pi_0}(\overrightarrow{\pi_1}(P))$ is prime