

DIRECTED ALGEBRAIC TOPOLOGY

AND

CONCURRENCY

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MPRI : Concurrency (2.3.1)

– Lecture 4 –

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INDEPENDENCE

Compatible programs

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By extension we define the parallel composition of P_1, \dots, P_N when the programs are **pairwise compatible**.

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$$\{j \in \{1, \dots, N\} \mid S_j \cap \text{dom}(\mu) \neq \emptyset\} \cap \{j \in \{1, \dots, N\} \mid S_j \cap \text{dom}(\mu') \neq \emptyset\} \neq \emptyset$$

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A permutation π of the set $\{0, \dots, q-1\}$ is said to be **compatible** with the sequence of multi-instructions μ_0, \dots, μ_{q-1} when it does not swap multi-instructions that should not be (it is order preserving on all pairs $\{k, k'\}$ such that μ_k and $\mu_{k'}$ should not be swapped).

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The permutation π is said to be **compatible** with the directed path γ when it is compatible with its associated sequence of multi-instructions.

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there exists an execution trace γ' whose associated sequence of multi-instructions is $\pi \cdot (\mu_0 \cdots \mu_{q-1})$, which has the same action on the system state than γ , that is to say

$$\sigma \cdot (\mu_0 \cdots \mu_{q-1}) = \sigma \cdot (\mu_{\pi^{-1}(0)} \cdots \mu_{\pi^{-1}(q-1)}) .$$

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Observational independence cannot be decided **statically**, moreover it is too **loose**.

Comparison

Main theorem

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ISOTHERMIC REGIONS

Boolean structure

One-dimensional regions

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Yet some infinite graphs may not enjoy the property e.g. when G is a graph with a single vertex and infinitely many arrows.

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- The collection of n -dimensional block coverings is denoted by $\text{Cov}_n G$, it is preordered by

$$C \preccurlyeq C' \quad \equiv \quad \forall b \in C \exists b' \in C', b \subseteq b'$$

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- $\alpha_n(X) = \{\emptyset\}$ if and only if $X = \emptyset$.

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We have a Galois connection (γ_n, α_n) between $\text{Cov}_n G$ and $\text{Pow}(|G|^n)$ with $\gamma_n(D) = \bigcup D$ for all $D \in \text{Cov}_n G$.

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Proof: any *connected* block is contained in a maximal *connected* block (by the Hausdorff maximal principle).

$$\bigcup_i^\uparrow (B_1^{(i)} \times \cdots \times B_n^{(i)}) = \left(\bigcup_i^\uparrow B_1^{(i)} \right) \times \cdots \times \left(\bigcup_i^\uparrow B_n^{(i)} \right)$$

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A subset $X \subseteq |G|^n$ is an isothetic region iff the collection of maximal subblocks of X is finite and covers X .

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Its maximal blocks are found among that of B^c therefore they have the form

$$D_1 \times \cdots \times D_{k-1} \times C_k \times D_{k+1} \times \cdots \times D_n$$

with $k \in \{1, \dots, n\}$, C_k ranging through the connected components of B_k^c and D_j , for $j \neq k$, ranging through the connected components of $|G|$.

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Moreover if \mathcal{B} and \mathcal{B}' are block coverings of X and X' containing all their maximal blocks, then the collection of maximal blocks of $B \cap B'$ for $B \in \mathcal{B}$ and $B' \in \mathcal{B}'$ is a block covering of $X \cap X'$ containing all its maximal blocks.

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- The maximal blocks of X^c thus form a finite block covering of X^c .

A result from directed topology

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For all directed paths γ on $|G|^n$ and all $X \in \mathcal{R}_n G$, the inverse image of X by γ has **finitely** many connected components.

Additional operators

Closure, interior, and boundary of an isothetic region

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The closure operator preserves finite products, therefore it preserves blocks.

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The interior of a set is the difference between its closure and its boundary. It follows that the interior operator also preserves isothetic regions.

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Theorem: if A, B , and X are isothetic regions, then so are $\text{frw}(A, B)$, $\text{cone}^f A$, \overline{A}^f , and their duals.

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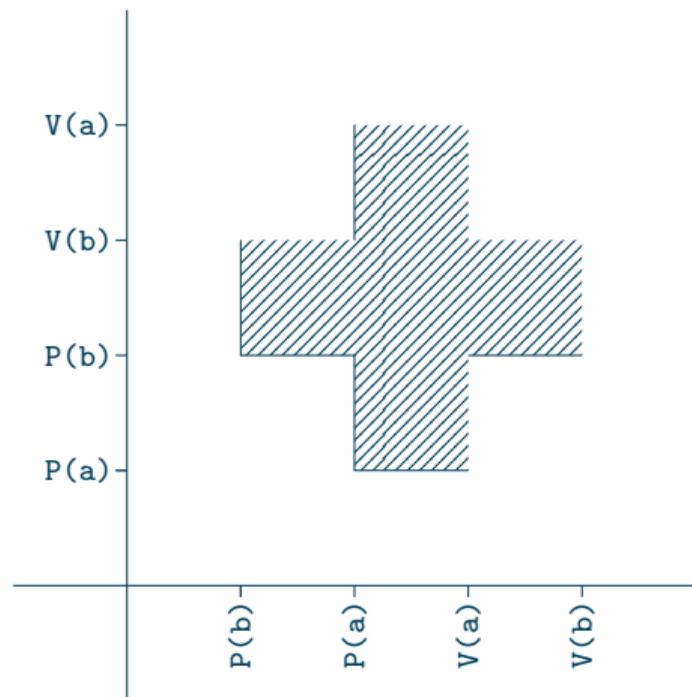
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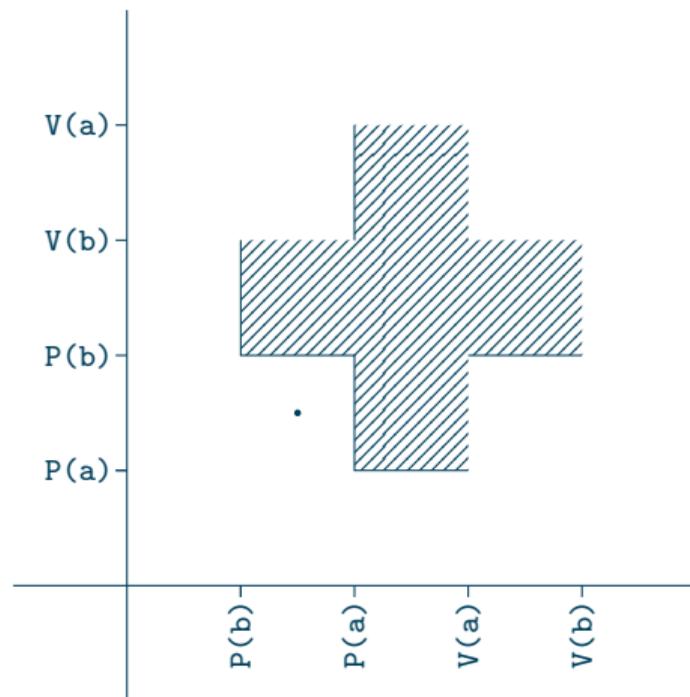
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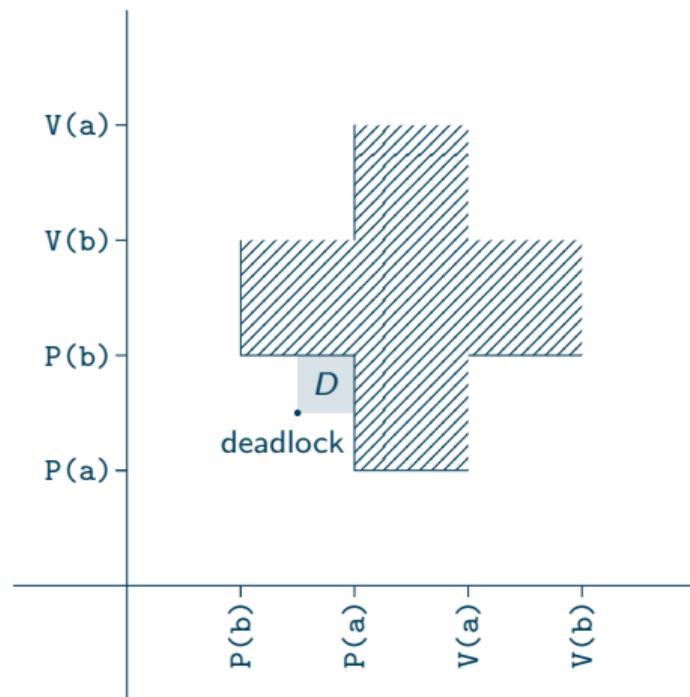
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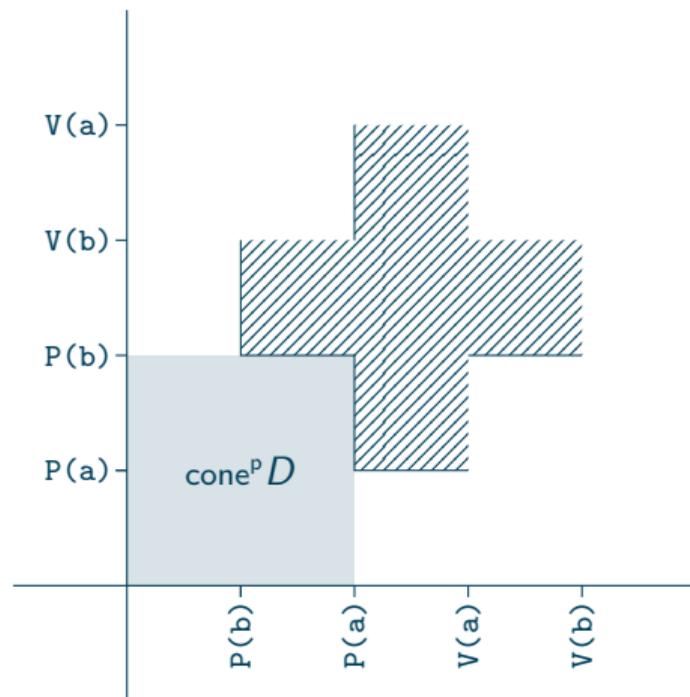
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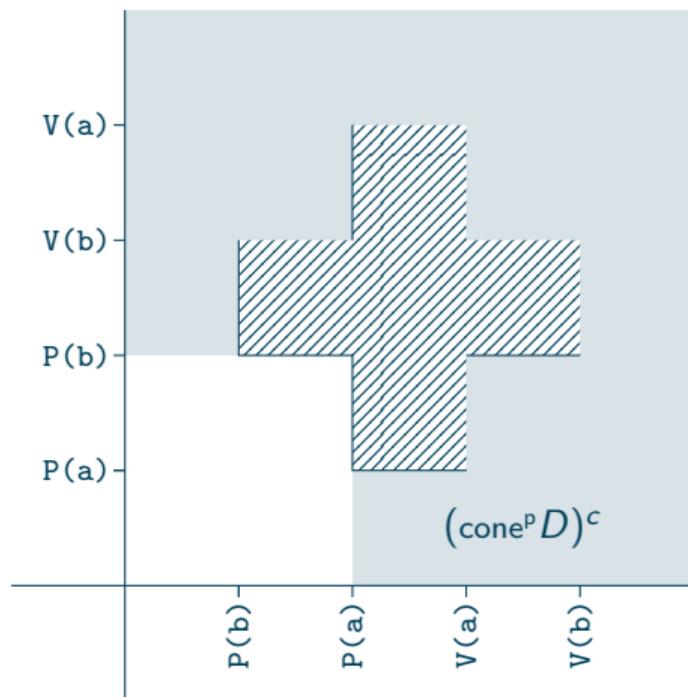
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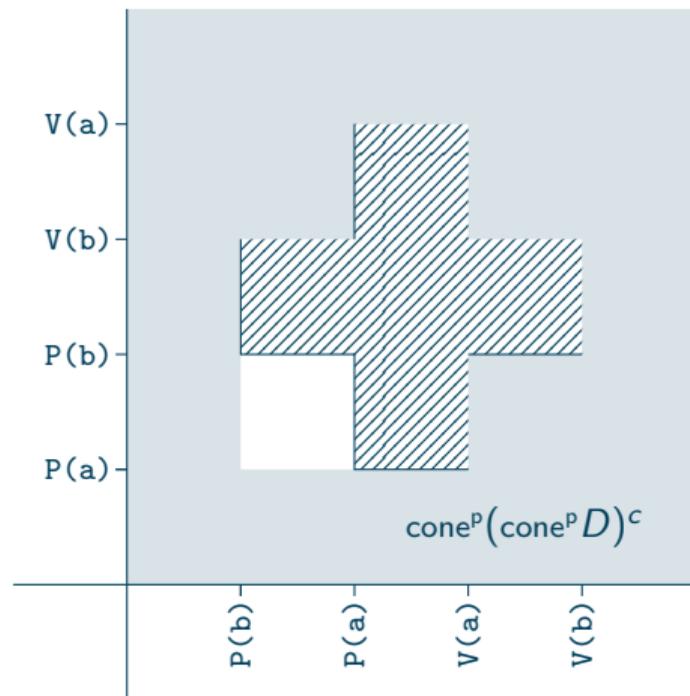
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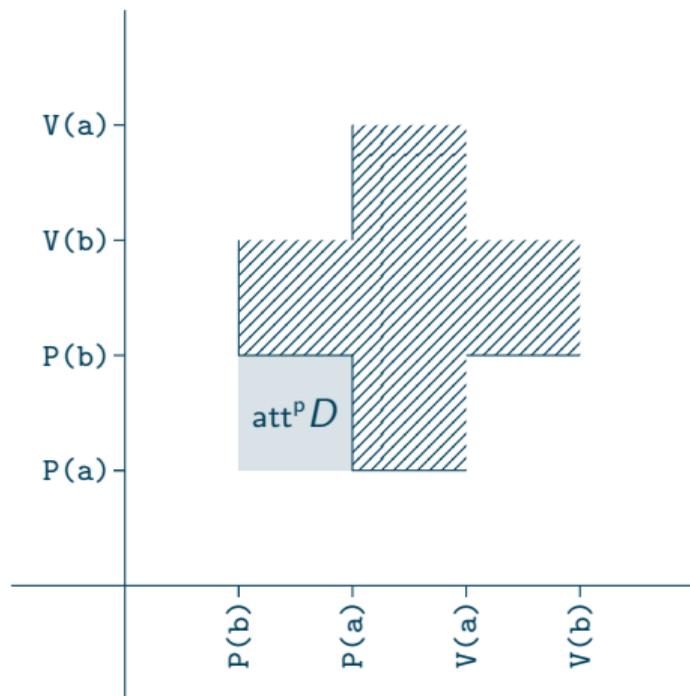
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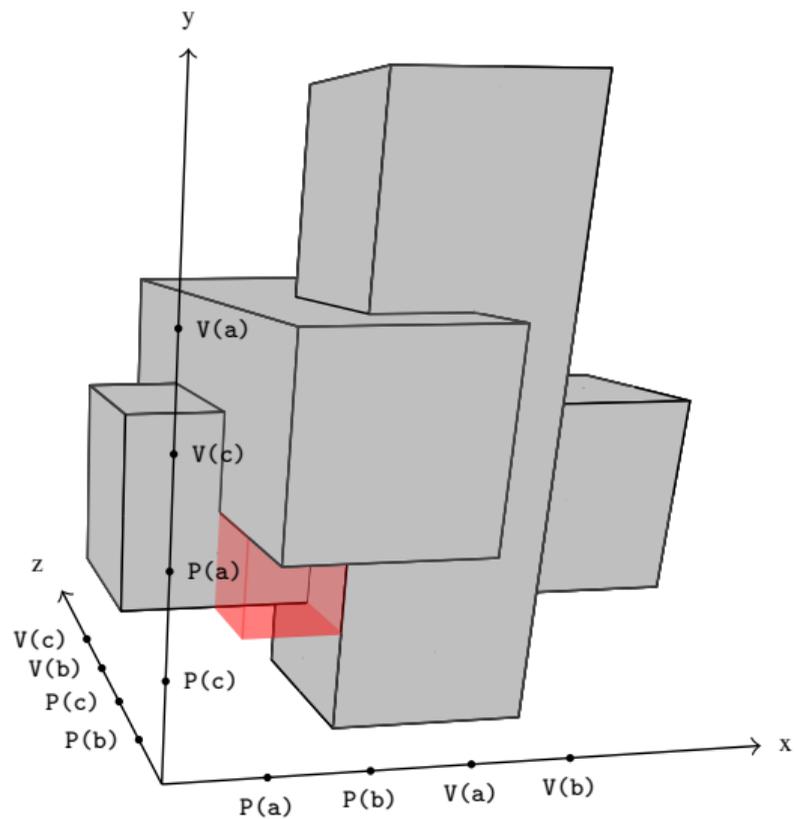


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Three dining philosophers



FACTORING ISOTHETIC REGIONS

Free commutative monoids

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- In particular, if $f : X \rightarrow Y$ is a set map, then

$$M(f)(\phi) = \sum_{x \in X} \phi(x)f(x)$$

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- If M contains nontrivial units, then one can consider the quotient monoid M/\sim where $x \sim y$ stands for: there exists a unit u s.t. $y = ux$

Examples

monoid	irreducibles	primes	units

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$\mathbb{Z}_6, \times, 1$	\emptyset	{2, 3, 4}	{1, 5}

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 - Note that two free commutative monoids are isomorphic in Cmon iff their set of prime elements have the same cardinality
e.g. $(\mathbb{N} \setminus \{0\}, \times, 1) \cong (\mathbb{Z}[X] \setminus \{0\}, \times, 1)$ in Cmon

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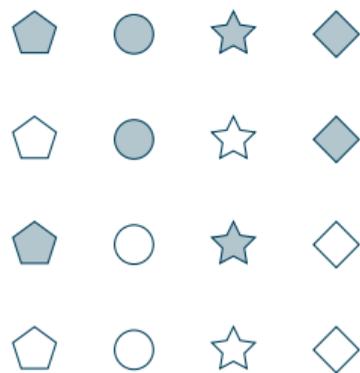
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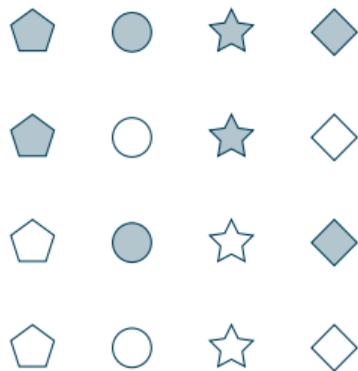
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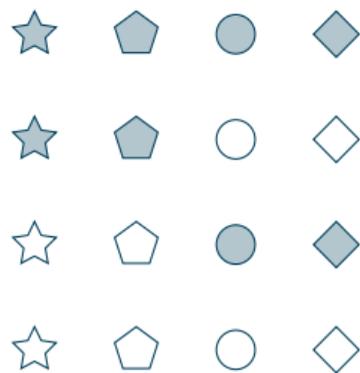
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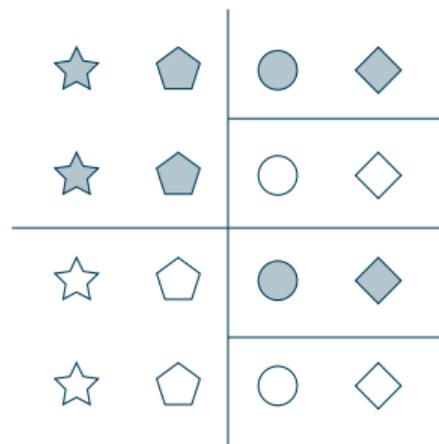
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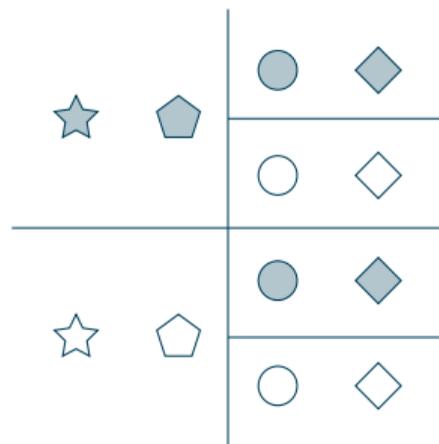
Monoids of homogeneous languages

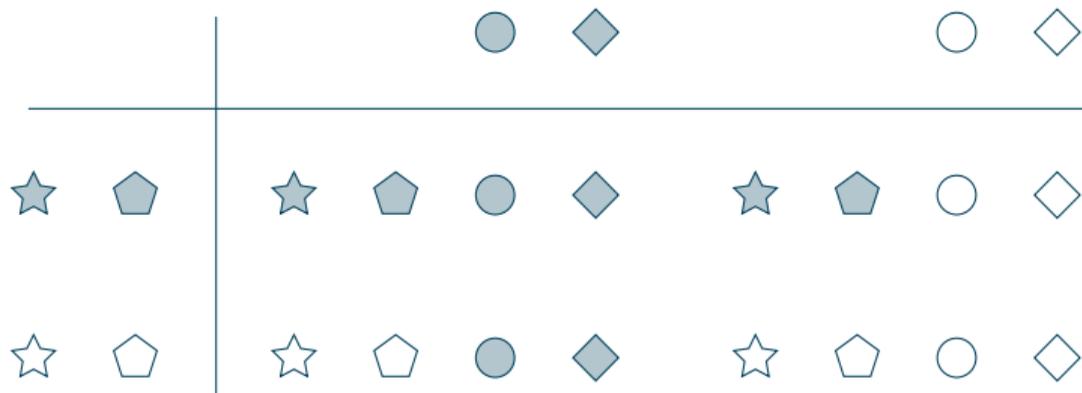












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on the left of the homogeneous languages

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- If $\sigma \in \mathfrak{S}_n$ and $\sigma' \in \mathfrak{S}_{n'}$ then define $\sigma \otimes \sigma' \in \mathfrak{S}_{n+n'}$ as:

$$\sigma \otimes \sigma'(k) := \begin{cases} \sigma(k) & \text{if } 1 \leq k \leq n \\ (\sigma'(k-n)) + n & \text{if } n+1 \leq k \leq n+n' \end{cases}$$

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- A Godement exchange law is satisfied, which ensures that \sim is actually a congruence:

$$(\sigma \cdot H) \cdot (\sigma' \cdot H') = (\sigma \otimes \sigma') \cdot (H \cdot H')$$

i.e. $H \sim K$ and $H' \sim K'$ implies $HH' \sim KK'$

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- For any homogeneous language H and $\sigma \in \mathfrak{S}_{\dim(H)}$, $\text{card}(H) = \text{card}(\sigma \cdot H)$ so we can define the cardinality of any element of $\mathcal{H}(\mathbb{A})$

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- The primality of H does not imply that of $\text{Card}(H)$
e.g. $H = \{a, b, c, d\}$ is prime though $\text{card}(H) = 4$

The brute force algorithm for factoring in $\mathcal{H}_f(\mathbb{A})$

Theory

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Given $w \in \mathbb{A}^n$ and $I \subseteq \{1, \dots, n\}$, we write $w|_I$ for the subword of w consisting of letters with indices in I .

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Denoting I^c for $\{1, \dots, n\} \setminus I$, we have

$$[H] = [H|_I] \cdot [H|_{I^c}]$$

in $\mathcal{H}_f(\mathbb{A})$ if and only if for all words $u, v \in H$ there exists a word $w \in H$ such that

$$w|_I = u|_I \quad \text{and} \quad w|_{I^c} = v|_{I^c}$$

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Practice

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For $I \subseteq \{1, \dots, n\}$ let π_I be the “projection” that sends $w \in H$ to $w|_I \in \mathbb{A}^{\text{card}(I)}$.

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For $I \subseteq \{1, \dots, n\}$ let $\pi_{|I}$ be the “projection” that sends $w \in H$ to $w_{|I} \in \mathbb{A}^{\text{card}(I)}$.

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2. if $\pi_{|I^c}(\pi_{|I}^{-1}(u))$ does not depend on $u \in H_{|I}$, then we have the factorization

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3. otherwise check whether there are still subsets of $\{1, \dots, n\}$ to check:
 - 3.1. yes: go to step 1
 - 3.2. no: $[H]$ is prime

Homogeneous languages and isothetic regions

Factoring a program

```
sem: 1 a b
```

```
sem: 2 c
```

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proc:
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```
init: p q p q
```

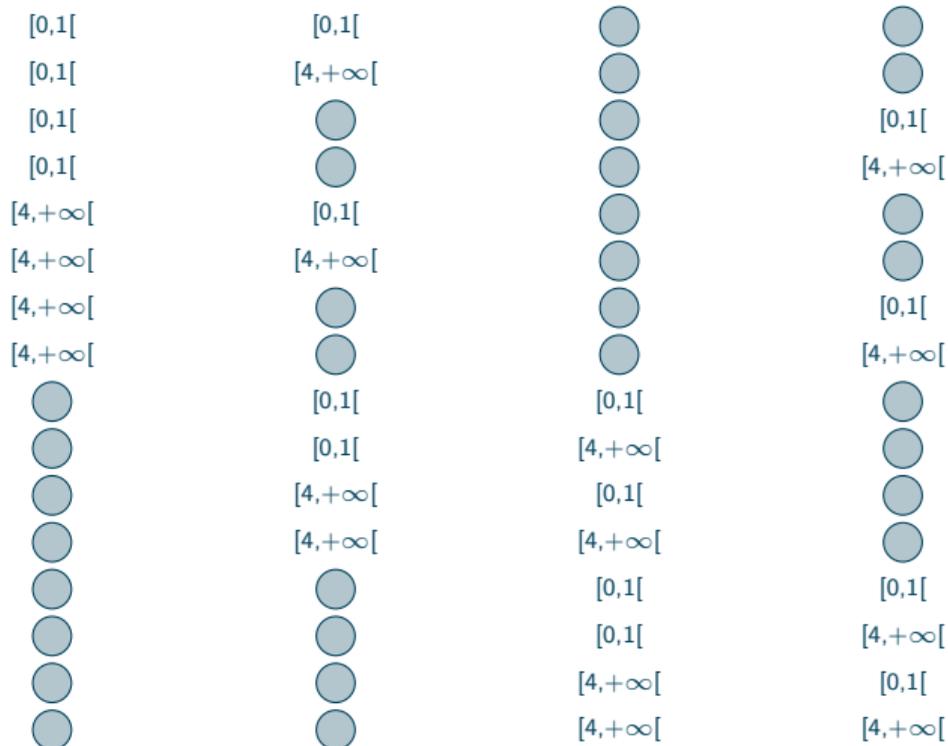
Factoring the space of states

brute force

$[0,1[$	$[0,1[$	$[0,+\infty[$	$[0,+\infty[$
$[0,1[$	$[4,+\infty[$	$[0,+\infty[$	$[0,+\infty[$
$[0,1[$	$[0,+\infty[$	$[0,+\infty[$	$[0,1[$
$[0,1[$	$[0,+\infty[$	$[0,+\infty[$	$[4,+\infty[$
$[4,+\infty[$	$[0,1[$	$[0,+\infty[$	$[0,+\infty[$
$[4,+\infty[$	$[4,+\infty[$	$[0,+\infty[$	$[0,+\infty[$
$[4,+\infty[$	$[0,+\infty[$	$[0,+\infty[$	$[0,1[$
$[4,+\infty[$	$[0,+\infty[$	$[0,+\infty[$	$[4,+\infty[$
$[0,+\infty[$	$[0,1[$	$[0,1[$	$[0,+\infty[$
$[0,+\infty[$	$[0,1[$	$[4,+\infty[$	$[0,+\infty[$
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$[0,+\infty[$	$[0,+\infty[$	$[0,1[$	$[0,1[$
$[0,+\infty[$	$[0,+\infty[$	$[0,1[$	$[4,+\infty[$
$[0,+\infty[$	$[0,+\infty[$	$[4,+\infty[$	$[0,1[$
$[0,+\infty[$	$[0,+\infty[$	$[4,+\infty[$	$[4,+\infty[$

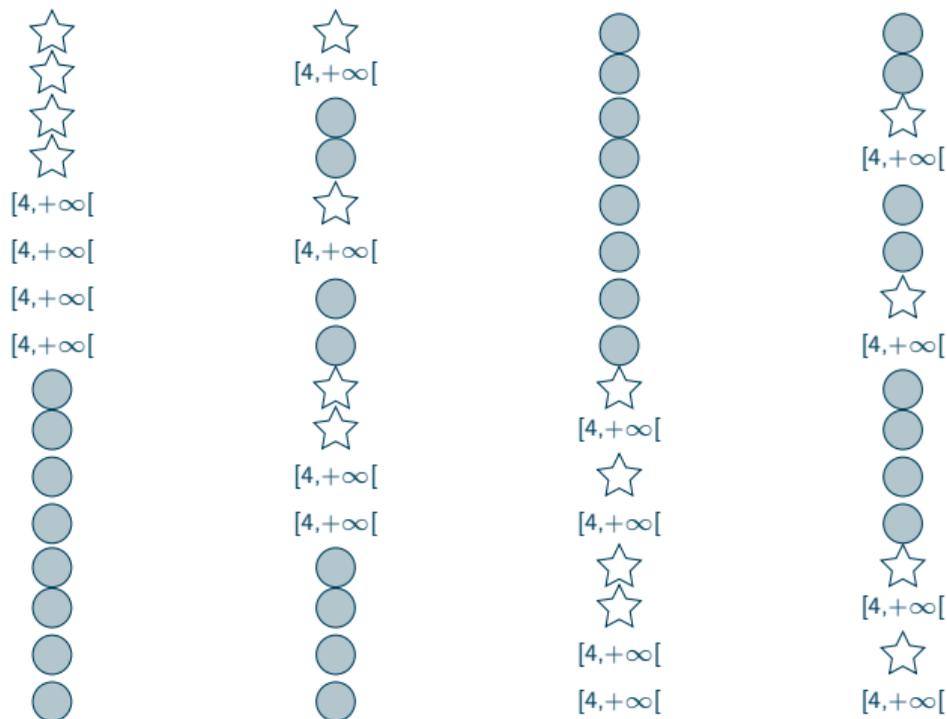
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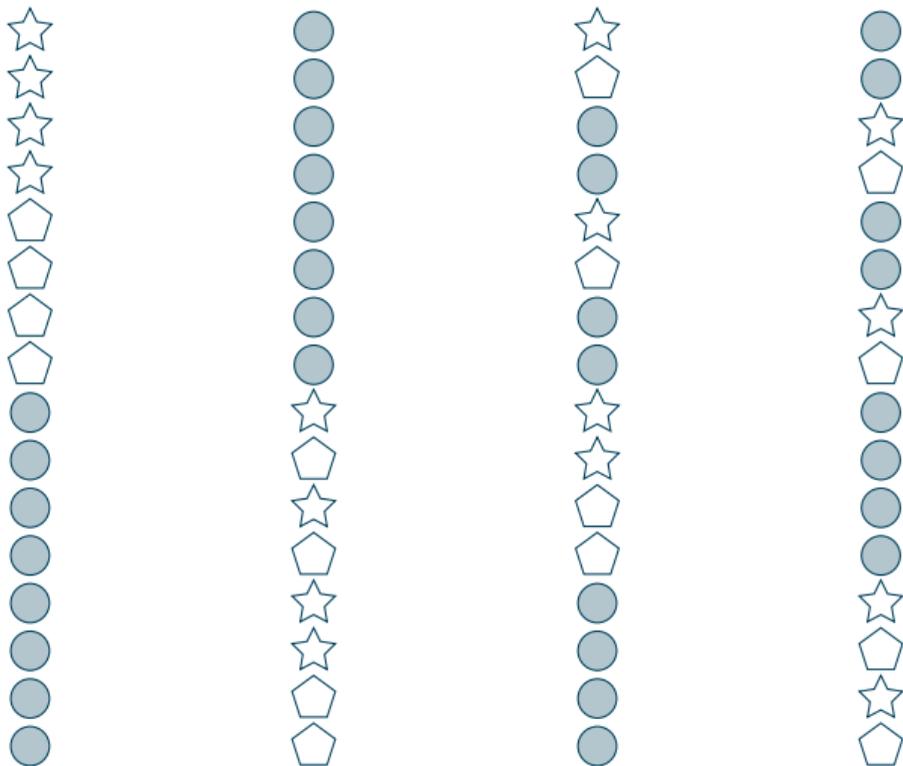
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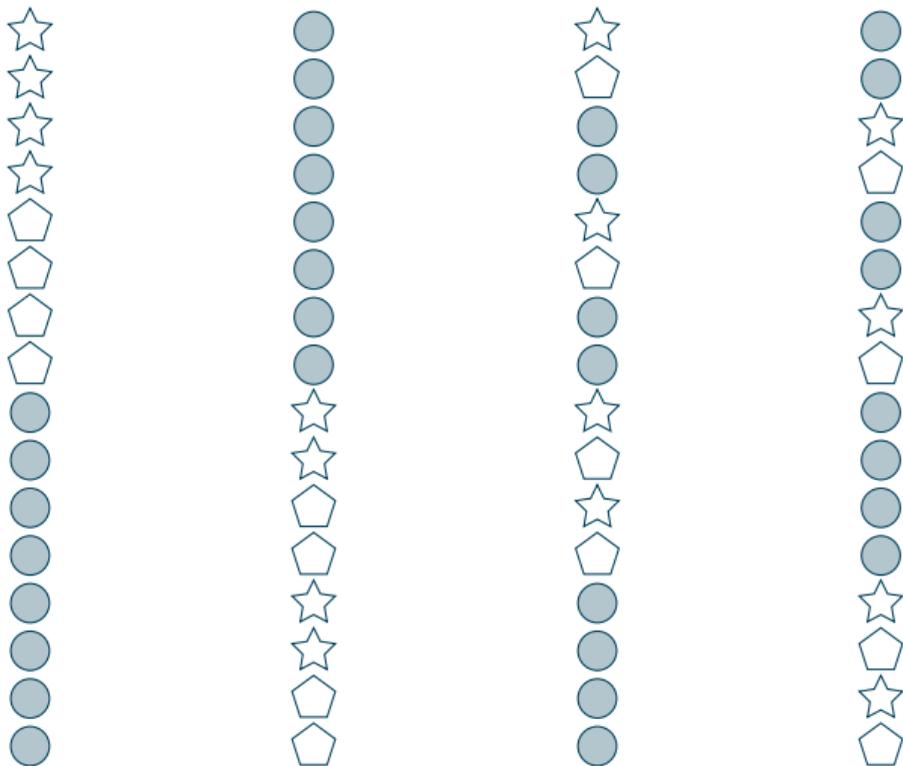
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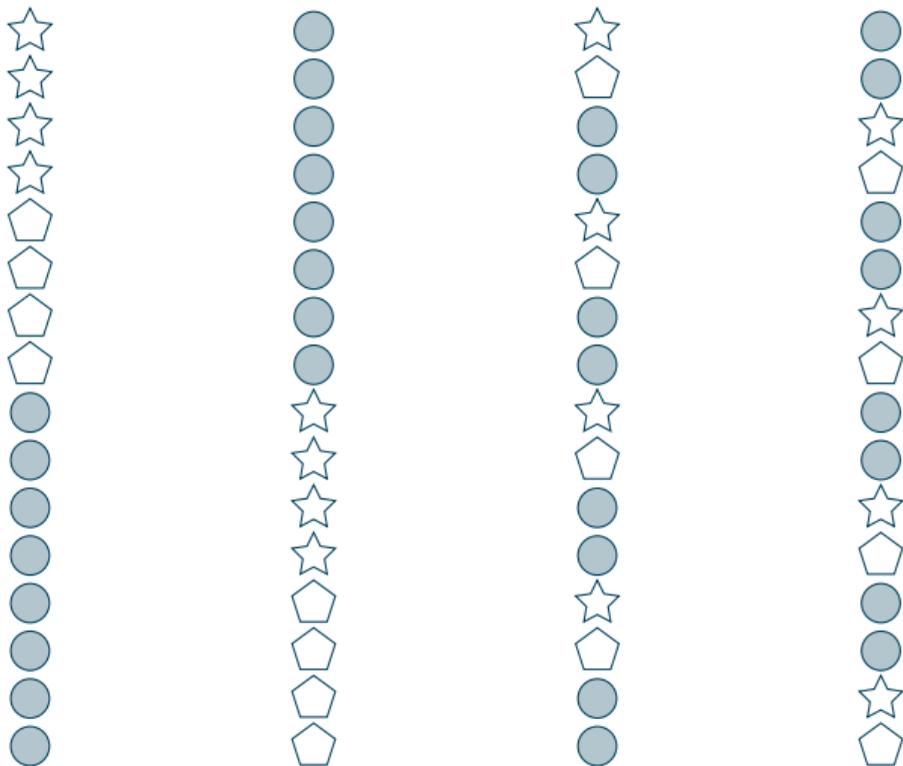
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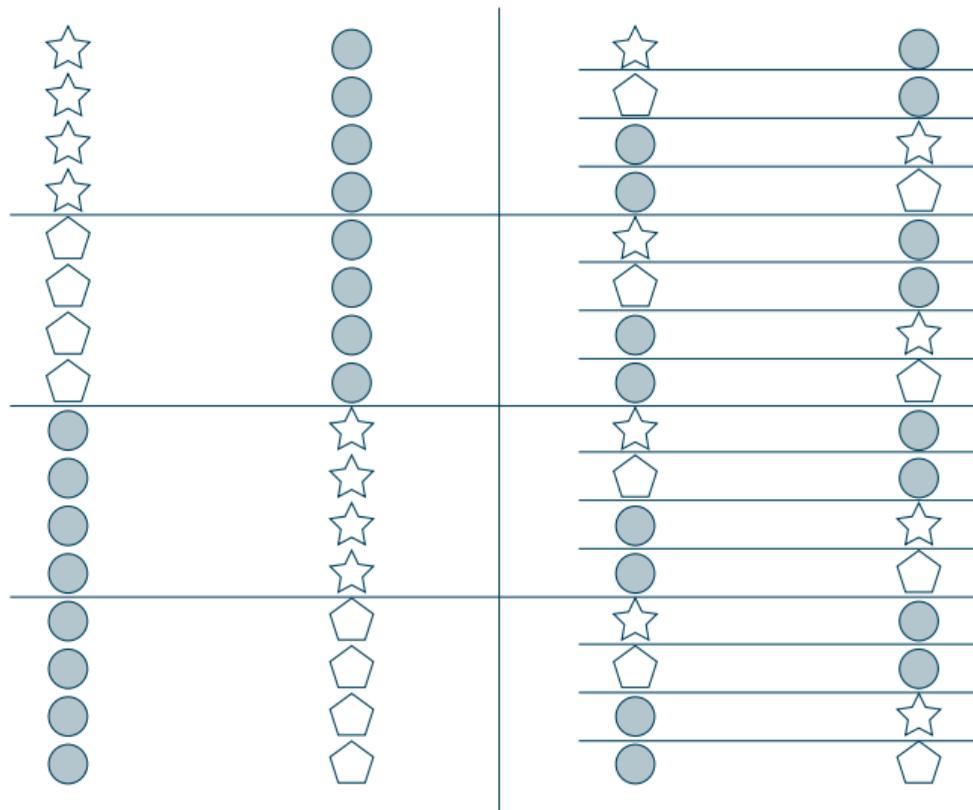
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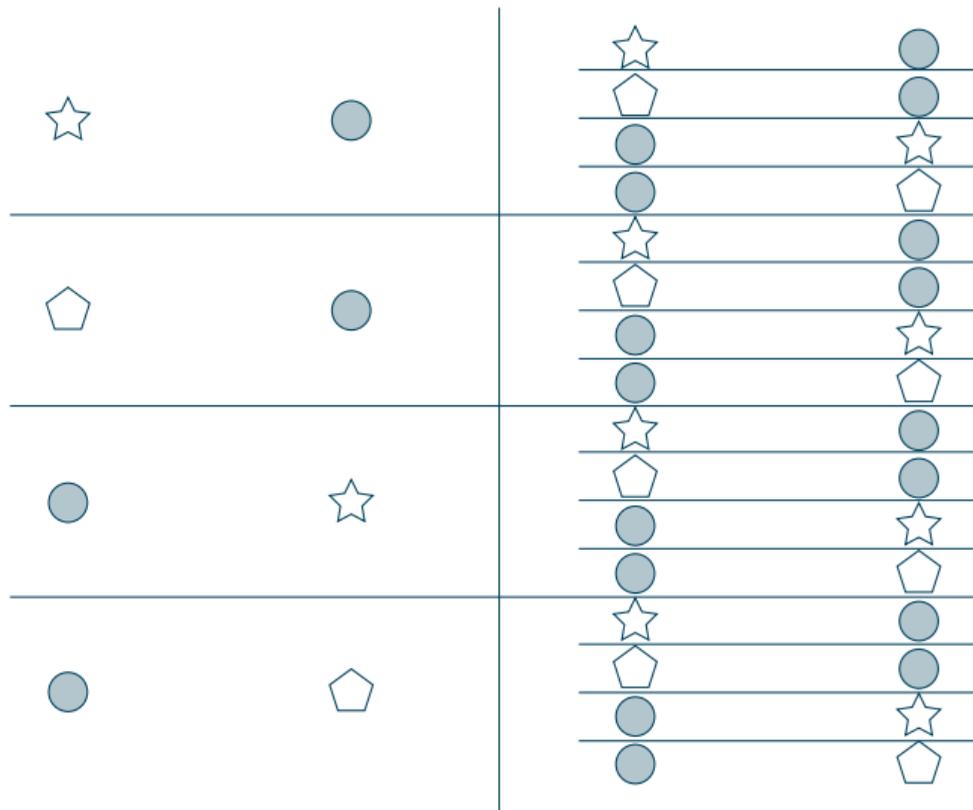
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☆ ●	☆ ● ☆ ●	☆ ● ⬠ ●	☆ ● ● ☆	☆ ● ● ⬠
⬠ ●	⬠ ● ☆ ●	⬠ ● ⬠ ●	⬠ ● ● ☆	⬠ ● ● ⬠
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- If \preceq is the equality relation, then $X \preceq Y$ amounts to $H_X \subseteq H_Y$ for some representatives H_X and H_Y of X and Y .

Homogeneous languages

over the alphabets $|G|$ and $\mathcal{R}_1 G \setminus \{\emptyset\}$ with G being a finite graph

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over the alphabets \mathbb{A} and $\mathcal{R}_1 G \setminus \{\emptyset\}$ with G being a finite graph

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- The induced morphism γ does not preserve the prime elements e.g. consider a covering of $[0, 1]^2$ with 3 distinct rectangles

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- therefore $\text{im}(\alpha)$ is commutative free

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If $\mathcal{F} = \alpha(X^c)$ then we obtain the prime factorization of X

Factoring a program

```
sem: 1 a b
```

```
sem: 2 c
```

```
proc:
```

```
  p = P(a);P(c);V(c);V(a)
```

```
  q = P(b);P(c);V(c);V(b)
```

```
init: p q p q
```

Factoring the space of states

subtle

$[2,3[$	$[2,3[$	$[2,3[$	$[0,+\infty[$
$[2,3[$	$[2,3[$	$[0,+\infty[$	$[2,3[$
$[1,4[$	$[0,+\infty[$	$[1,4[$	$[0,+\infty[$
$[2,3[$	$[0,+\infty[$	$[2,3[$	$[2,3[$
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