DIRECTED ALGEBRAIC TOPOLOGY

AND

CONCURRENCY

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MPRI : Concurrency (2.3.1)
– Lecture 3 –

2024 – 2025

THE BIG PICTURE

```
sem 1 a
proc:  p = P(a);V(a)
init:  2p
```

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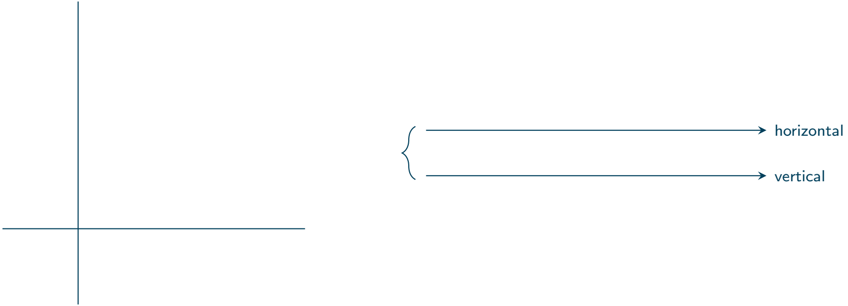

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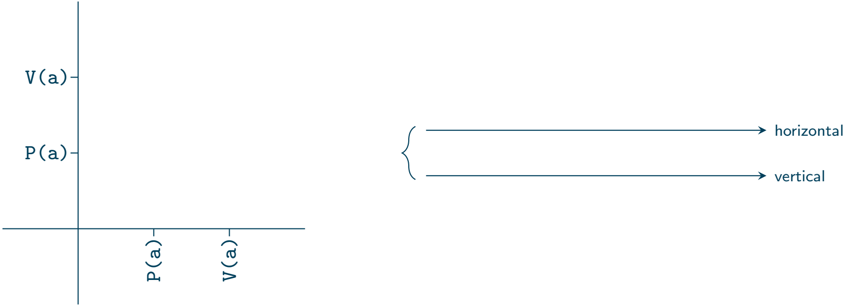

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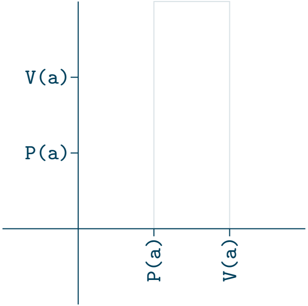

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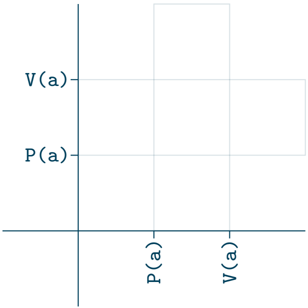

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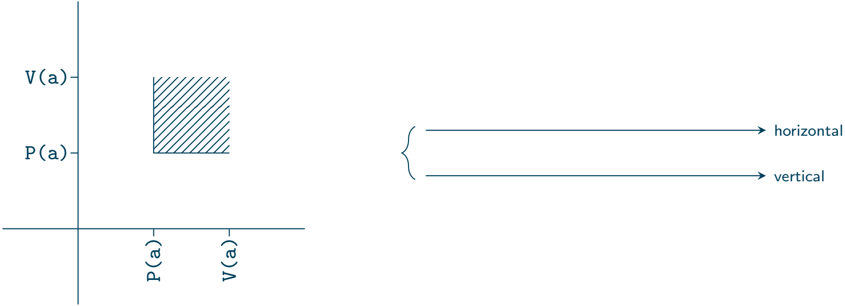

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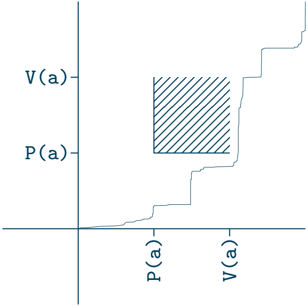

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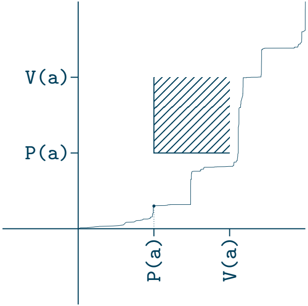

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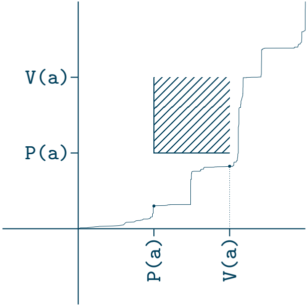

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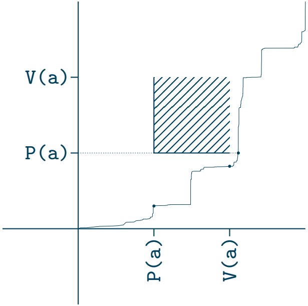

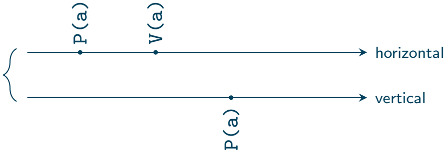

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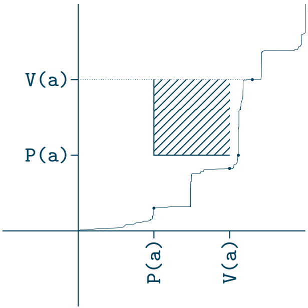

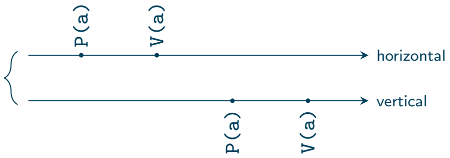

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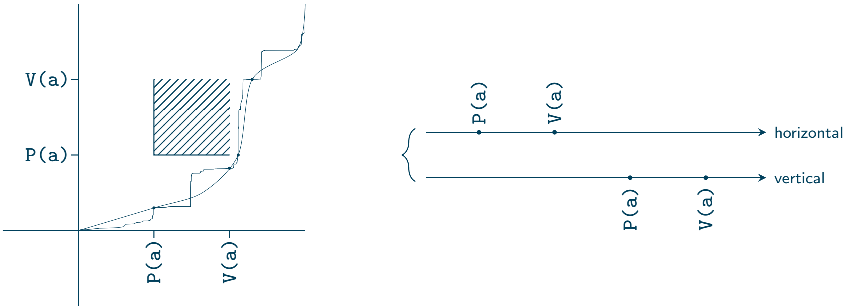

$$\underbrace{P_1 \quad | \quad \cdots \quad | \quad P_n}_{\text{program } P}$$

$$\underbrace{G_1 \quad , \quad \ldots \quad , \quad G_n}_{\text{graphs}}$$

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$$\underbrace{|G_1| \times \cdots \times |G_n|}_{\text{sets}}$$

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$$\underbrace{|G_1| \times \cdots \times |G_n|}_{\text{sets}} \qquad \underbrace{\mathcal{X}_1 \times \cdots \times \mathcal{X}_n}_{\text{ordered bases}}$$

$$\underbrace{G_1 \;,\; \ldots \;,\; G_n}_{\text{graphs}}$$

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$$\lvert P \rvert \subseteq \underbrace{\lvert G_1 \rvert \times \cdots \times \lvert G_n \rvert}_{\text{sets}} \qquad \underbrace{\mathcal{X}_1 \times \cdots \times \mathcal{X}_n}_{\text{ordered bases}}$$

$$\underbrace{G_1 \;,\; \ldots \;,\; G_n}_{\text{graphs}}$$

$$\underbrace{P_1 \;\mid\; \cdots \;\mid\; P_n}_{\text{program } P}$$

 The big picture

$$\overbrace{\mathcal{E}_1 \times \cdots \times \mathcal{E}_n}^{\text{euclidean ordered bases}}$$

$$\text{blowup}\left\{\begin{array}{ccc} \|P\| \subseteq & \|G_1\| \times \cdots \times \|G_n\| & \mathcal{E}_1 \times \cdots \times \mathcal{E}_n \\ & \beta_1 \downarrow \times \cdots \times \downarrow \beta_n & \beta_1 \downarrow \times \cdots \times \downarrow \beta_n \\ |P| \subseteq & |G_1| \times \cdots \times |G_n| & \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \end{array}\right.$$

$$\underbrace{|G_1| \times \cdots \times |G_n|}_{\text{sets}} \qquad \underbrace{\mathcal{X}_1 \times \cdots \times \mathcal{X}_n}_{\text{ordered bases}}$$

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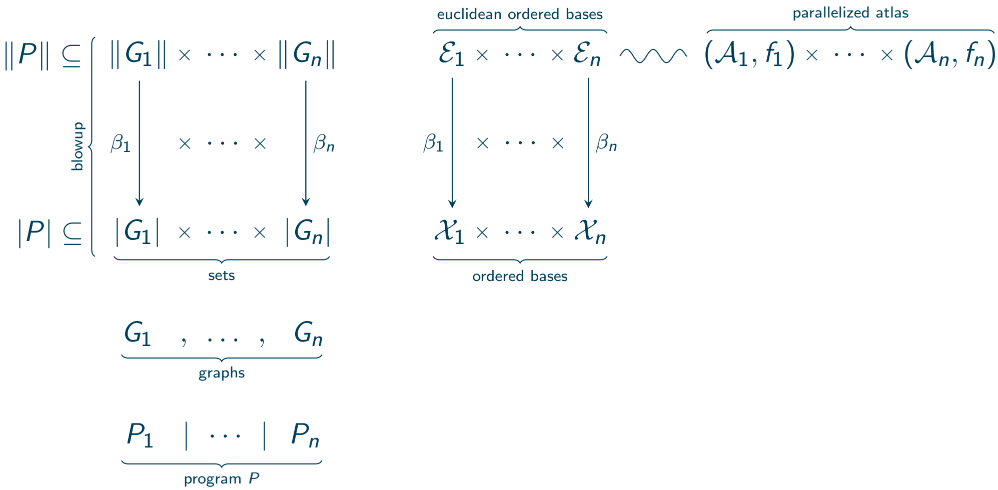
euclidean ordered bases
parallelized atlas
$\|P\| \subseteq \|G_1\| \times \cdots \times \|G_n\|$
$\mathcal{E}_1 \times \cdots \times \mathcal{E}_n \rightsquigarrow (\mathcal{A}_1, f_1) \times \cdots \times (\mathcal{A}_n, f_n)$
blowup
$\beta_1 \times \cdots \times \beta_n$
$|P| \subseteq |G_1| \times \cdots \times |G_n|$
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sets
ordered bases
$G_1, \ldots, G_n$
graphs
$P_1 \mid \cdots \mid P_n$
program P

GEOMETRIC MODELS

Cartesian product

Cartesian product in $\mathcal{S}et$

Cartesian product in $\mathcal{Set}$

$$A \times B := \left\{ (a, b) \;\middle|\; a \in A \text{ and } b \in B \right\}$$

Cartesian product in $\mathcal{S}et$

$$A \times B := \left\{ (a, b) \;\middle|\; a \in A \text{ and } b \in B \right\}$$

There exist two mappings π_A and π_B

$$\begin{aligned} \pi_A \; : \; A \times B &\longrightarrow A \\ (a, b) &\longmapsto a \end{aligned} \qquad \begin{aligned} \pi_B \; : \; A \times B &\longrightarrow B \\ (a, b) &\longmapsto b \end{aligned}$$

Cartesian product in $\mathcal{S}et$

$$A \times B := \{(a,b) \mid a \in A \text{ and } b \in B\}$$

There exist two mappings π_A and π_B

$$\pi_A : A \times B \longrightarrow A \qquad (a,b) \longmapsto a$$

$$\pi_B : A \times B \longrightarrow B \qquad (a,b) \longmapsto b$$

such that for all sets X the following map is a bijection

$$\mathcal{S}et[X, A \times B] \longrightarrow \mathcal{S}et[X, A] \times \mathcal{S}et[X, B]$$

$$h \longmapsto (\pi_A \circ h, \pi_B \circ h)$$

Cartesian product in a category $\mathcal{C}$

The object c is the Cartesian product (in $\mathcal{C}$) of a and b when there exist two morphisms $\pi_a : c \to a$ and $\pi_b : c \to b$ such that for all objects x of $\mathcal{C}$ the following map is a bijection

$$\begin{array}{ccc} \mathcal{C}[x, c] & \longrightarrow & \mathcal{C}[x, a] \times \mathcal{C}[x, b] \\ h & \longmapsto & (\ \pi_a \circ h\ ,\ \pi_b \circ h\ ) \end{array}$$

When such an object c exists we write $c = a \times b$

Cartesian product in the category of graphs ($\mathcal{Grph}$)

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$$\left(\begin{array}{c} A \\ \mathsf{t}\Big\downarrow\Big\downarrow\mathsf{s} \\ V \end{array}\right) \times \left(\begin{array}{c} A' \\ \mathsf{t}'\Big\downarrow\Big\downarrow\mathsf{s}' \\ V' \end{array}\right) \cong$$

Cartesian product in the category of graphs ($\mathcal{Grph}$)

$$\left(\begin{array}{c} A \\ t \downarrow\downarrow s \\ V \end{array}\right) \times \left(\begin{array}{c} A' \\ t' \downarrow\downarrow s' \\ V' \end{array}\right) \cong \left(\begin{array}{c} A \times A' \\ t\times t' \downarrow\downarrow s\times s' \\ V \times V' \end{array}\right)$$

The Cartesian product in $\mathcal{Grph}$ is deduced form the Cartesian product in $\mathcal{Set}$

Examples of Cartesian products

Examples of Cartesian products

- The product of (X, Ω_X) and (Y, Ω_Y) in $\mathcal{Top}$

Examples of Cartesian products

- The product of (X, Ω_X) and (Y, Ω_Y) in $\mathcal{T}op$ is given by $X \times Y$ together with unions of subsets of the form $U \times V$ with $U \in \Omega_X$ and $V \in \Omega_Y$.

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- The product of $(X, \sqsubseteq_X)$ and $(Y, \sqsubseteq_Y)$ in $\mathcal{Pos}$

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- The product of (X, d_X) and (Y, d_Y) in $\mathcal{Met}_{emb}$

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- The product of (X, d_X) and (Y, d_Y) in $\mathcal{Met}_{emb}$ does not exist.

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- The product of (X, d_X) and (Y, d_Y) in $\mathcal{Met}_{emb}$ does not exist.
- The product of (X, d_X) and (Y, d_Y) in $\mathcal{Met}_{ctr}$ is given by $X \times Y$ together with $d\big((x, y), (x', y')\big) = \max\{d_X(x, x'), d_Y(y, y')\}$.

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- The product of (X, d_X) and (Y, d_Y) in $\mathcal{Met}_{top}$

Examples of Cartesian products

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- The product of (X, d_X) and (Y, d_Y) in $\mathcal{Met}_{top}$ can also be given by $X \times Y$ together with the Euclidean product

$$d((x, y), (x', y')) = \sqrt{d_X^2(x, x') + d_Y^2(y, y')}$$

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- The product of $(X, [\mathcal{U}]_\sim)$ and $(Y, [\mathcal{V}]_\sim)$ in $\mathcal{Lpo}$ is given by $X \times Y$ together with the collection of ordered charts $U \times V$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$.
- The product of (X, d_X) and (Y, d_Y) in $\mathcal{Met}_{emb}$ does not exist.
- The product of (X, d_X) and (Y, d_Y) in $\mathcal{Met}_{ctr}$ is given by $X \times Y$ together with $d\big((x, y), (x', y')\big) = \max\{d_X(x, x'), d_Y(y, y')\}$.
- The product of (X, d_X) and (Y, d_Y) in $\mathcal{Met}_{top}$ can also be given by $X \times Y$ together with the Euclidean product

$$d\big((x, y), (x', y')\big) = \sqrt{d_X^2(x, x') + d_Y^2(y, y')}$$

- Categories of models of algebraic theories.

Infinite Cartesian product

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Infinite products of directed circle does not exist in $\mathcal{L}po$.

Turning discrete models into geometric ones

Canonical partition

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where $p = (p_1, \ldots, p_n)$, $p_i \in V_i \sqcup A_i$, and $\dim p = \#\{ i \in \{1, \ldots, n\} \mid p_i \in A_i \}$

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The collection of canonical blocks forms the canonical partition of $\upharpoonleft G_1 \downharpoonright \times \cdots \times \upharpoonleft G_n \downharpoonright$.

The geometric model of a conservative program

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The forbidden region of a conservative program $\Pi = (G_1, \ldots, G_n)$ is the disjoint union of canonical blocks

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the distance being given by

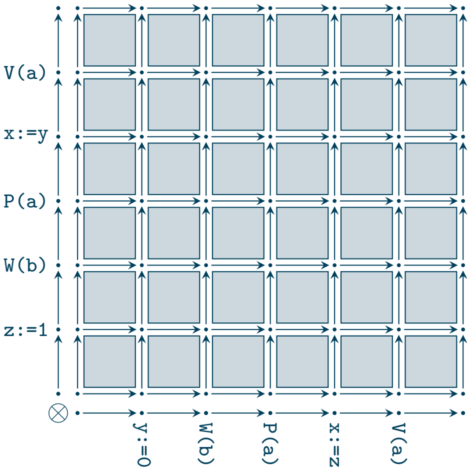
$$d(p, p') \quad = \quad \max\left\{ d_{\upharpoonleft G_i \downharpoonright}(p_i, p'_i) \mid i \in \{1, \ldots, n\} \right\}$$

in accordance with the fact that the execution time of the simultaneous execution of many processes is the longest execution time among that of the processes considered individually.

Gallery of examples

From discrete to continuous

sem: 1 a   sync: 1 b



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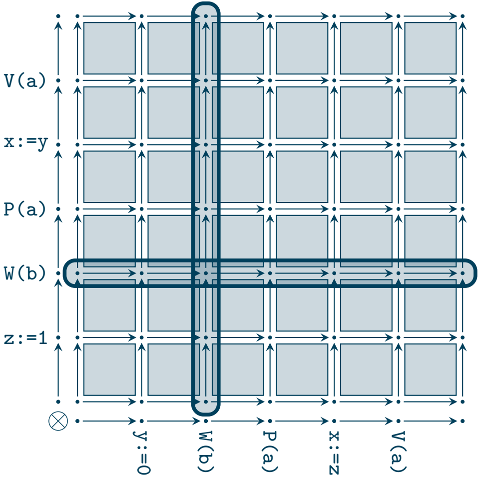

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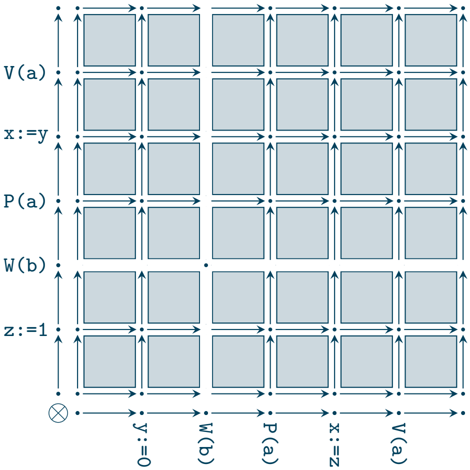

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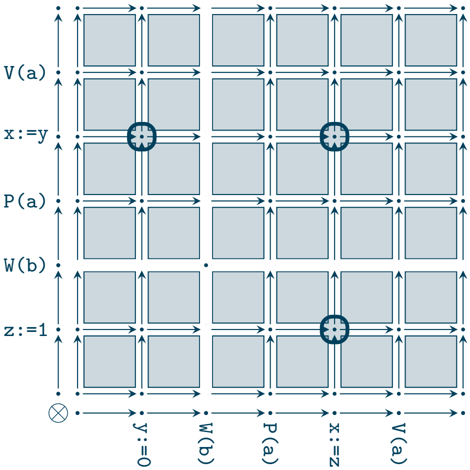

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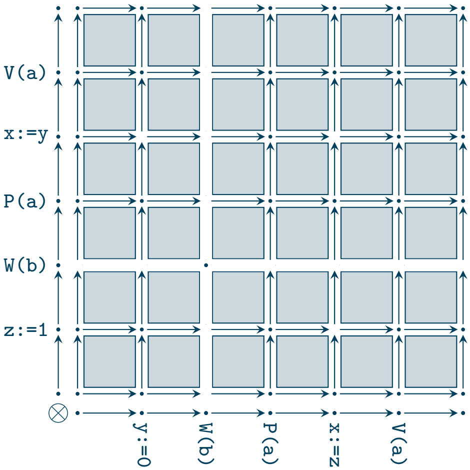

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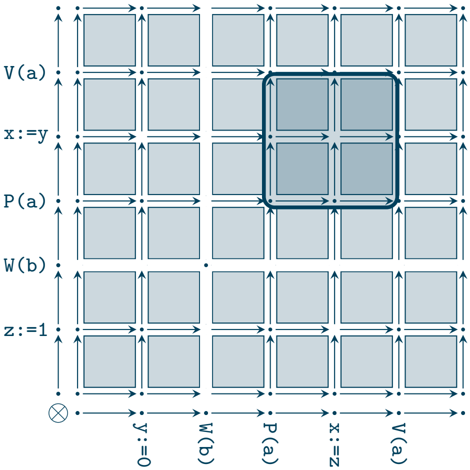

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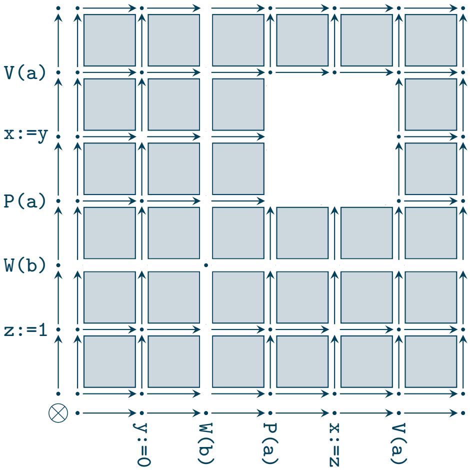

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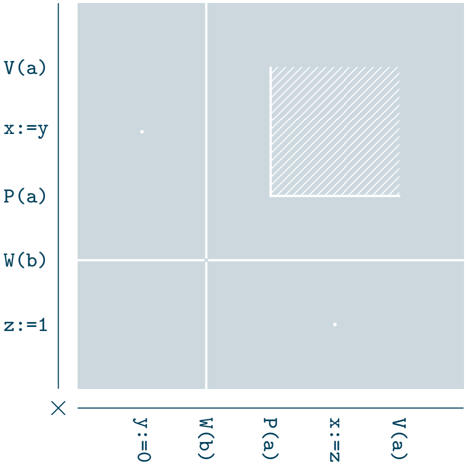

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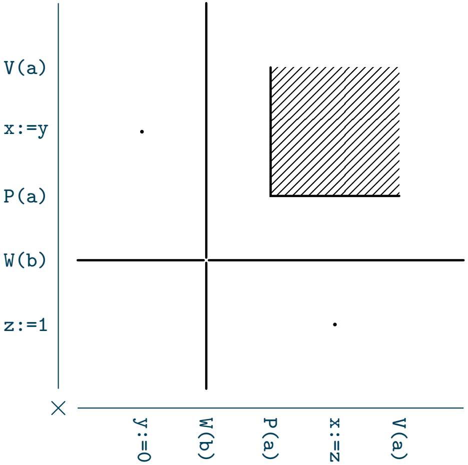

Square

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sem 1 a
proc:  p = P(a);V(a)
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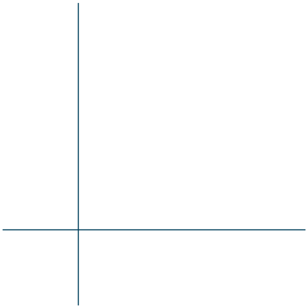

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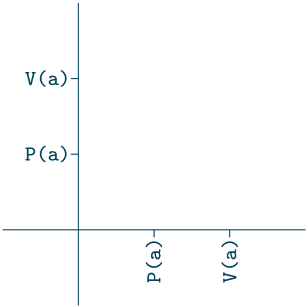

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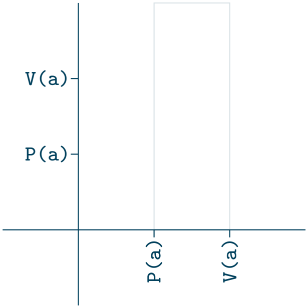

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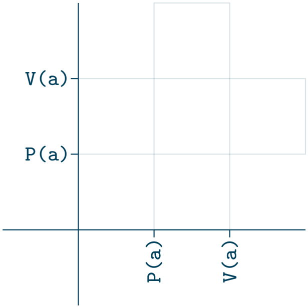

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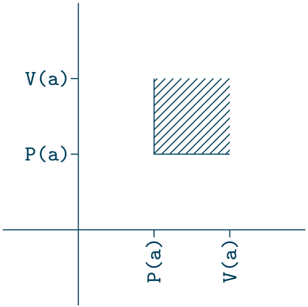

Swiss Cross

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```
sem 1 a b
proc:
p = P(a);P(b);V(b);V(a)
q = P(b);P(a);V(a);V(b)
init:  p q
```

Swiss Cross

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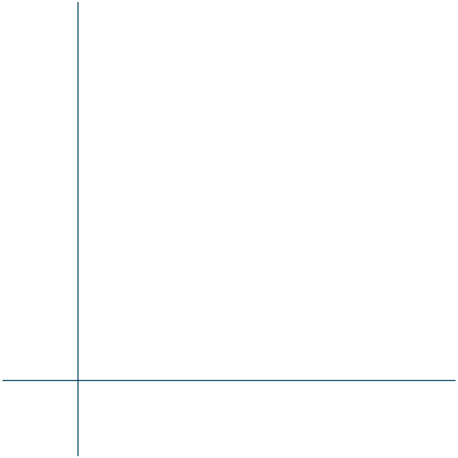

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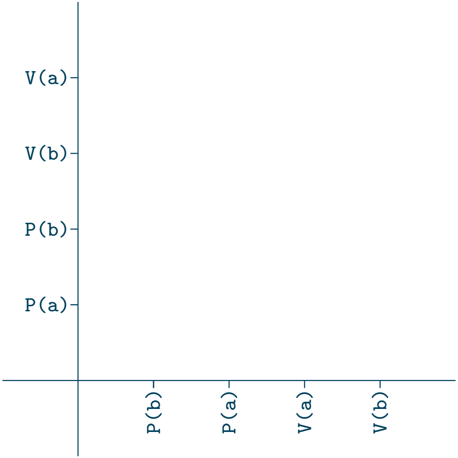

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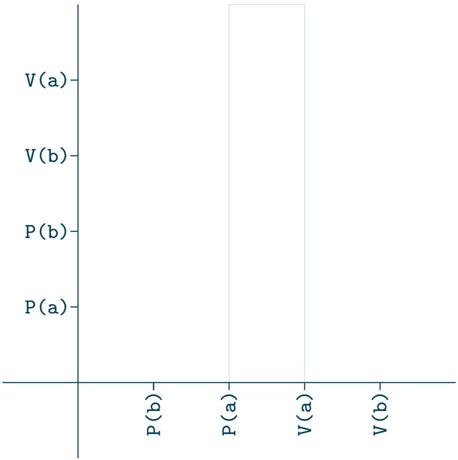

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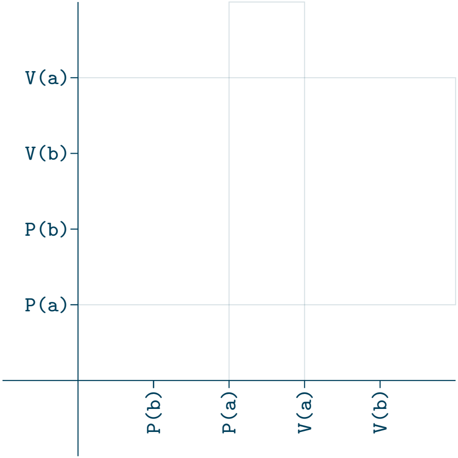

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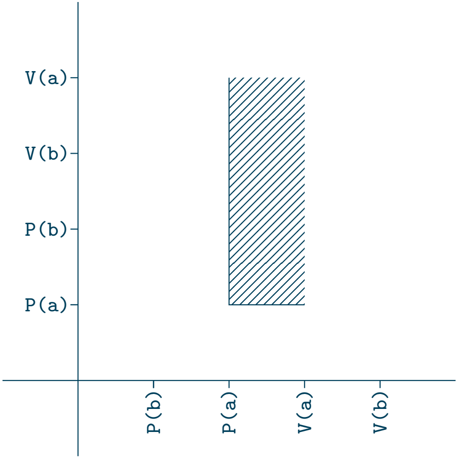

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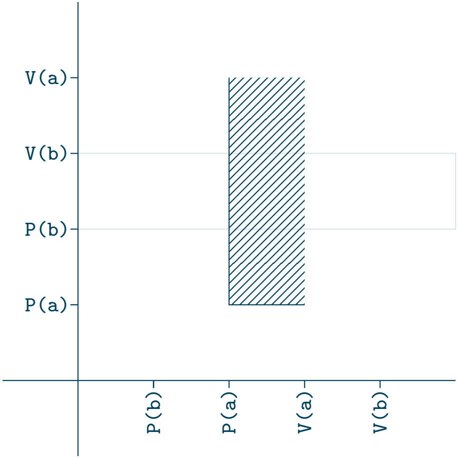

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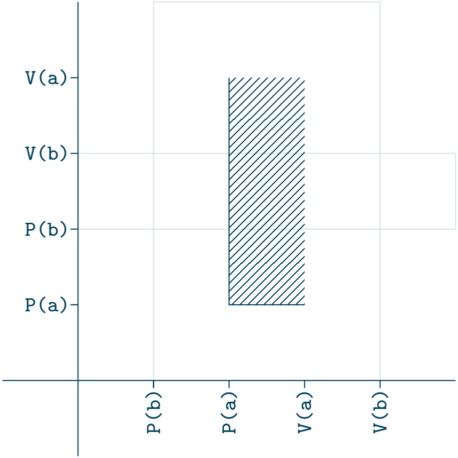

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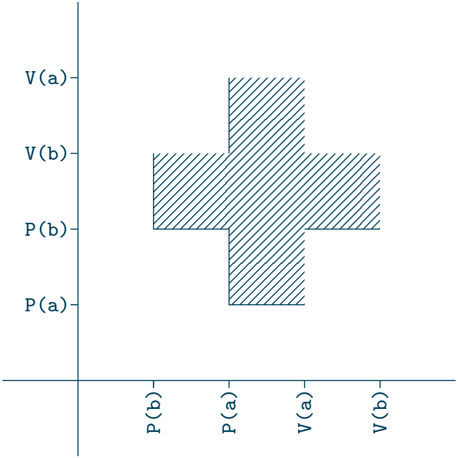

Binary synchronization

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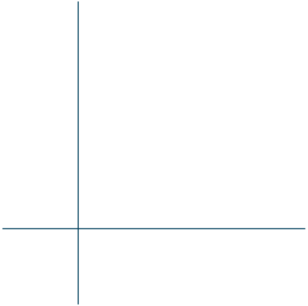

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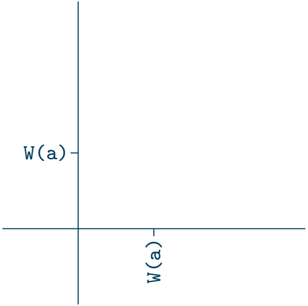

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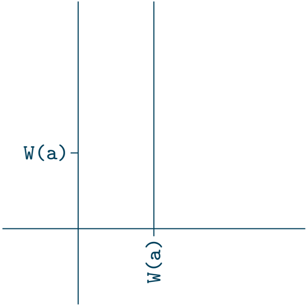

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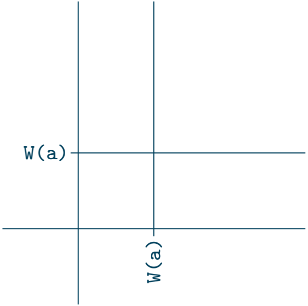

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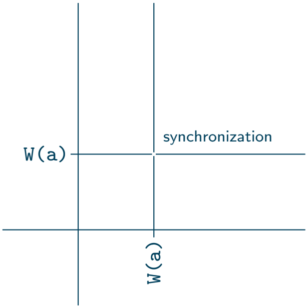

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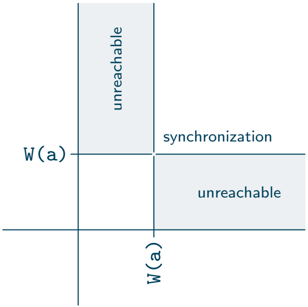

Producer/Consumer

nonlooping

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```
sync 1 a
proc:
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  c = W(a)   ; x:=x-1
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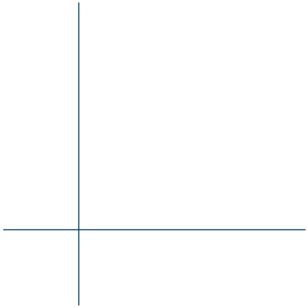

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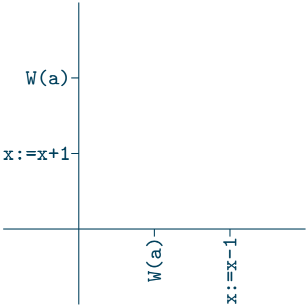

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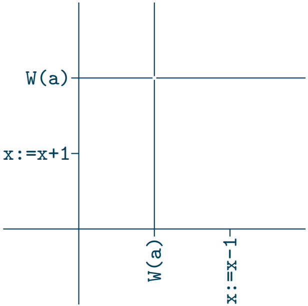

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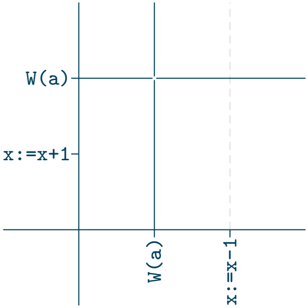

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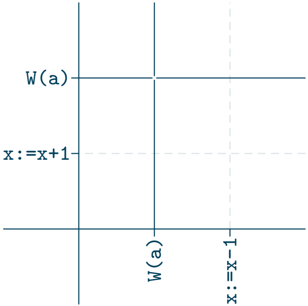

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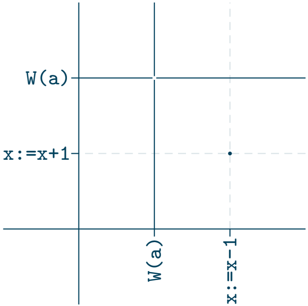

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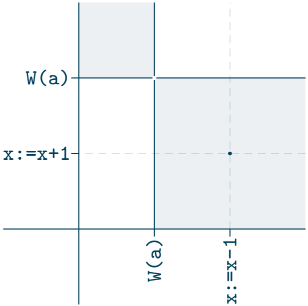

Producer/Consumer

looping

Producer/Consumer

looping

```
sync 1 a b
proc:
  p = x:=x+1 ; W(a)   ; W(b) ; J(p)
  c = W(a)   ; x:=x-1 ; W(b) ; J(c)
init:  p c
```

Producer/Consumer

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sync 1 a b
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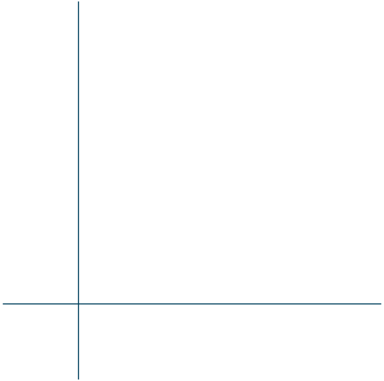

Producer/Consumer

looping

```
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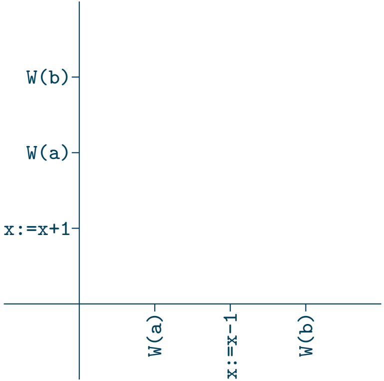

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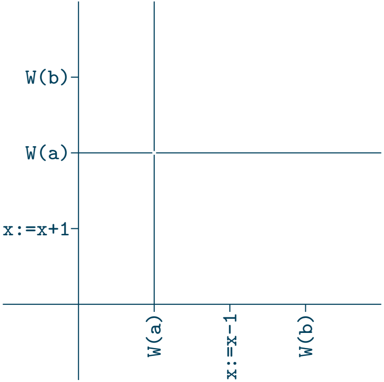

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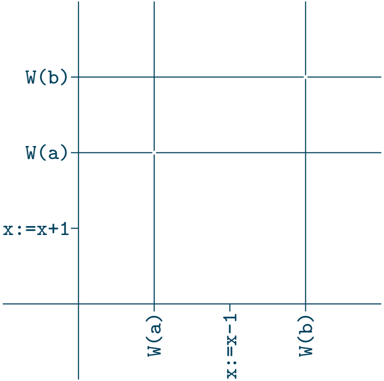

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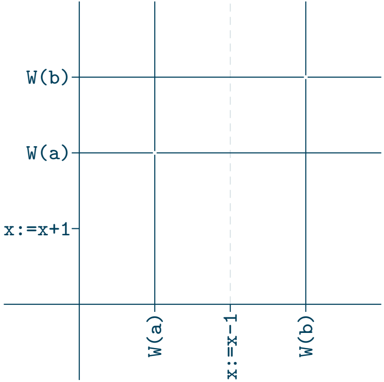

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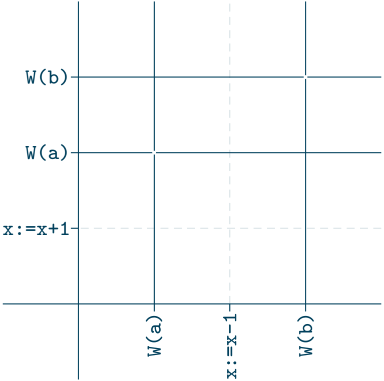

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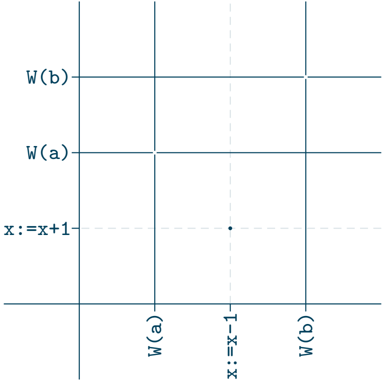

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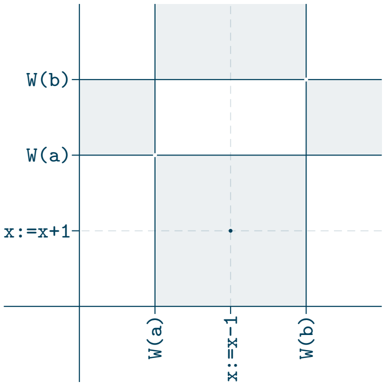
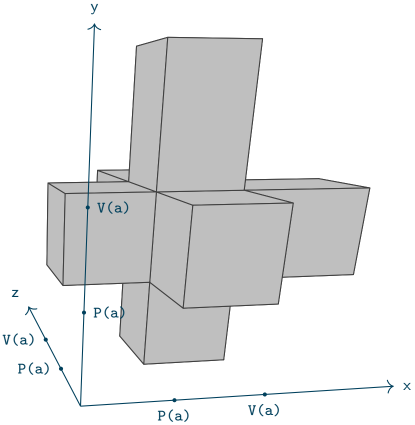
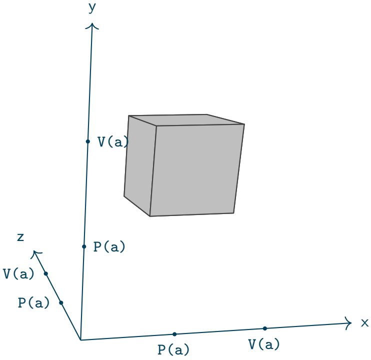
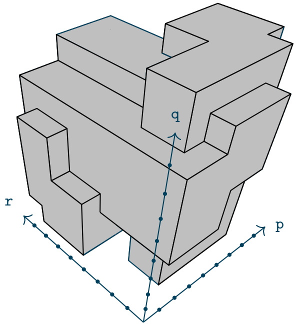


3D Swiss Cross (tetrahemihexacron) and floating cube





The Lipski algorithm




```
sem 1:  u v w x y z
proc:
  p = P(x);P(y);P(z);V(x);P(w);V(z);V(y);V(w)
  q = P(u);P(v);P(x);V(u);P(z);V(v);V(x);V(z)
  r = P(y);P(w);V(y);P(u);V(w);P(v);V(u);V(v)
init:  p q r
```

Geometric vs Discrete

Justifying the definition of discrete directed paths

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Discretization and lifting

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- Given a directed path γ on the local pospace $\upharpoonleft G_1 \downharpoonright \times \cdots \times \upharpoonleft G_n \downharpoonright$ we have a finite partition $I_0 < \cdots < I_N$ of $\mathrm{dom}(\gamma)$ such that for all $k \in \{0, \ldots, N\}$, there exists a (necessarily unique) point p^k such that $\gamma(I_k) \subseteq B_{p^k}$.

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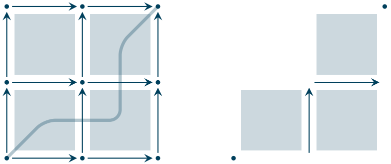
- The sequence $p^0, \ldots, p^N$ is a directed path on $(G_1, \ldots, G_n)$, it is called the discretization of γ and denoted by $D(\gamma)$.
- Given a directed path δ on $(G_1, \ldots, G_n)$ there exists a directed path γ on $\upharpoonleft G_1 \downharpoonleft \times \cdots \times \upharpoonleft G_n \downharpoonleft$ whose discretization is δ, such a directed path γ is said to be a lifting of δ.

Example of discretization



Admissible directed paths and execution traces

on $\upharpoonleft G_1 \downharpoonleft \times \cdots \times \upharpoonleft G_n \downharpoonleft$

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The action of a directed path γ on $\upharpoonleft G_1 \downharpoonleft \times \cdots \times \upharpoonleft G_n \downharpoonleft$ on the right of a state σ is that of its discretization of $D(\gamma)$.

Example

```
var x = 0
var y = 0
var z = 0
sync 1 b
sem 1 a
```

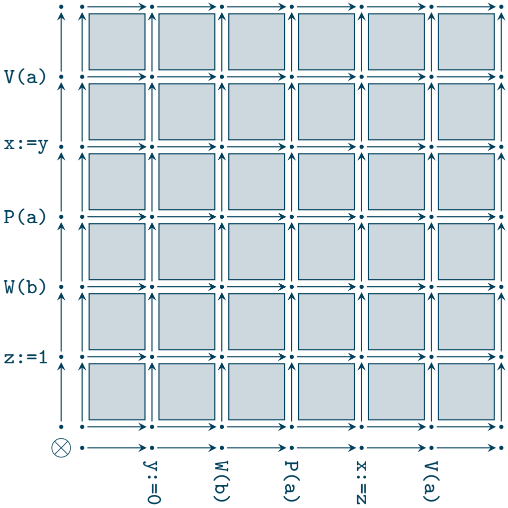
```
proc p = y:=0 ; W(b) ; P(a) ; x:=z ; V(a)

proc q = z:=1 ; W(b) ; P(a) ; x:=y ; V(a)
```

```
init p q
```

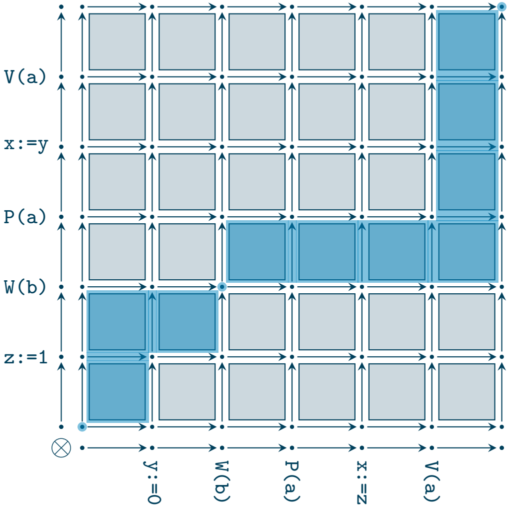
Discretization of an execution trace

`sem: 1 a` `sync: 1 b`



Discretization of an execution trace

`sem:  1 a        sync:  1 b`



Discretization of an execution trace

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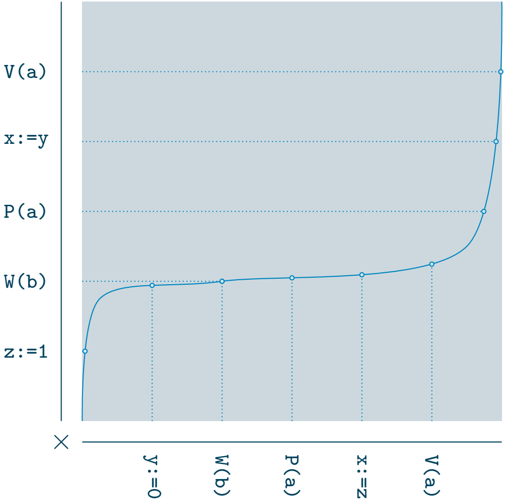

Potential function on $\upharpoonleft G_1 \downharpoonright \times \cdots \times \upharpoonleft G_n \downharpoonright$

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If the program under consideration is conservative, then we have the potential function

$$F : \upharpoonleft G_1 \downharpoonright \times \cdots \times \upharpoonleft G_n \downharpoonright \times \mathcal{S} \to \{\text{multisets over } \{1, \ldots, n\}\}$$

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The function F is constant on each canonical block B_p, its value is given by $\tilde{F}(p)$ where $\tilde{F}$ denotes the "discrete" potential function.

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- Any directed path on a continuous model is admissible.

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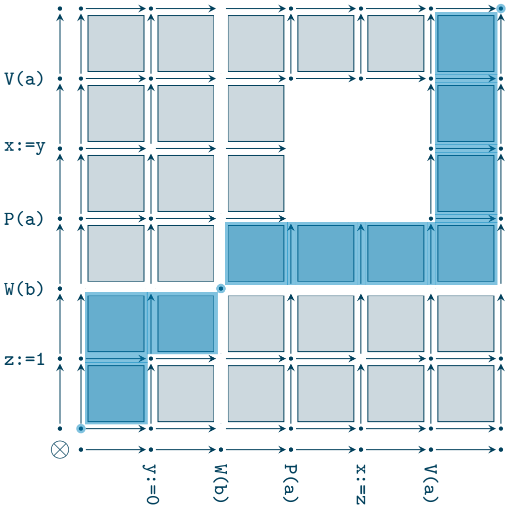
- Any directed path on a continuous model is admissible.
- Conversely, for each admissible path on a continuous model which meets a forbidden point, there exists a directed path which avoids them and such that both directed paths induce the same sequence of multi-instructions.

Directed paths on the geometric model are admissible

sem: 1 a sync: 1 b



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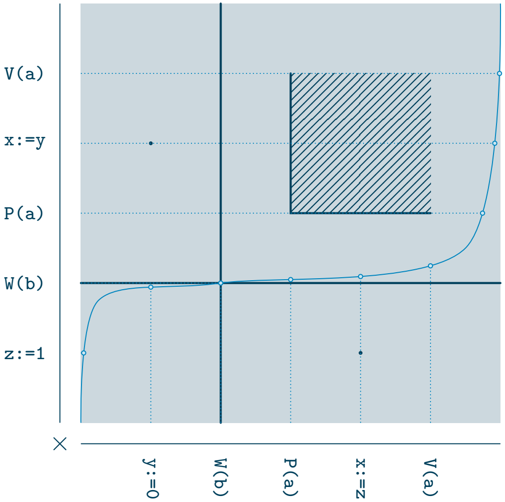
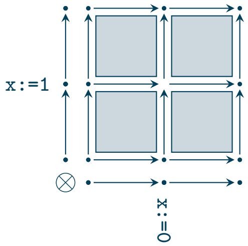


Continuous replacement

sem: 1 a sync: 1 b



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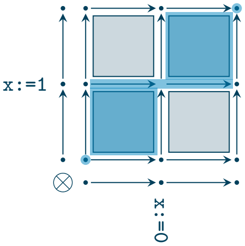

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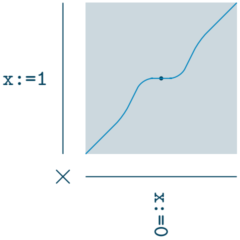

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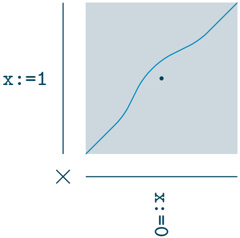

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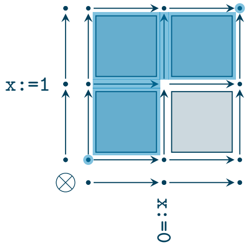

The motivating theorem

Trade off

More mathematics for more properties?

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- Both discrete and geometric models are sound and complete.

Trade off

More mathematics for more properties?

- Both discrete and geometric models are sound and complete.
- The continuous models satisfy extra properties that are “naturally” expressed in terms of metrics.

Uniform distance between directed paths

Uniform distance between directed paths

Given a compact Hausdorff space K and a metric space (X, d_X), the set of continuous maps from K to X can be equipped with the uniform distance

$$d(f,g) \quad = \quad \max\{d_X(f(k),g(k)) \mid k \in K\} \ .$$

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We consider the case where $K = [0, r]$ is the domain of definition of a directed path and (X, d_X) is the geometric model of a conservative program.

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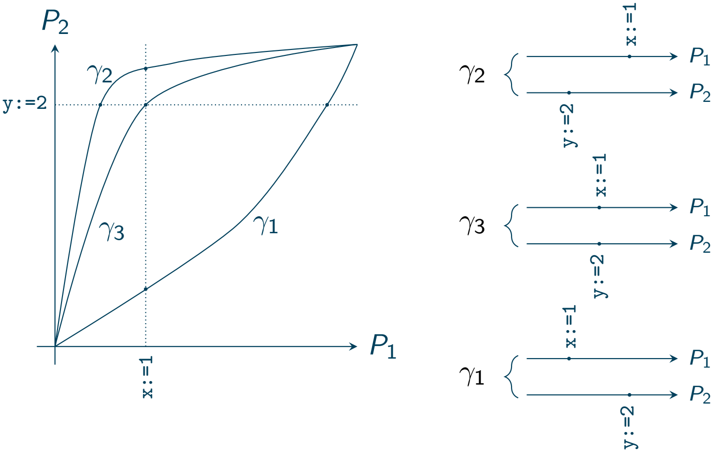
There exists an open ball Ω of $dX^{[0,r]}(B_p, B_{p'})$, centred in γ, such that all the elements of Ω induce the same action on valuations. Moreover, if γ is an execution trace, then so are all the elements of Ω.

Illustration



HOMOTOPY OF PATHS

The undirected case

Homotopy of paths

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As a consequence we have $\gamma(0) = \delta(0)$ and $\gamma(r) = \delta(r)$.

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The Curryfication $(\hat{\_})$ induces a homeomorphism from $X^{[0,r]\times[0,q]}$ to $\left(X^{[0,r]}\right)^{[0,q]}$

$$(h : [0, r] \times [0, q] \to X) \to (\hat{h} : [0, q] \to X^{[0,r]})$$

The two faces of homotopies

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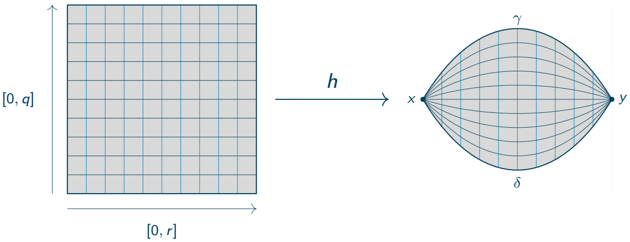
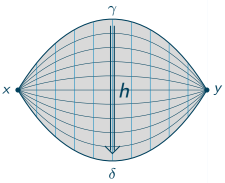
h is a continuous map from $[0,r]\times[0,q]$ to X i.e. $h \in \mathcal{Top}\big[[0,r]\times[0,q],X\big]$



The two faces of homotopies

but is also a path from γ to δ in the space $X^{[0,r]}$ i.e. $h \in \mathcal{Top}\big[[0,q], X^{[0,r]}\big]$



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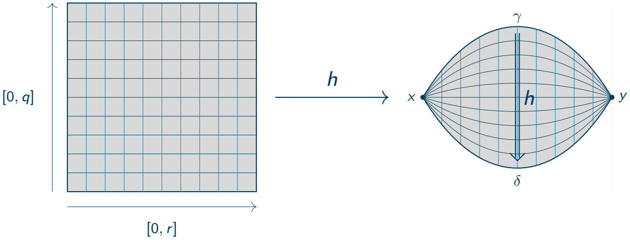

We introduce the following notation

Concatenation of homotopies

vertical composition

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Let $g : [0, r] \times [0, q'] \to X$ and $h : [0, r] \times [0, q] \to X$ be homotopies from γ to ξ and from ξ to δ.

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Let $g : [0, r] \times [0, q'] \to X$ and $h : [0, r] \times [0, q] \to X$ be homotopies from γ to ξ and from ξ to δ.

The mapping $h * g : [0, r] \times [0, q + q'] \to X$ defined by

$$h * g(t, s) \quad = \quad \begin{cases} g(t, s) & \text{if } 0 \leqslant s \leqslant q \\ h(t, s - q) & \text{if } q \leqslant s \leqslant q + q' \end{cases}$$

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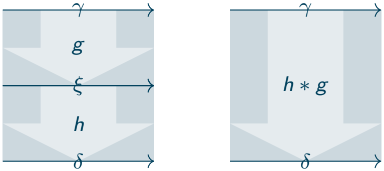

The directed case

Directed homotopy on a locally ordered space

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Let $\gamma, \delta \in \mathcal{L}po([0, r], X)$ such that $\partial^- \gamma = \partial^- \delta$ and $\partial^+ \gamma = \partial^+ \delta$.

Directed homotopy on a locally ordered space

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- Each of the preceding class of homotopies is stable under concatenation.

Homotopy and dihomotopy relations

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Therefore γ and $\gamma \circ \theta$ are dihomotopic.

Images of directed paths on a pospace

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Hence $\phi \circ h$ is an elementary homotopy from γ and γ'.

Relation to geometric models

Main theorem

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Two weakly dihomotopic paths on the geometric model of a conservative program induce the same action on valuations. Moreover, if one of them is an execution trace, then so is the other.

Proof

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By a standard result from general topology, the Curryfication of h

$$\hat{h} : s \in [0, q] \mapsto (t \in [0, r] \mapsto h(t, s) \in X)$$

is a continuous path on $dX^{[0,r]}(p, p')$.

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The conclusion follows considering the sequence

$$\hat{h}(0), \hat{h}(\varepsilon), \hat{h}(2\varepsilon), \hat{h}(3\varepsilon), \cdots, \hat{h}(n\varepsilon), \hat{h}(q)$$

where n is the greatest natural number such that $n\varepsilon \leqslant q$.

Programs with mutex only

Directed Homotopy in Non-Positively Curved Spaces, *É. Goubault and S. Mimram*, LMCS 2020

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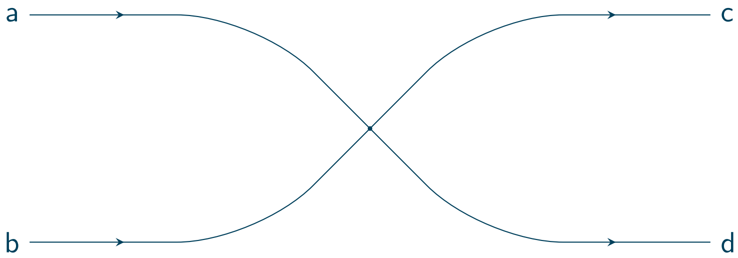
Let X be the geometric model of a conservative program whose semaphores have arity 1 (mutex), then two directed paths on X are dihomotopic if and only if they are homotopic.

SMOOTH MODELS

Removing singularities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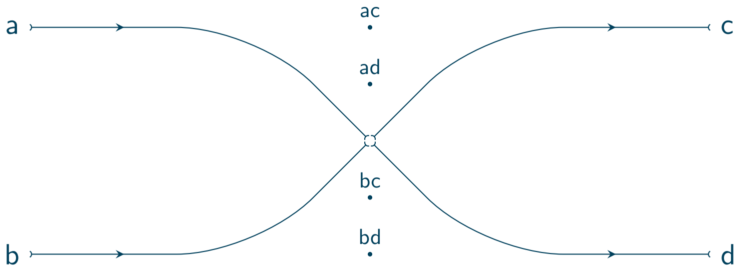
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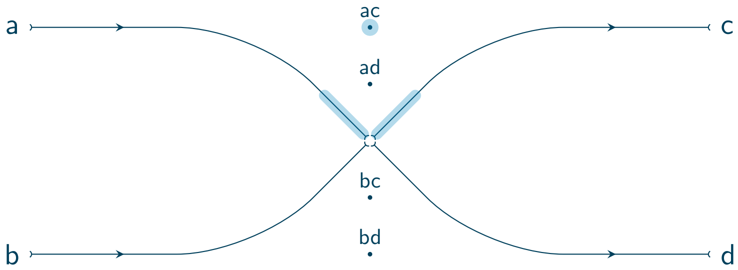
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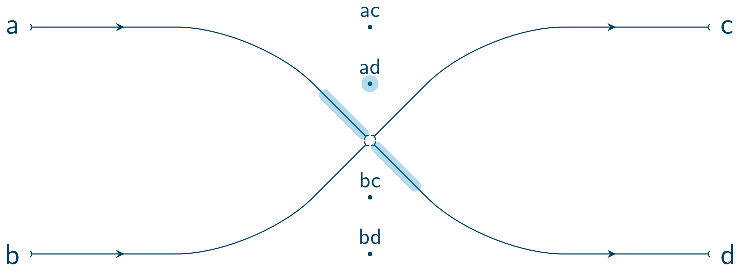
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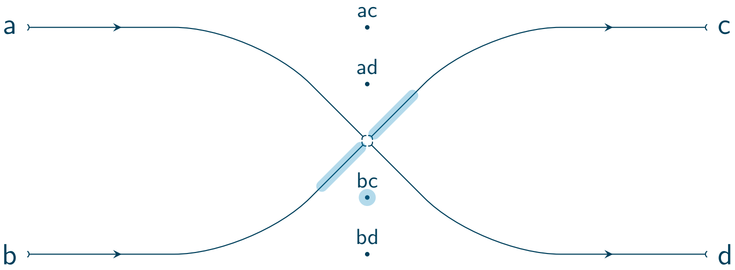
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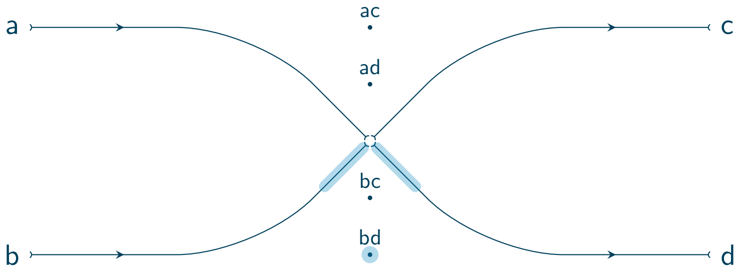
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For small $\varepsilon > 0$, the ε-neighborhoods of (a,t) and (a,b) are

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The *standard ordered base* $\mathcal{E}_G$ of G is the collection of ε-neighborhoods (each of them being equipped with the obvious total order).

The *blowup* of G is the map

$$\begin{array}{cccc} \beta_G & : & \|G\| & \to & |G| \\ & & (a,b) & \mapsto & \partial^+(a)(=\partial^-(b)) \\ & & (a,t) & \mapsto & (a,t) \end{array}$$

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The blowup β_G is locally order-preserving from $\mathcal{E}_G$ to $\mathcal{X}_G$.

Universal property of graph blowups

An ordered base $\mathcal{E}$ is said to be *euclidean* of dimension $n \in \mathbb{N}$ when every point p of $\mathcal{E}$ is contained in some $E \in \mathcal{E}$ with $E \cong \mathbb{R}^n$ (as ordered spaces).

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A locally order-preserving map $f : \mathcal{E} \to \mathcal{X}$ is a *local embedding* when for every point p of $\mathcal{E}$ and $X \in \mathcal{X}$ containing $f(p)$, there exists $E \in \mathcal{E}$ containing p such that $f : E \to X$ is an ordered space embedding.

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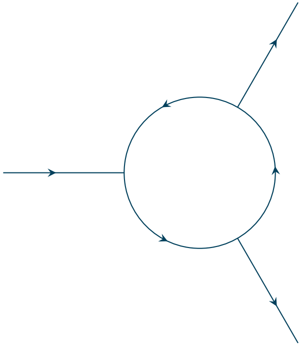
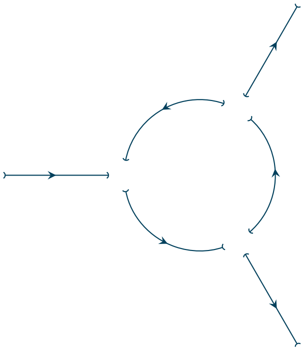
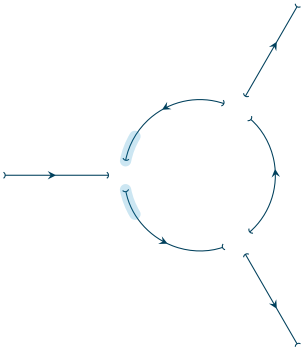
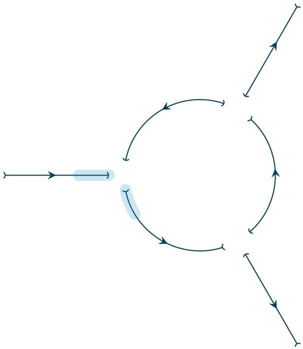
Theorem (Universal property of graph blowups)

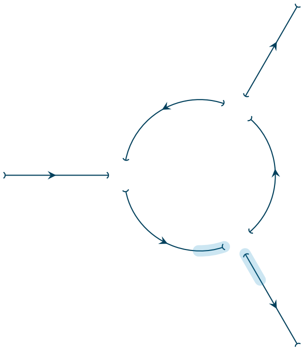
For every euclidean ordered base $\mathcal{E}$, and every local embedding $f : \mathcal{E} \to \mathcal{X}_{G_1} \times \cdots \times \mathcal{X}_{G_n}$ of dimension n, there is a unique continuous map $g : \mathcal{E} \to \mathcal{E}_{G_1} \times \cdots \times \mathcal{E}_{G_n}$ such that $f = \bar{\beta} \circ g$ with $\bar{\beta} = \beta_{G_1} \times \cdots \times \beta_{G_n}$; moreover g is a local embedding of dimension n.

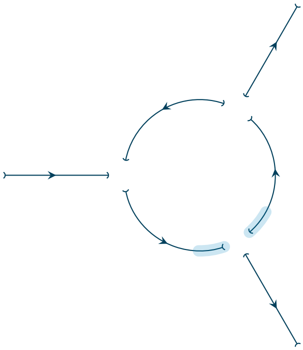


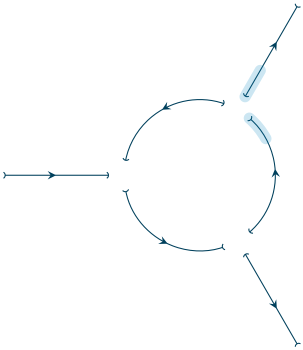


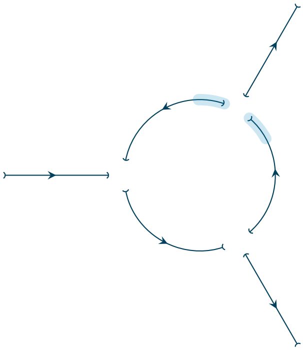












Local orders and Vector fields

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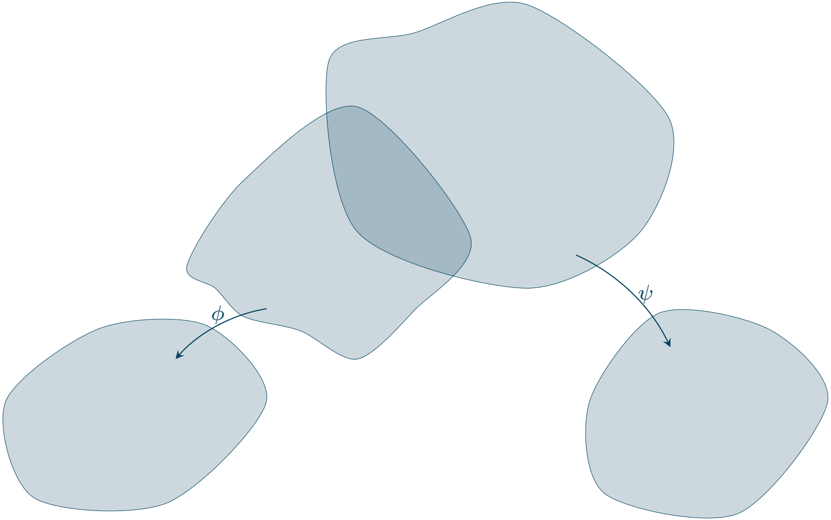
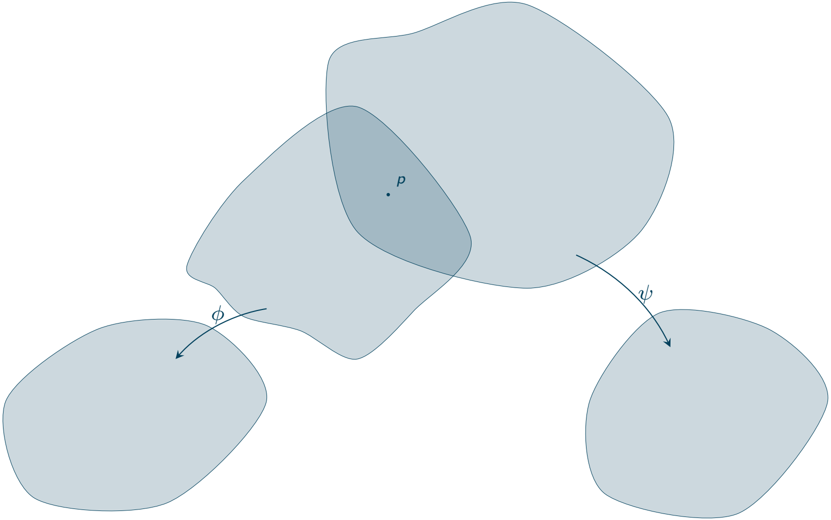
$U \subseteq \mathrm{dom}(\phi)$ is said to be *open* when so is $\phi(U)$ in $\mathbb{R}^n$; we deduce $\phi_U : U \to \phi(U)$.

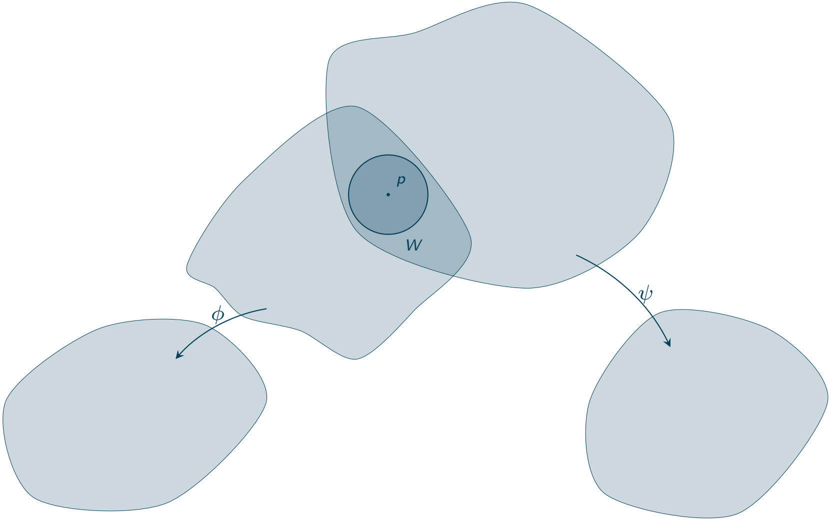
The n-charts ϕ and ψ are



The n-charts ϕ and ψ are *compatible at p* $\in \text{dom}(\phi) \cap \text{dom}(\psi)$ when



The n-charts ϕ and ψ are *compatible at $p \in \mathrm{dom}(\phi) \cap \mathrm{dom}(\psi)$* when there exists W open in $\mathrm{dom}(\phi)$ and in $\mathrm{dom}(\psi)$ such that



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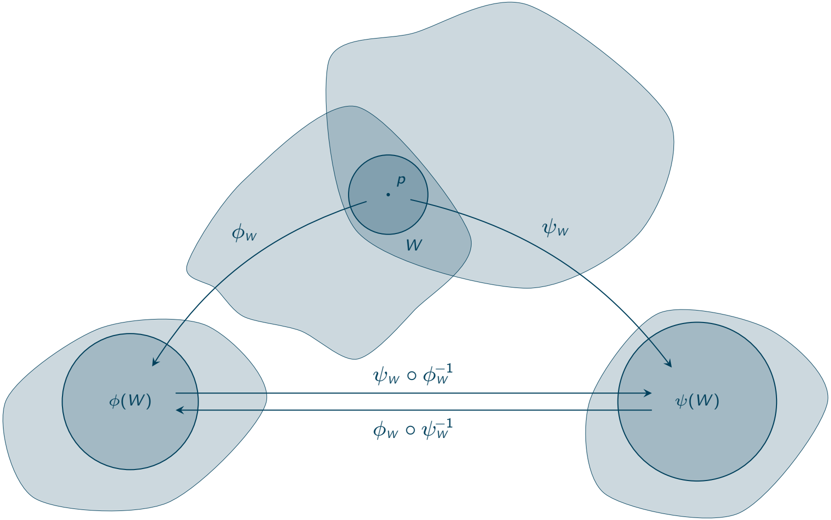

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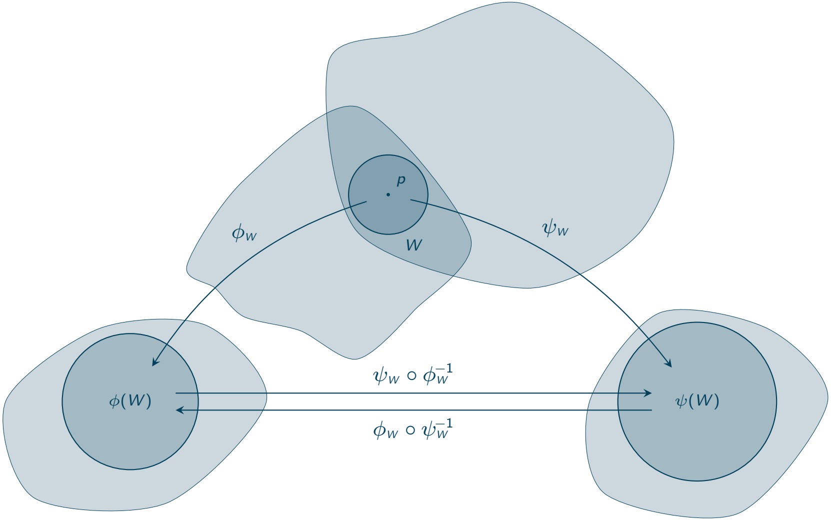
We say that W is a *witness of compatibility* of ϕ and ψ at p.



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$$\Updownarrow$$

$W = \mathrm{dom}(\phi) \cap \mathrm{dom}(\psi)$ is open in $\mathrm{dom}(\phi)$ and in $\mathrm{dom}(\psi)$ and the maps $\phi_W \circ \psi_W^{-1}$ and $\psi_W \circ \phi_W^{-1}$ are smooth.

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Given atlases $\mathcal{A}$, $\mathcal{B}$, map $f : \mathcal{A} \to \mathcal{B}$ is said to be *smooth* when for all $\phi \in \mathcal{A}$, $p \in \mathrm{dom}(\phi)$, $\psi \in \mathcal{B}$ with $f(p) \in \mathrm{dom}(\psi)$, $\psi \circ f \circ \phi^{-1}$ is smooth (as a map between open subsets of euclidean spaces).

The *standard charts* of G are the following bijections

$$\begin{aligned}
\phi_a &: \{a\} \times]0,1[\;\to\;]0,1[\;, \quad \text{and} \\
\phi_{ab} &: \{a\} \times]\tfrac{1}{2},1[\,\cup\, \{(a,b)\} \,\cup\, \{b\} \times]0,\tfrac{1}{2}[\;\to\;]-\tfrac{1}{2},\tfrac{1}{2}[\\
&\text{with} \quad (a,t) \mapsto t-1 \,, \quad (a,b) \mapsto 0 \,, \quad (b,t) \mapsto t
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The *transition maps* are translations:

$$\begin{aligned} \phi_{ab} \circ \phi_a^{-1} &: t \in]\tfrac{1}{2},1[\;\mapsto\; t-1 \;\in\;]-\tfrac{1}{2},0[\\ \phi_{ab} \circ \phi_b^{-1} &: t \in]0,\tfrac{1}{2}[\;\mapsto\; t \;\in\;]\,0,\tfrac{1}{2}[\end{aligned}$$

The set of *tangent vectors* of $\mathcal{A}$ is the quotient

$$\{(p, \phi, u) \mid \phi \in \mathcal{A};\; p \in \mathrm{dom}(\phi);\; u \in \mathbb{R}^n\}/ \sim$$

with $(p, \phi, u) \sim (q, \psi, v)$ when $p = q$ and $d(\psi_W \circ \phi_W^{-1})_{\phi(p)}(u) = v$ (with W a witness of compatibility of ϕ and ψ at p).
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The *tangent space* at p is $T_p\mathcal{A} = \pi_{\mathcal{A}}^{-1}(\{p\})$; it is a vector space with

$$[\![p, \phi, u]\!] + \lambda[\![p, \phi, v]\!] \quad = \quad [\![p, \phi, u + \lambda v]\!]\ .$$

A *vector field* on $\mathcal{A}$ is a smooth map $f : \mathcal{A} \to T\mathcal{A}$ such that $\pi_{\mathcal{A}} \circ f = \mathrm{id}_{\mathcal{A}}$,
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The *standard vector field* on the standard atlas is

$$\begin{array}{rcl} \mathcal{A}_G & \to & T\mathcal{A}_G \\ p & \mapsto & (p, 1) \end{array}$$

For every smooth map $f : \mathcal{A} \to \mathcal{B}$ we have $Tf : T\mathcal{A} \to T\mathcal{B}$ defined by

$$Tf[\![p, \phi, u]\!] = [\![fp, \psi, d(\psi \circ f \circ \phi^{-1})_{\phi(p)}(u)]\!]$$

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For every smooth path γ on $\mathcal{A}_G$, every $\phi \in \mathcal{A}_G$ we have

$$T\gamma(t, u) = T\gamma[\![t, \mathrm{id}_I, u]\!] = [\![\gamma(t), \phi, d(\phi \circ \gamma \circ \mathrm{id}_I^{-1})_t(u)]\!] = (\gamma(t), \gamma'(t) \cdot u)\ .$$

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The tangent vector to γ at t is of the form $(\gamma(t), \gamma'(t))$; γ is locally order-preserving iff $\gamma'(t) \geqslant 0$ for every t.

Proposition (standard vector field vs standard ordered base)

For every $\phi \in \mathcal{A}_G$, *for all* $p, q \in \mathrm{dom}(\phi)$, *we have* $p \leqslant q$ *(with* $(\mathrm{dom}(\phi), \leqslant) \in \mathcal{A}_G$*) iff there exists a smooth path* γ *on* $\mathcal{A}_G$ *from* p *to* q *with* $\mathrm{im}(\gamma) \subseteq \mathrm{dom}(\phi)$ *and* $\gamma' \geqslant 0$, *i.e.* $\phi \circ \gamma$ *is a smooth map between open intervals of* $\mathbb{R}$ *with nonnegative derivative,* $\min(\phi \circ \gamma) = \phi(p)$, *and* $\max(\phi \circ \gamma) = \phi(q)$.

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The above result is a special instance of Lawson's correspondence:

Ordered manifolds, invariant cone fields, and semigroups. Lawson, J. D., Forum Mathematicum, 1989.

Approximation

From every norm $|\_|$ on $\mathbb{R}^n$ one defines the length of a smooth path $\gamma = (\gamma_1, \ldots, \gamma_n)$ on $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$ by

$$\mathcal{L}(\gamma) \quad = \quad \int_{t \in I} |\gamma'(t)| dt$$

with $\gamma'(t) = (\gamma_1'(t), \ldots, \gamma_n'(t))$ the coordinates of the tangent vector to γ at t in the standard base $((\gamma_1(t), 1), \ldots, (\gamma_n(t), 1))$ of the tangent space at $\gamma(t)$.

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We also define the distance between p, $q \in |G_1| \times \cdots \times |G_n|$ as $d(p, q) = |d_{G_1}(p_1, q_1), \dots, d_{G_n}(p_n, q_n)|$ from which we deduce the length $L(\gamma)$ of any path γ on $|G_1| \times \cdots \times |G_n|$.

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If δ is a smooth path on $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$ then $\mathcal{L}(\delta) = L((\beta_{G_1} \times \cdots \times \beta_{G_n}) \circ \delta)$.

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$\lvert x_1, \ldots, x_n \rvert_2$	$=$	$\sqrt{\sum_{i=1}^n x_i^2}$	Riemannian
$\lvert x_1, \ldots, x_n \rvert_1$	$=$	$\sum_{i=1}^n \lvert x_i \rvert$	cumulative execution time
$\lvert x_1, \ldots, x_n \rvert_\infty$	$=$	$\max\{x_1, \ldots, x_n\}$	parallel execution time

A subset X of $|G_1| \times \cdots \times |G_n|$ is said to be *tile compatible* when for all $p, q \in |G_1| \times \cdots \times |G_n|$ such that $(\pi_{G_1}, \ldots, \pi_{G_n})(p) = (\pi_{G_1}, \ldots, \pi_{G_n})(q)$, we have $p \in X$ iff $q \in X$.

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The *standard cone* of $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$ at $p = (p_1, \ldots, p_n)$ is the cone $C_p = \{ \sum_{i=1}^{n} (p_i, \lambda_i) \mid \lambda_i \geqslant 0 \} \subseteq T_p(\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n})$.

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A *conal path* on a subset Y of $\|G_1\| \times \cdots \times \|G_n\|$ is a smooth path δ on $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$ such that $\delta(t) \in Y$ and $T\delta(t) \in C_{\delta(t)}$ for every $t \in \mathrm{dom}(\delta)$.

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Theorem (Approximation)

For every directed path $\gamma = (\gamma_1, \ldots, \gamma_n)$ on a tile compatible subset X of $|G_1| \times \cdots \times |G_n|$, and every $\varepsilon > 0$, there exists a conal path $\delta = (\delta_1, \ldots, \delta_n)$ on $(\beta_{G_1} \times \cdots \times \beta_{G_n})^{-1}(X)$ such that:

- *γ and $(\beta_{G_1} \times \cdots \times \beta_{G_n}) \circ \delta$ start (resp. finish) at the same point,*
- $\max \left\{ d_i(\gamma_i(t), \beta_i(\delta_i(t))) \mid t \in \mathrm{dom}(\gamma);\ i \in \{1, \ldots, n\} \right\} < \varepsilon$*, and*
- $\mathcal{L}_\infty(\delta) < L_\infty(\gamma)$.