DIRECTED ALGEBRAIC TOPOLOGY

AND

CONCURRENCY

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MPRI : Concurrency (2.3.1)
– Lecture 2 –

2024 – 2025

A BIT OF CATEGORY THEORY

Categories

Category $\mathcal{C}$

Definition (the "underlying graph" part)

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- We define the homset $\mathcal{C}(x, y) := \left\{ \gamma \in \mathrm{Mo}(\mathcal{C}) \,\middle|\, \partial^-\gamma = x \text{ and } \partial^+\gamma = y \right\}$

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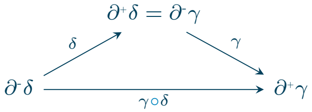
- The binary composition is a partially defined and often denoted by $\circ$

$$\left\{(\gamma,\delta) \;\middle|\; \gamma,\delta \text{ morphisms of } \mathcal{C} \text{ s.t. } \partial^-\gamma = \partial^+\delta\right\} \xrightarrow{\text{composition}} \mathsf{Mo}(\mathcal{C})$$



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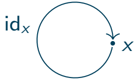

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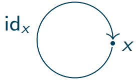

- For all morphisms γ one has $\mathrm{id}_{\partial^{+}\gamma} \circ \gamma = \gamma = \gamma \circ \mathrm{id}_{\partial^{-}\gamma}$

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- The opposite of a category is obtained by reversing all its arrows (i.e. by swapping the roles of the source and the target)

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- if $r \circ s = \mathrm{id}$ then r is called a retract/split epimorphism and s is called a section/split monomorphism.

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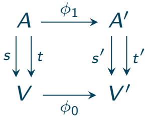

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$$\begin{array}{ccc} A & \xrightarrow{\phi_1} & A' \\ s\Big\downarrow\Big\downarrow t & & s'\Big\downarrow\Big\downarrow t' \\ V & \xrightarrow[\phi_0]{} & V' \end{array}$$

Composition

$$\begin{array}{ccccc} A & \xrightarrow{\phi_1} & A' & \xrightarrow{\psi_1} & A'' \\ s\Big\downarrow\Big\downarrow t & & s'\Big\downarrow\Big\downarrow t' & & s''\Big\downarrow\Big\downarrow t'' \\ V & \xrightarrow[\phi_0]{} & V' & \xrightarrow[\psi_0]{} & V'' \end{array}$$

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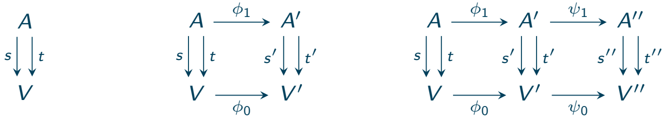

with $s'(\phi_1(\alpha)) = \phi_0(\partial^- \alpha)$ and $t'(\phi_1(\alpha)) = \phi_0(\partial^+ \alpha)$

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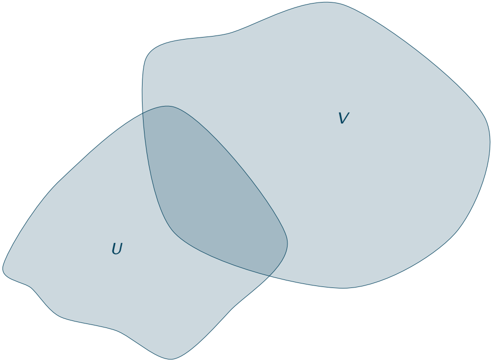

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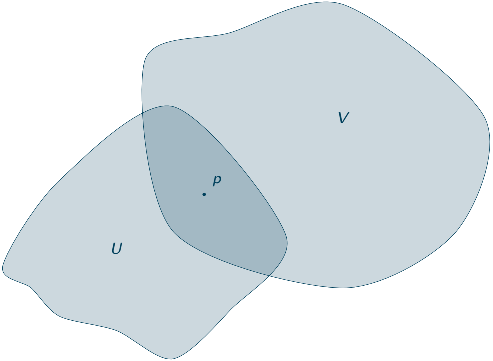

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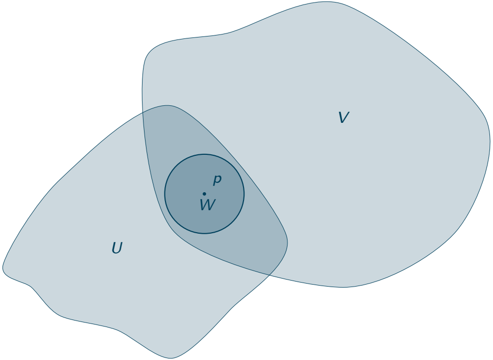

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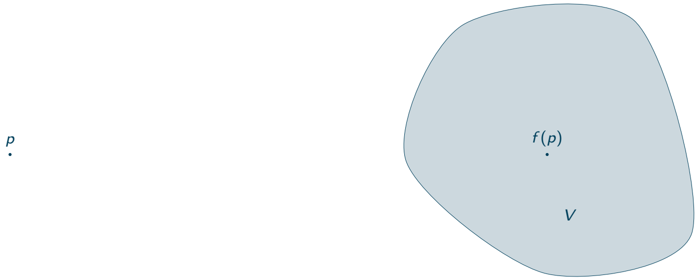

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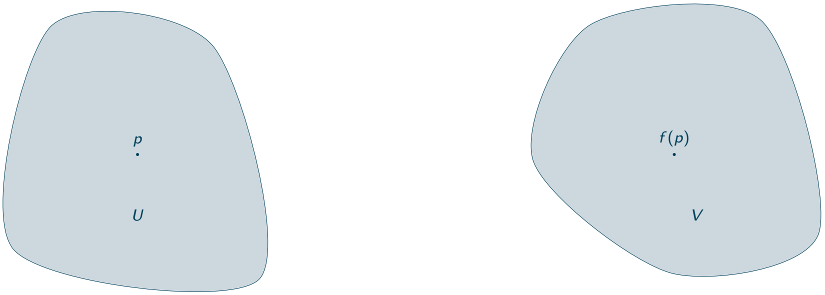

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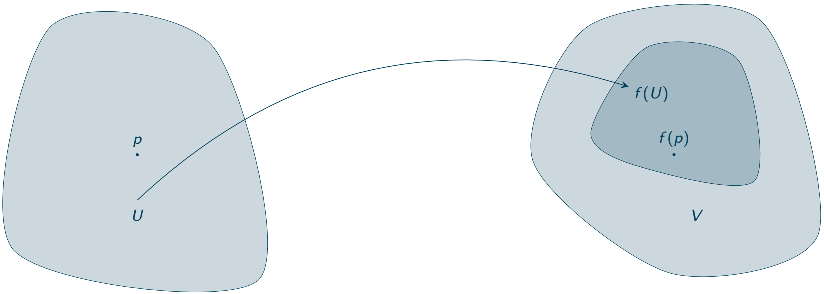

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The complement of an open subsets is said to be closed.

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with $\partial^{-\prime}(\mathsf{Mo}(f)(\alpha)) = \mathsf{Ob}(f)(\partial^- \alpha)$ and $\partial^{+\prime}(\mathsf{Mo}(f)(\alpha)) = \mathsf{Ob}(f)(\partial^+ \alpha)$

Hence it is in particular a morphism of graphs.

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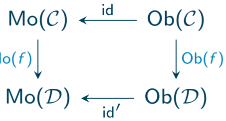
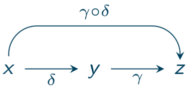
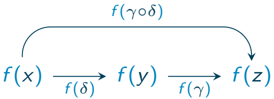


and satisfies $\mathrm{Mo}(f)(\gamma \circ \delta) = \mathrm{Mo}(f)(\gamma) \circ \mathrm{Mo}(f)(\delta)$





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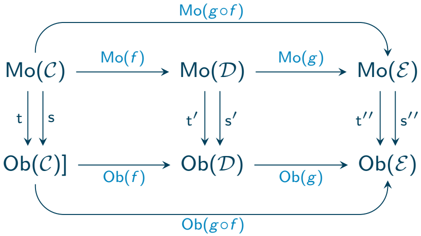

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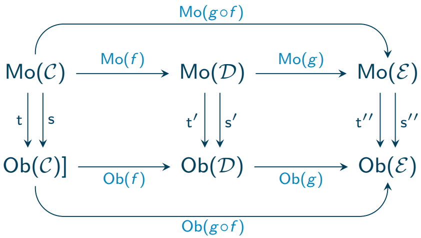

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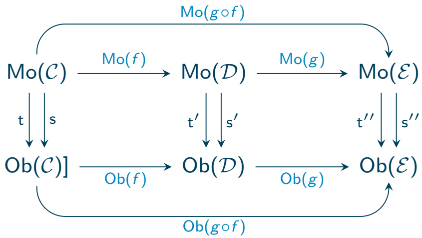

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The small categories and their funtors form a (large) category denoted by $\mathcal{C}at$

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- f is fully faithful when it is full and faithful
- f is an embedding when it is faithful and $\mathrm{Ob}(f)$ is one-to-one
- f is an equivalence when it is fully faithful and every object of $\mathcal{D}$ is isomorphic to an object of the form $f(C)$ with $C \in \mathcal{C}$.

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Remark : The full embeddings compose

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The functors I and Sp are equivalences of categories.

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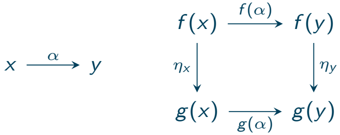

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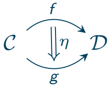

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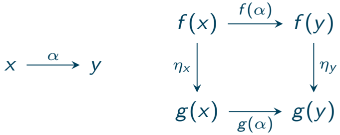

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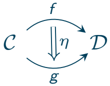

If every η_x is an isomorphism of $\mathcal{D}$, then η is said to be a natural isomorphism, its inverse η^{-1} is $(\eta_x^{-1})_{x \in \mathrm{Ob}(\mathcal{C})}$.

A functor $f : \mathcal{C} \to \mathcal{D}$ is an equivalence iff there exists a functor $g : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\mathrm{id}_{\mathcal{C}} \cong g \circ f$ and $\mathrm{id}_{\mathcal{D}} \cong f \circ g$.

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E.g.: we have $\mathrm{id}_{\mathcal{T}op} = I \circ Sp$ and the collection $B \Rrightarrow Sp(B)$ for $B \in \mathcal{B}as$ is a natural isomorphism from $\mathrm{id}_{\mathcal{B}as}$ to $Sp \circ I$.

AN ALGEBRAIC TOPOLOGY TEASER

The overall idea of algebraic topology

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Problem: prove the topological spaces X and Y are *not* the same

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Strategy: find a functor F defined over $\mathcal{Top}$ such that $F(X) \ncong F(Y)$

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Every subset of a Hausdorff space is saturated.

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A Hausdorff space is locally compact iff each of its points admits a compact neighbourhood.

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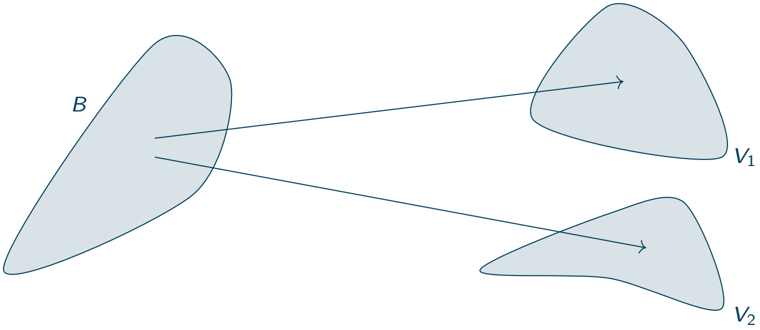
$$\mathcal{T}op \xrightarrow{\pi_0} \mathcal{S}et$$

$$\begin{array}{ccc} X & & \pi_0(X) \\ \downarrow f & \longmapsto & \pi_0(f) \downarrow \\ Y & & \pi_0(Y) \end{array}$$

An application

The continuous image of a connected space is connected

The image of the space B is entirely contained in a connected component of the space V.



This situation is abstracted by classifying continuous maps from B to V according to which connected component (V_1 or V_2) the single connected components of B (namely B itself) is sent to. There are exactly two set theoretic maps from the singleton $\{B\}$ to the pair $\{V_1, V_2\}$ hence there is at most (in fact exactly) two kinds of continuous maps from B to V.

$$\{B\} \rightrightarrows \{V_1, V_2\}$$

In particular B and V are not homeomorphic.

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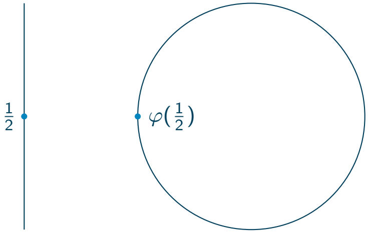
Then φ induces a homeomorphism

$$[0, \tfrac{1}{2}[\cup]\tfrac{1}{2}, 1] \;\to\; \mathbb{S}^1 \setminus \{\varphi(\tfrac{1}{2})\}$$

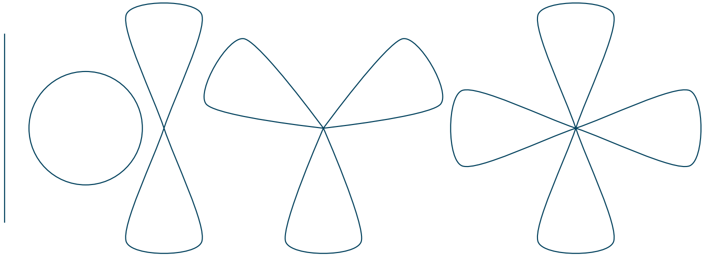
which does not exist!



Generalization

Bouquets of circles

These topological spaces are pairwise not homeomorphic. Why ?



Distributed Computing *through* Combinatorial Topology

MK
MORGAN KAUFMANN

Maurice Herlihy
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Sergio Rajsbaum

METRIC SPACES

Categories of Metric Spaces

Metric spaces

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Goal: turn any graph into metric space in a functorial way.

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A path γ from x to x' such that $\ell(\gamma) = d(x, x')$ is said to be geodesic.

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The metric space (X, d) is a length space when the distance between two points $x, x' \in X$ is the following greatest lower bound

$$\inf\left\{\ell(\gamma) \mid \gamma \text{ is a path from } x \text{ to } x'\right\}$$

A path γ from x to x' such that $\ell(\gamma) = d(x, x')$ is said to be geodesic.
The space is said to be geodesic when any two points are related by a geodesic path.

The Hopf-Rinow theorem

Metric Spaces of Non-Positive Curvature, *M. R. Bridson, and A. Haefliger*, 1999

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Metric Graphs

Neighbours

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Distance between two neighbours

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- If $\partial^- a \neq \partial^+ a$ there is a canonical bijection

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Itinerary

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An itinerary on $A\times]0,1[\sqcup V$ is a (finite) sequence $p_0,\ldots,p_q$ of points such that p_k and p_{k+1} are neighbours for $k \in \{0,\ldots,q-1\}$.

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The metric graph associated with G is the metric space

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Open balls

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The open ball of radius $r < 1$ centered at the vertex v is the set

$$\{v\} \quad \cup \quad \{a \mid \partial^{-} a = v\} \times]0, r[\quad \cup \quad \{a \mid \partial^{+} a = v\} \times]1 - r, 1[$$

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If $r \leqslant \frac{1}{4}$ then $B(c, r)$ is geodesically stable, i.e. for all p, $q \in B(c, r)$

$$\{p, q\} \quad \subseteq \quad \bigcup \{\mathrm{im}(\gamma) \mid \gamma \text{ geodesic from } p \text{ to } q\} \quad \subseteq \quad B(c, r)\,.$$

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Every finite graph with weighted arrows (in $\mathbb{R}_+ \setminus \{0\}$) with can be embedded in $\mathbb{R}^3$.

LOCALLY ORDERED METRIC GRAPHS

Ordered Bases

The category of ordered bases ($\mathcal{OB}$)

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We write that $(X, \leqslant_X)$ is a subposet of $(Y, \leqslant_Y)$, or $(X, \leqslant_X) \hookrightarrow (Y, \leqslant_Y)$, when $X \subseteq Y$ and $a \leqslant_X b \Leftrightarrow a \leqslant_Y b$ for all $a, b \in X$.

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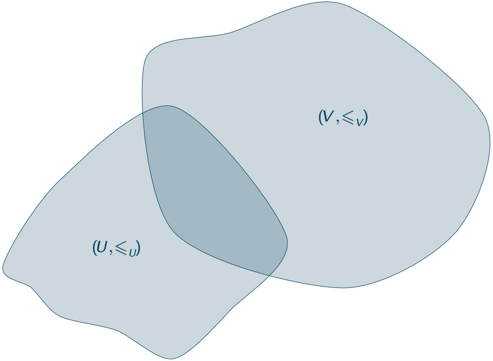

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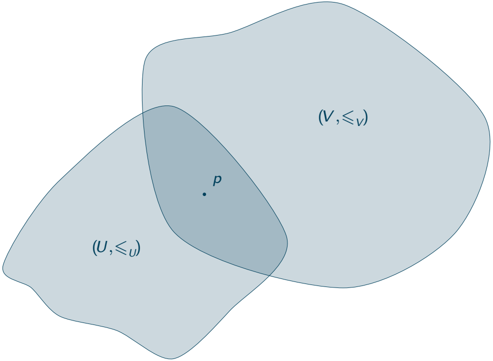

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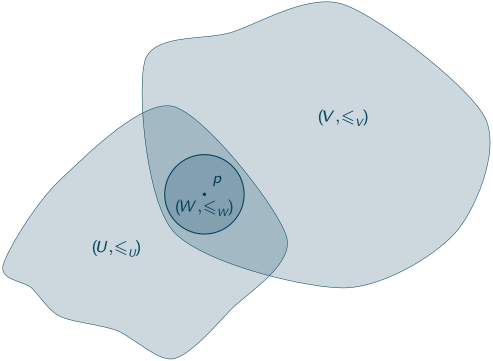

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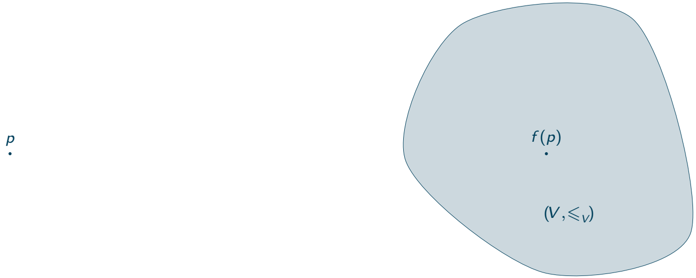

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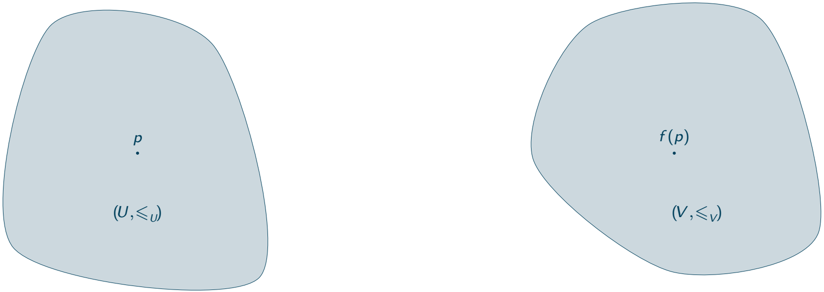

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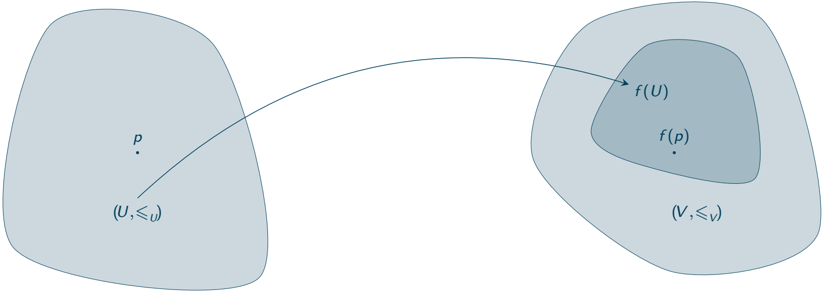

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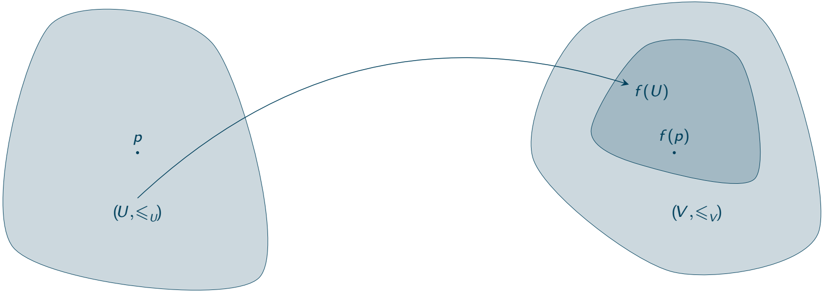

Ordered bases and locally order-preserving maps form the category $\mathcal{OB}$.

The underling topology of an ordered base

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If $f : \mathcal{B} \to \mathcal{B}'$ is locally order-preserving, then $Uf : U\mathcal{B} \to U\mathcal{B}'$ is continuous; we have a forgetful functor $\mathcal{OB} \to \mathcal{B}as$.

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We have a functor $U : \mathcal{OB} \to \mathcal{Set}$ obtained as the composite $\mathcal{OB} \to \mathcal{Bas} \to \mathcal{Set}$.

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We have a functor $U : \mathcal{OB} \to \mathcal{S}et$ obtained as the composite $\mathcal{OB} \to \mathcal{B}as \to \mathcal{S}et$.

The underlying space functor $Sp : \mathcal{OB} \to \mathcal{T}op$ is the composite $\mathcal{OB} \to \mathcal{B}as \to \mathcal{T}op$.

We write $\mathcal{B} \sim \mathcal{B}'$ when $Sp(\mathcal{B}) = Sp(\mathcal{B}')$ and $\mathcal{B} \cup \mathcal{B}'$ is still an ordered base; and we say that $\mathcal{B}$ and $\mathcal{B}'$ are equivalent.

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The relation $\sim$ is an equivalence relation on the collection of ordered bases over a given set.

If $\mathcal{A} \sim \mathcal{A}'$ and $\mathcal{B} \sim \mathcal{B}'$, then any map $f : U\mathcal{A} \to U\mathcal{B}$ is locally order-preserving from $\mathcal{A}$ to $\mathcal{B}$ iff it is so from $\mathcal{A}'$ to $\mathcal{B}'$.

Locally ordered spaces

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Lemma: Every ordered base is contained in a unique maximal ordered base.

Proposition: the full embedding $\mathcal{LoSp} \to \mathcal{OB}$ is an equivalence of categories whose quasi-inverse is the functor that assigns its locally ordered space to every ordered base.

The locally ordered line

Examples of equivalent ordered bases on $\mathbb{R}$

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- $\{(I, \leqslant) \mid I$ open interval of $\mathbb{R}\}$,

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Examples of equivalent ordered bases on $\mathbb{R}$

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- $\{(U, \leqslant) \mid U$ open subset of $\mathbb{R}\}$,
- $\{(U, \sqsubseteq_U) \mid U$ open subset of $\mathbb{R}\}$ where $x \sqsubseteq_U y$ stands for $x \leqslant y$ and $[x, y] \subseteq U$,

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- $\{(U, \sqsubseteq_U) \mid U$ open subset of $\mathbb{R}\}$ where $x \sqsubseteq_U y$ stands for $x \leqslant y$ and $[x, y] \subseteq U$,
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The locally ordered line

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The locally ordered circle

Examples of equivalent ordered bases on $\mathbb{S}^1$

The locally ordered circle

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Ordered spaces

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Topology and Order, L. Nachbin, 1965

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- Problem: there is no pospace on the circle whose collection of directed paths is

$$\left\{e^{i\theta(t)} \mid \theta : [0, r] \to \mathbb{R} \text{ increasing}\right\}$$

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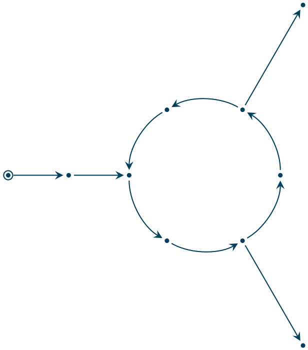
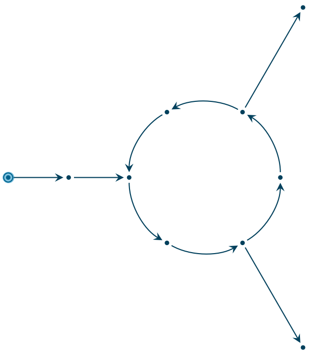
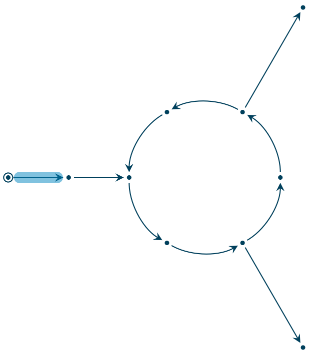
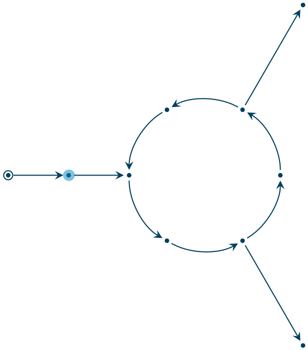
A local pospace has no vortex.

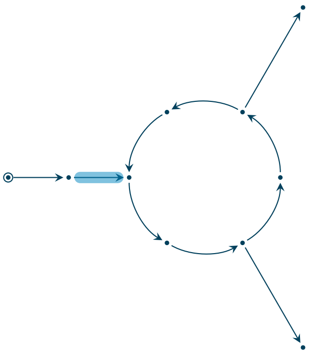
Ordered bases on metric graphs

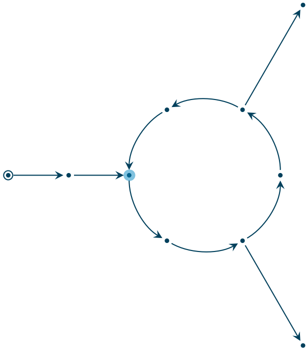


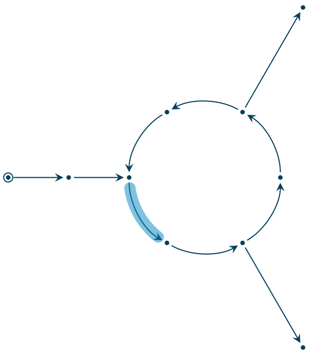


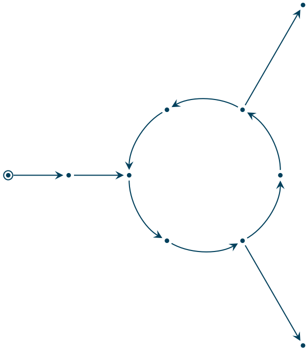


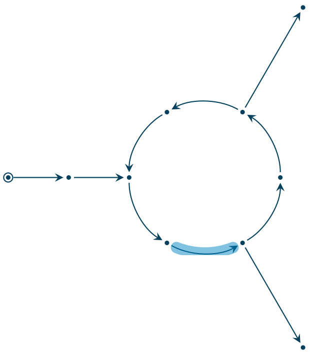


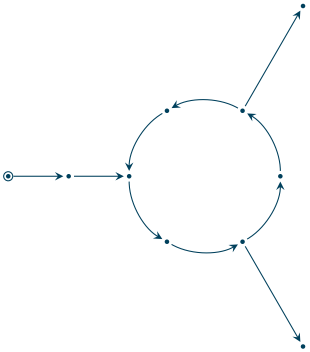


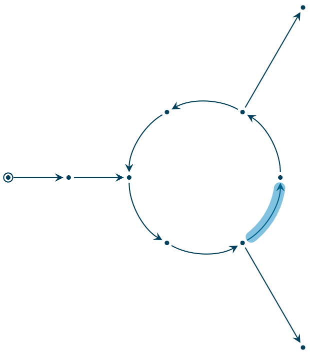


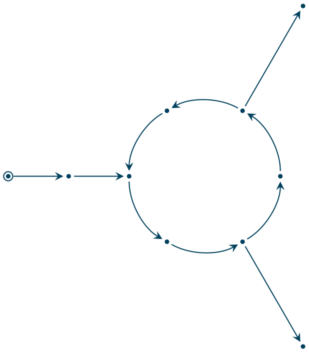


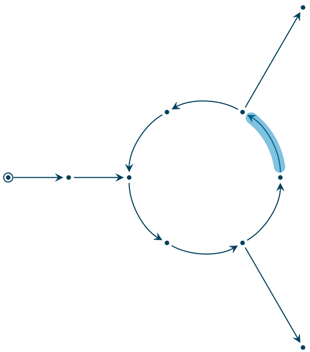


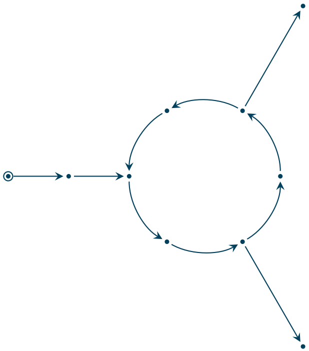


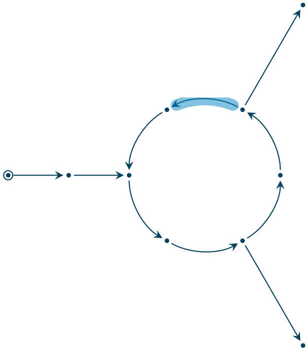


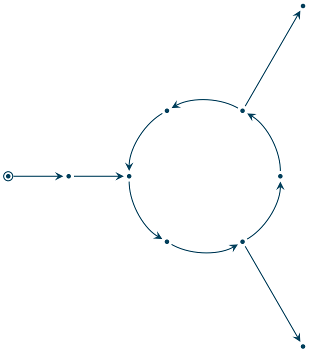


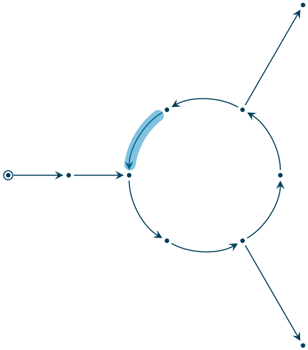


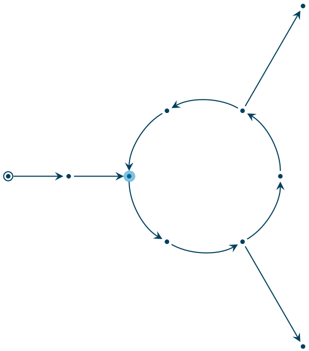


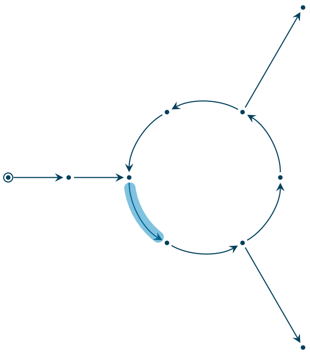


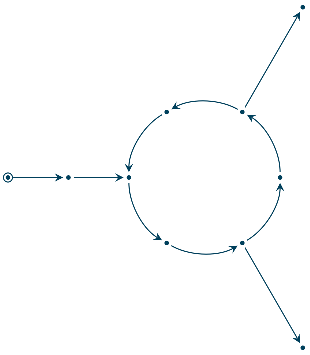


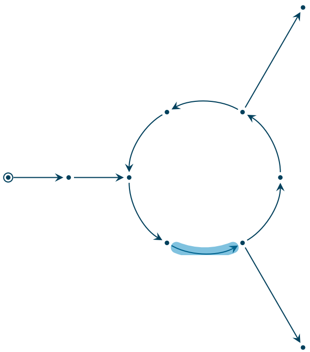


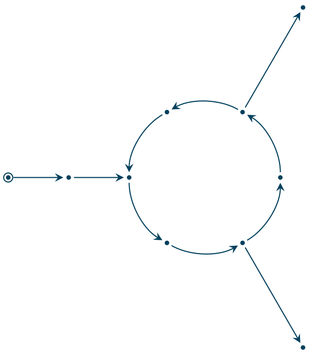


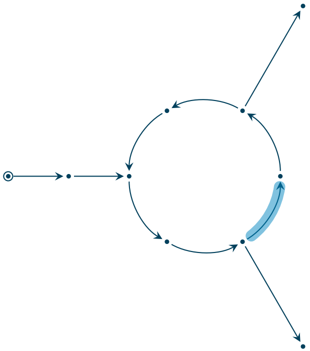


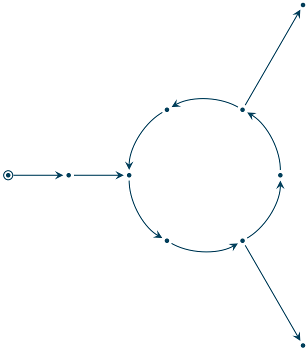


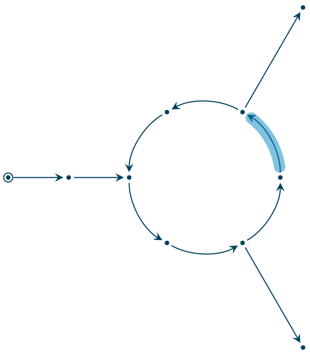


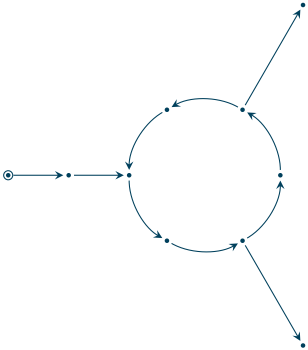


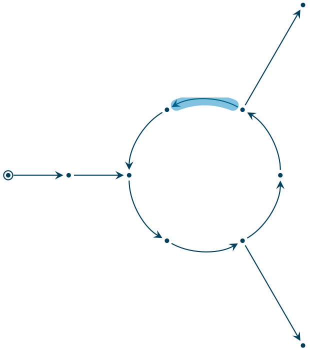


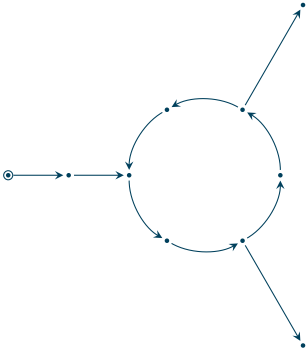


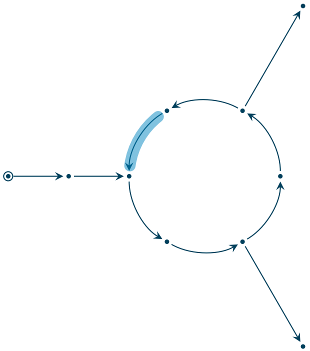


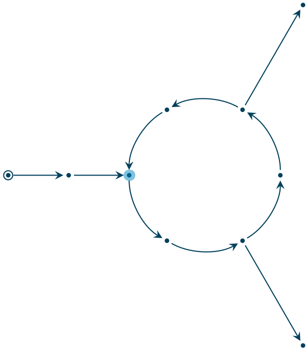


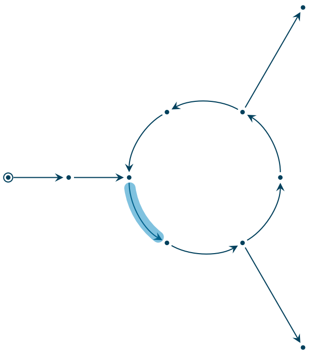


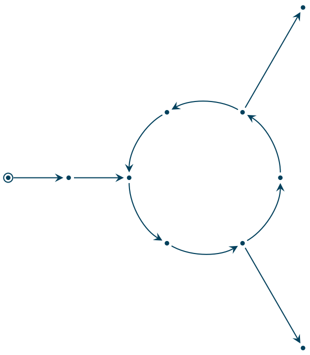


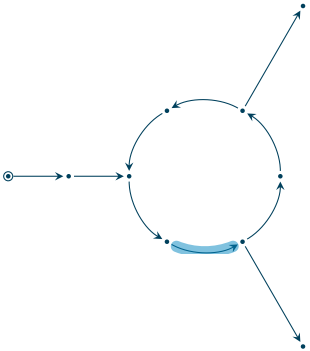


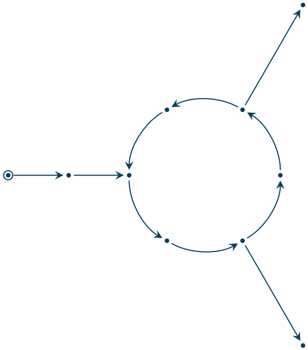


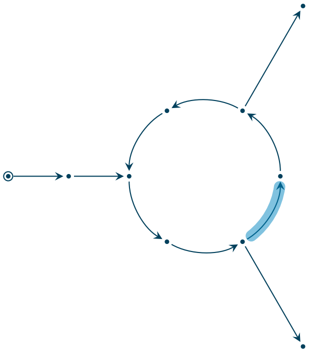


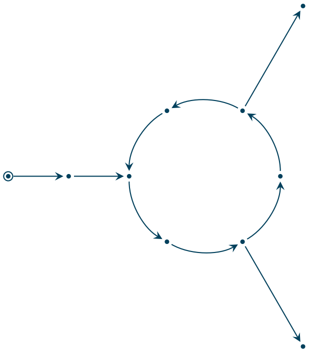


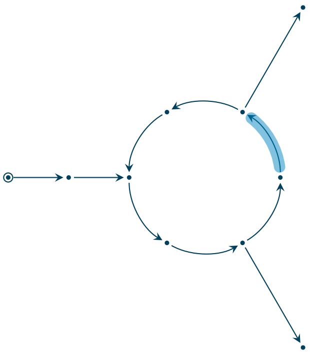


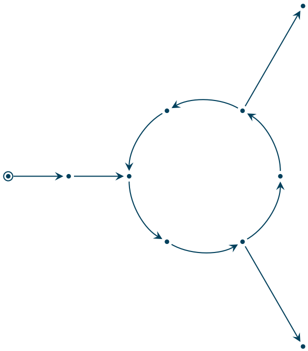


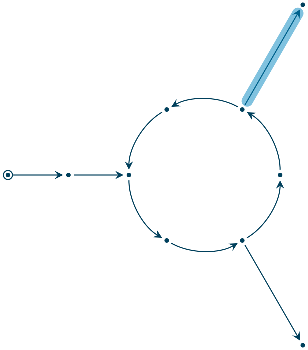


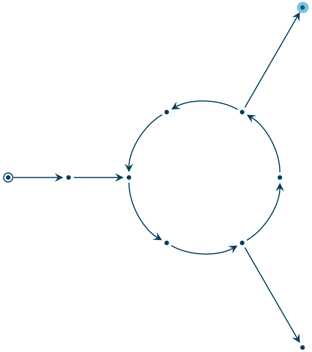


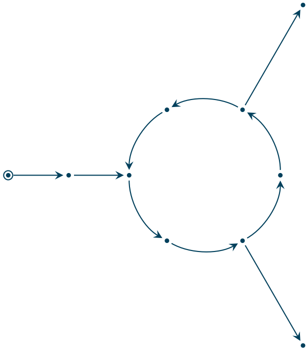


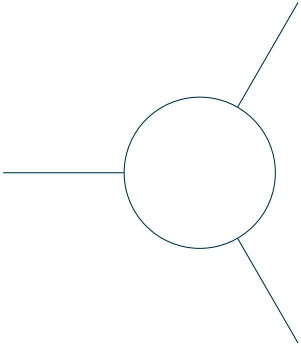


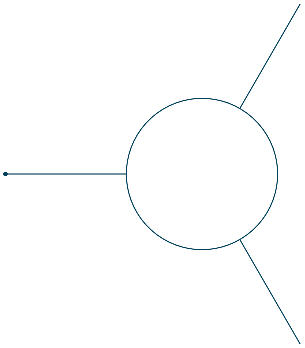


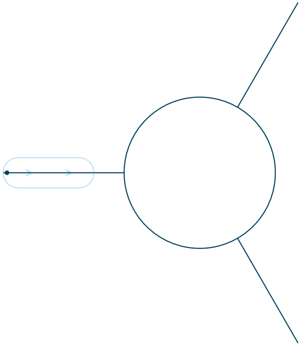


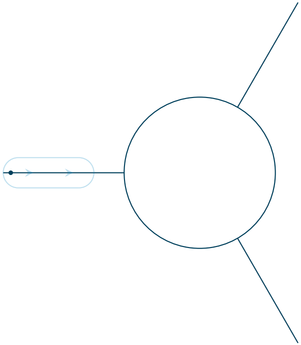


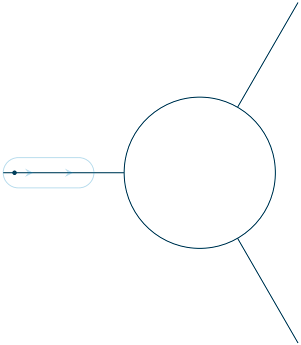


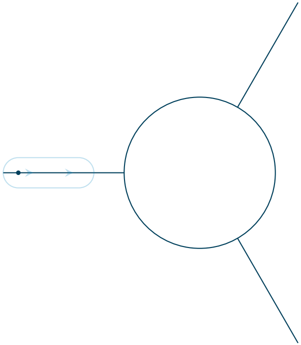


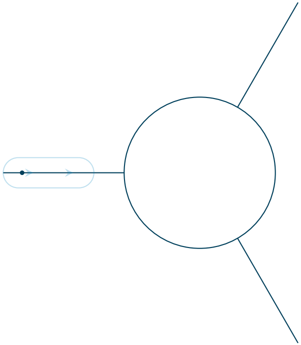


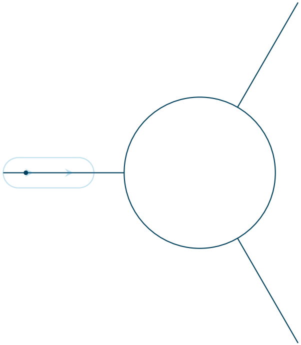


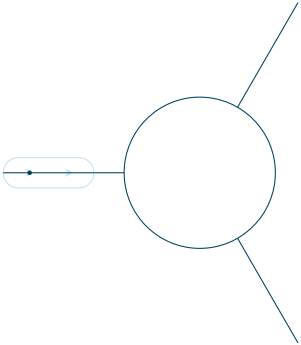


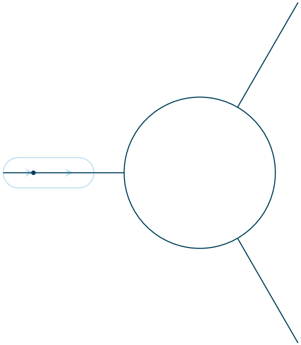


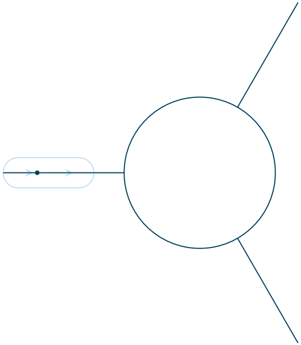


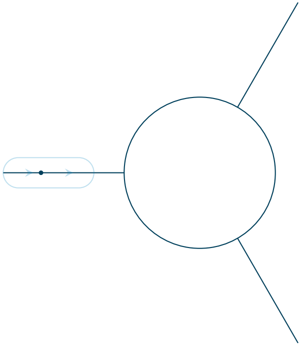


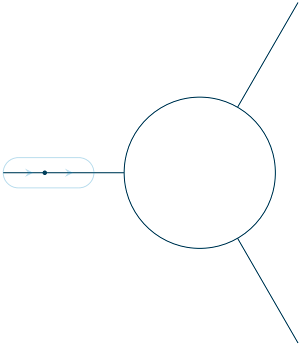


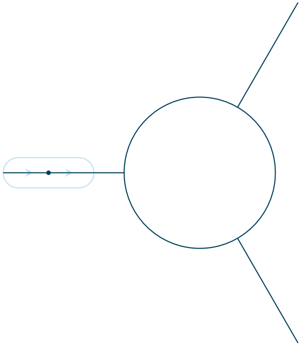


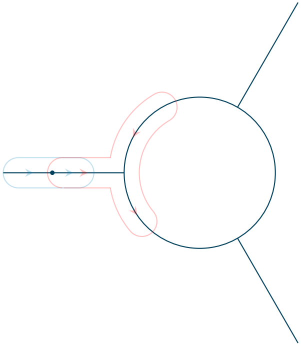


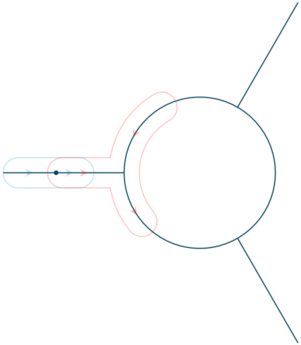


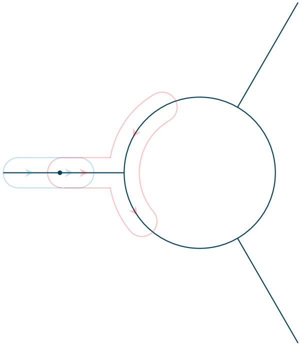


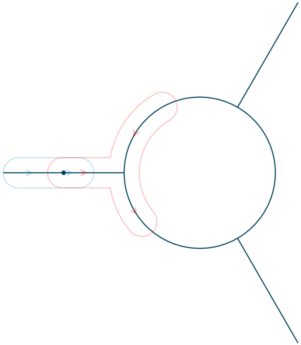


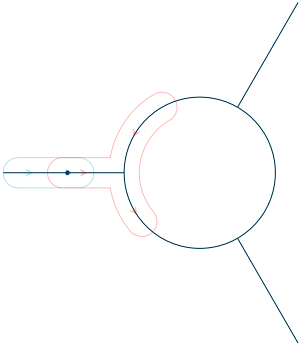


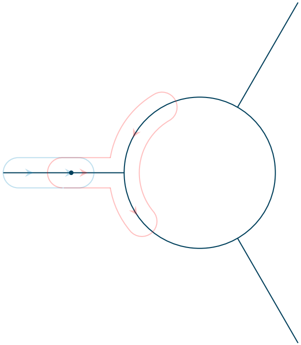


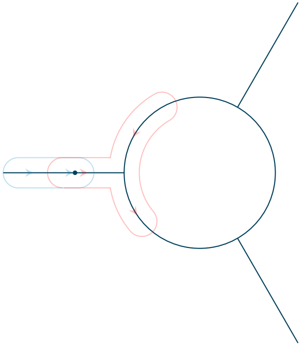


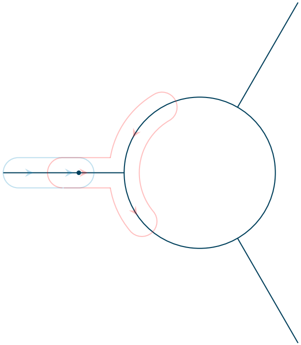


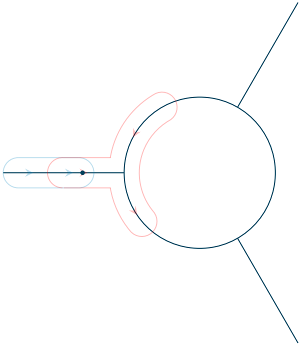


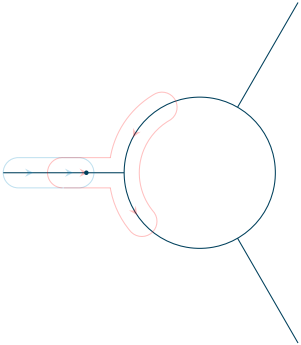


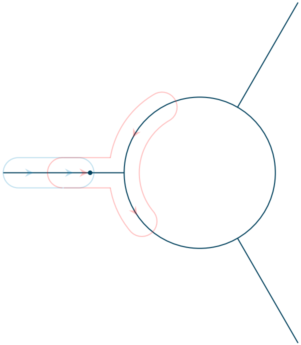


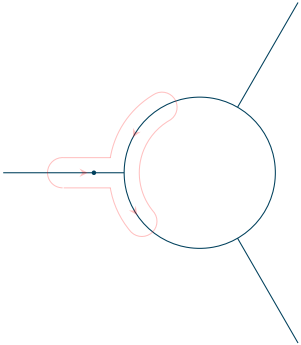


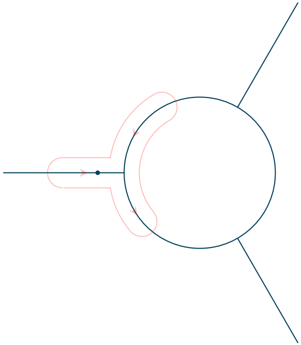


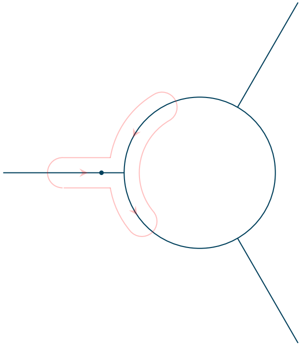


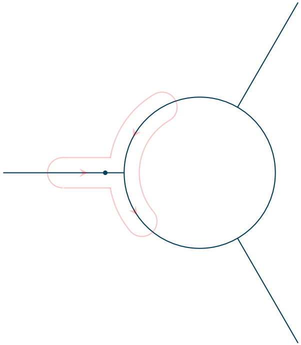


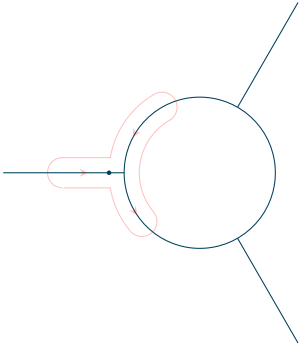


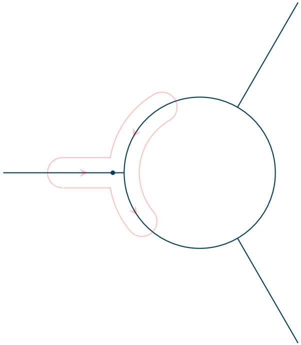


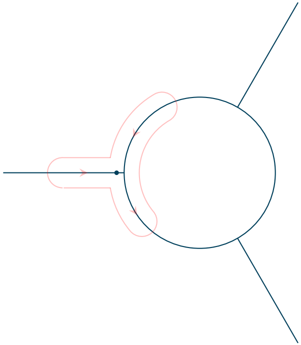


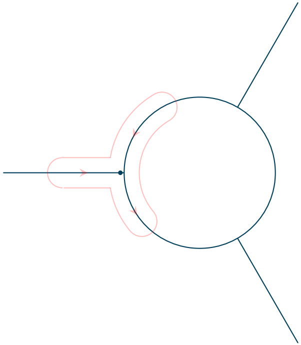


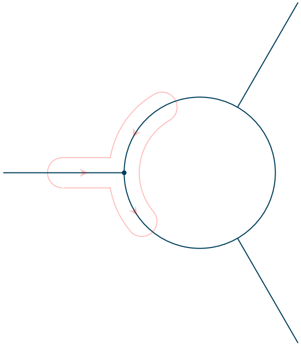


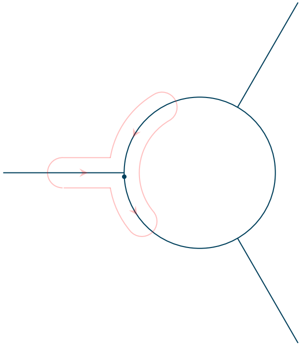


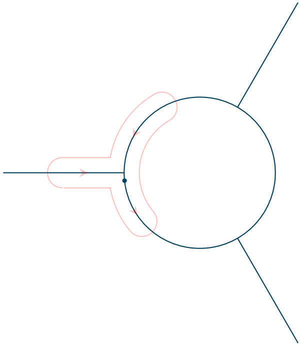


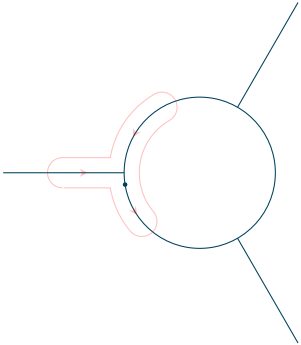


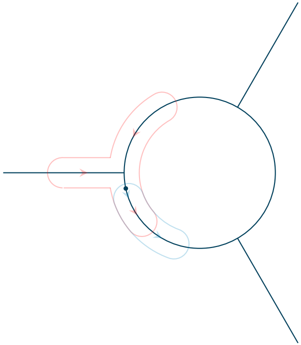


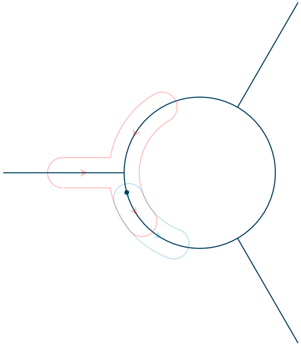


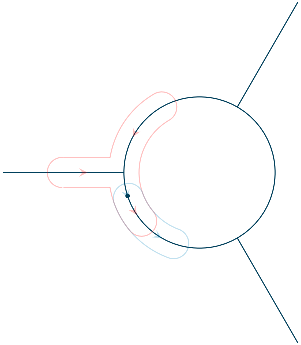


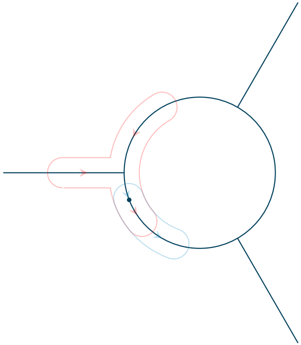


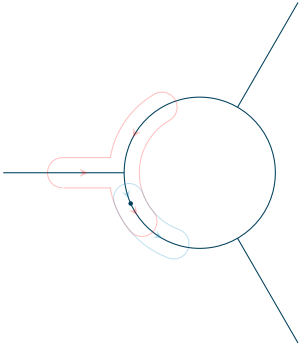


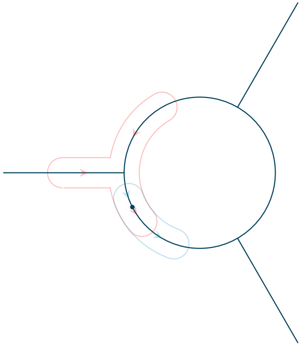


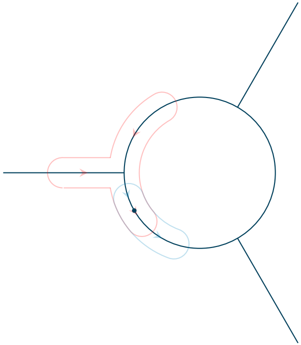


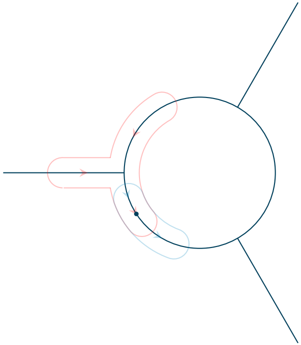


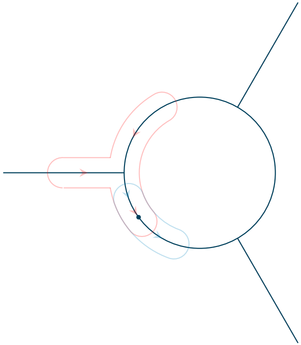


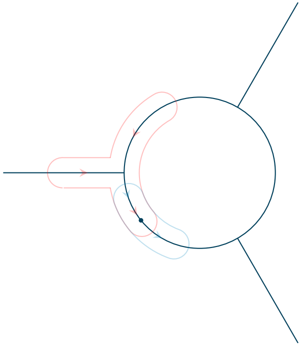


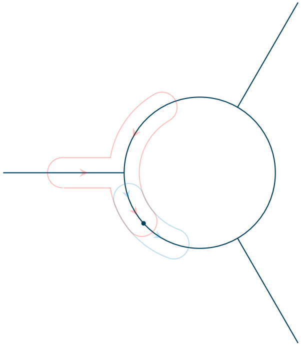


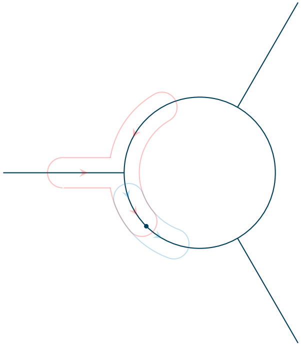


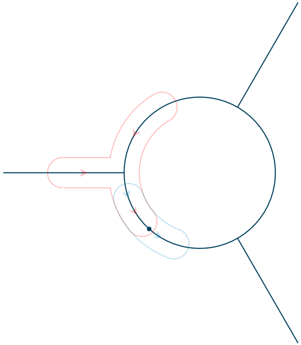


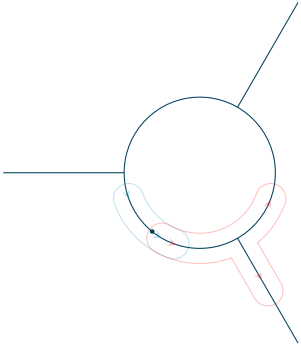


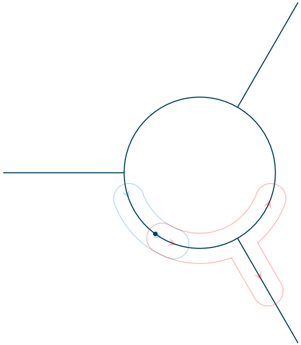


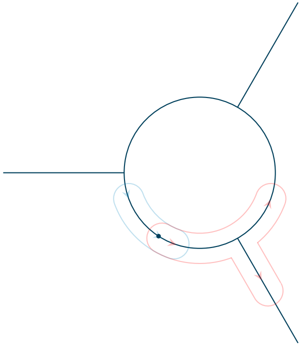


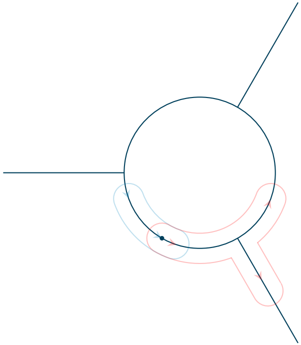


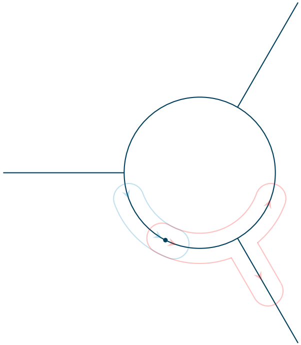


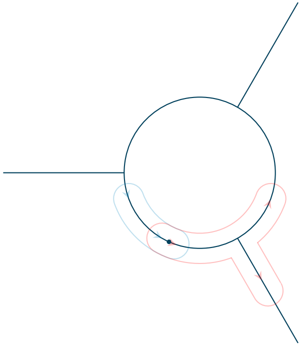


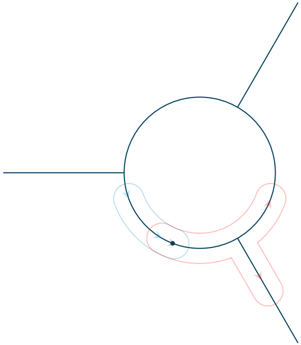


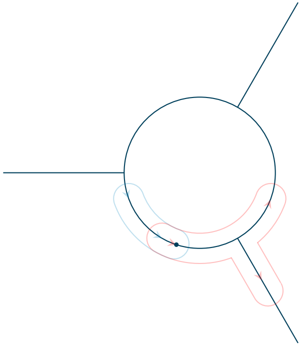


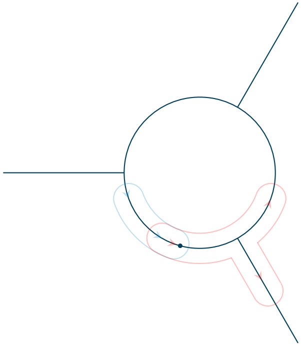


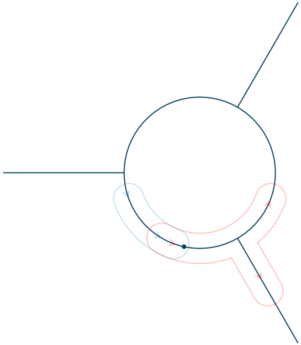


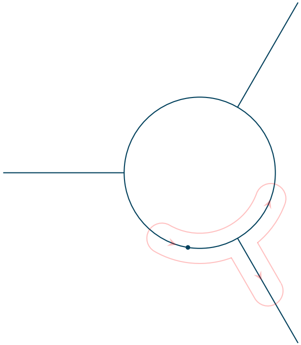


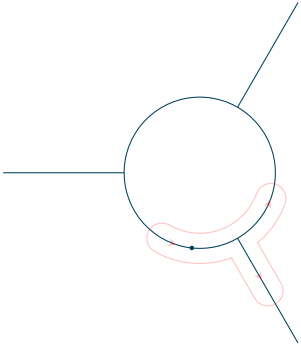


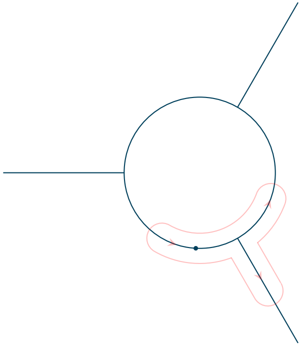


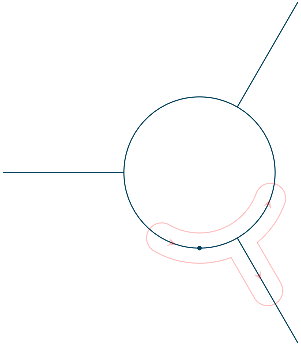


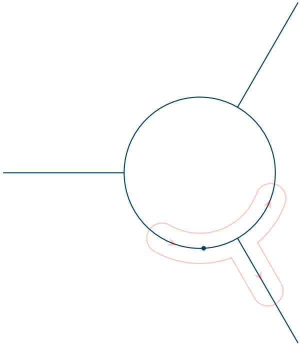


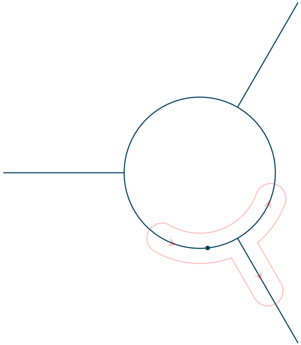


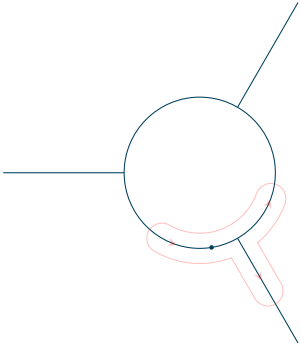


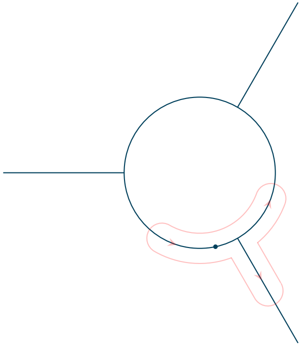


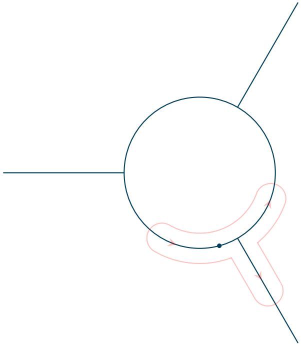


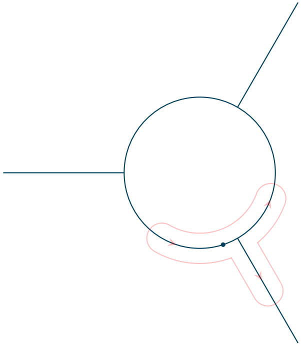


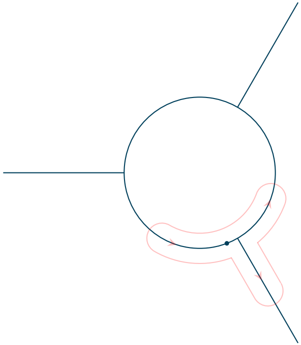


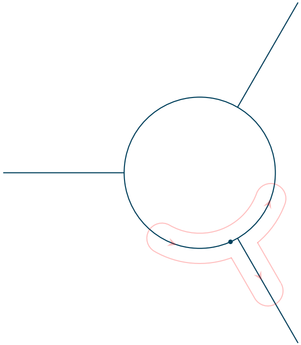


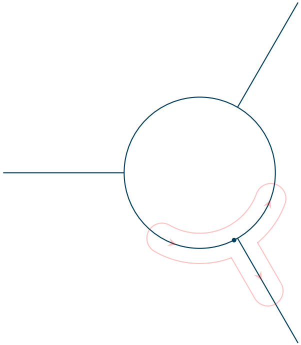


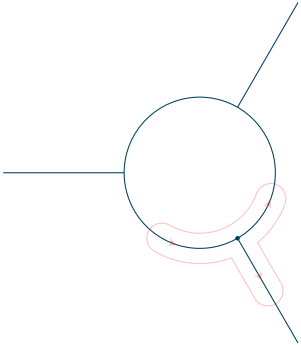


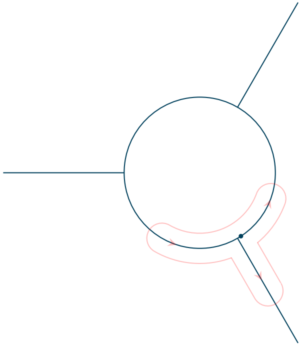


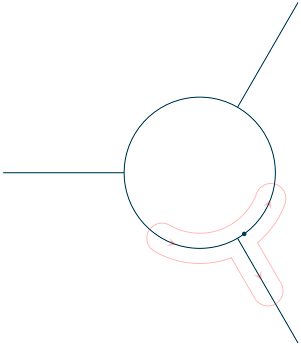


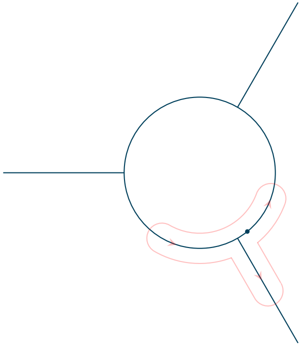


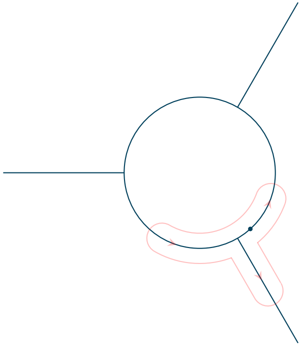


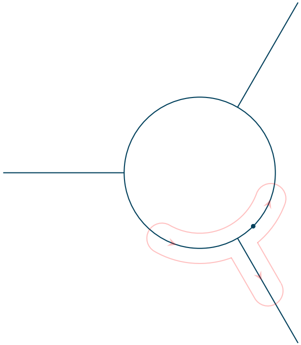


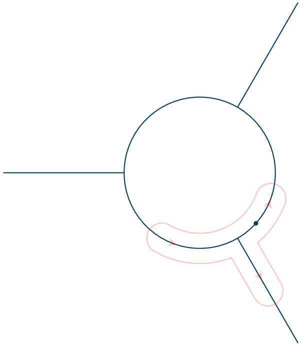


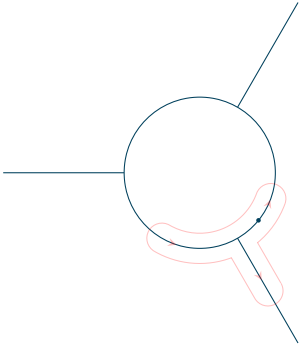


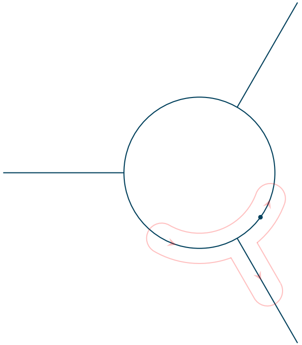


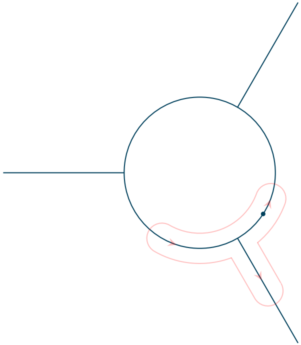


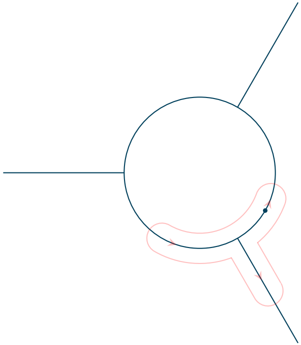


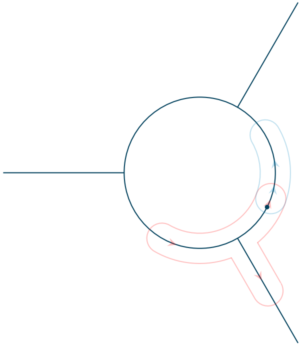


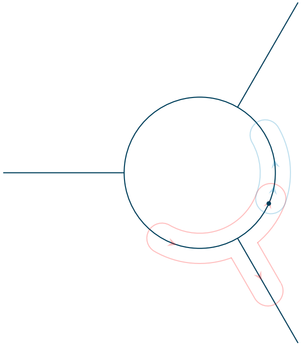


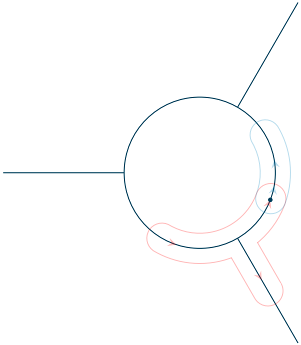


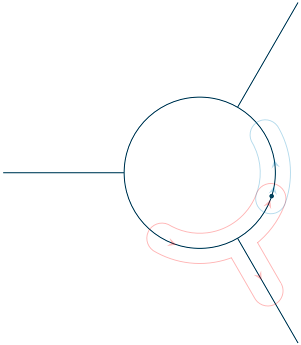


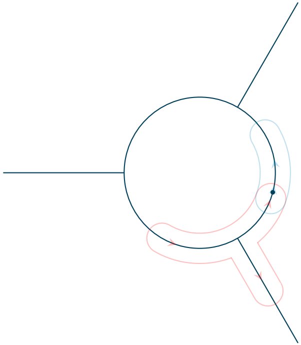


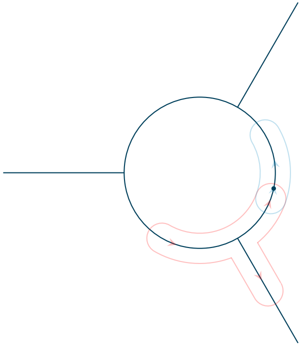


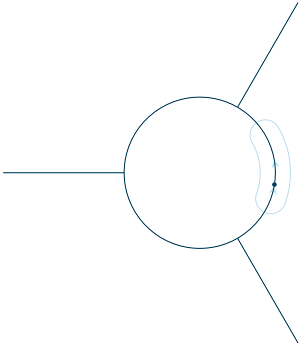


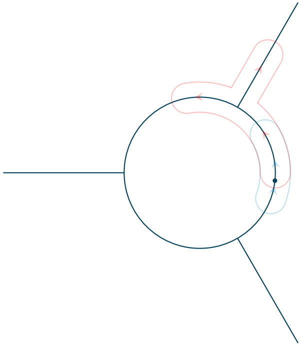


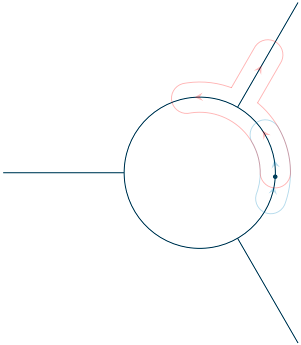


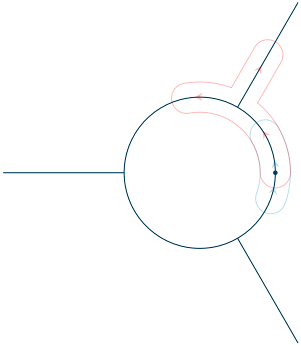


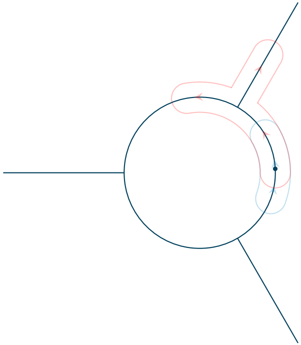


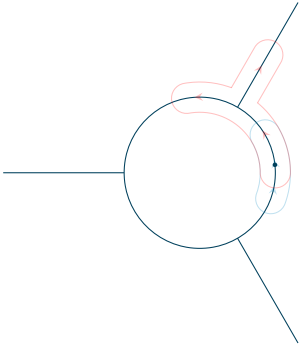


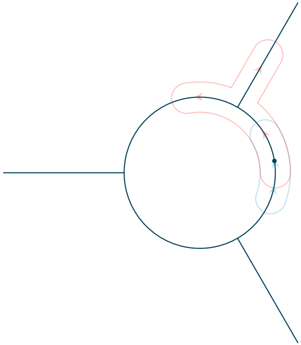


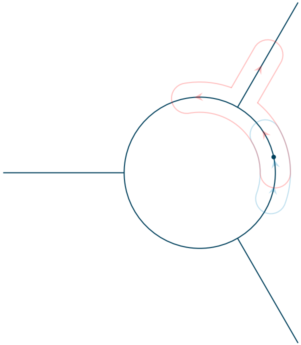


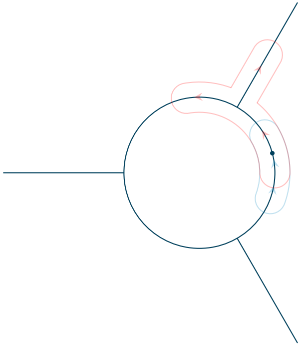


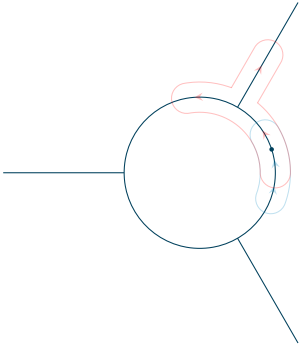


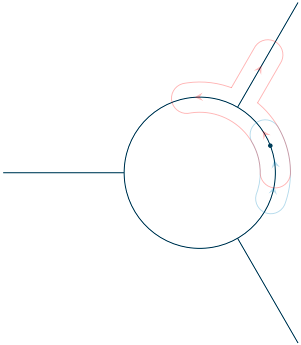


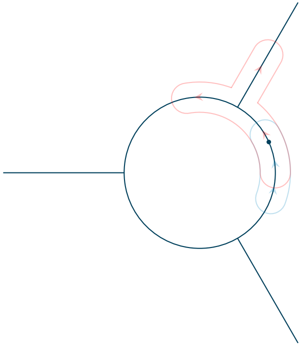


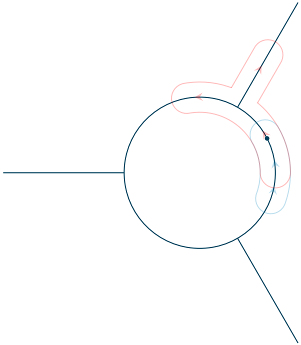


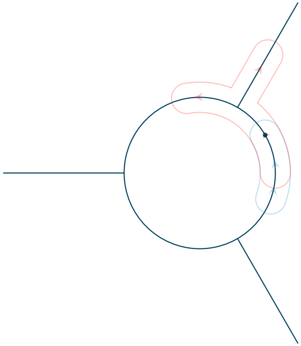


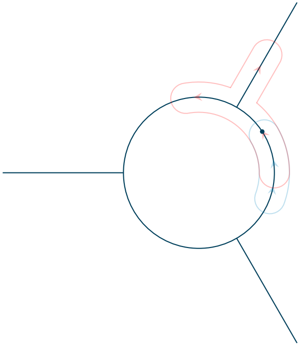


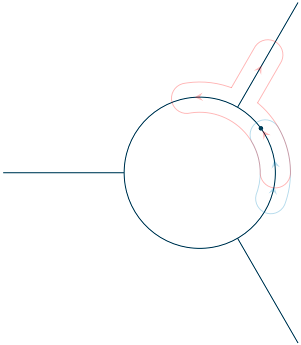


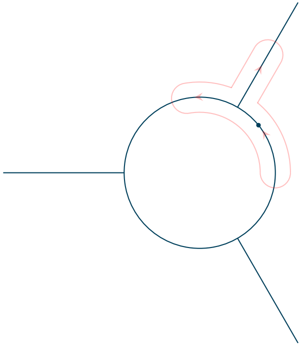


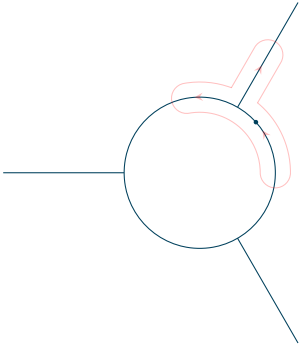


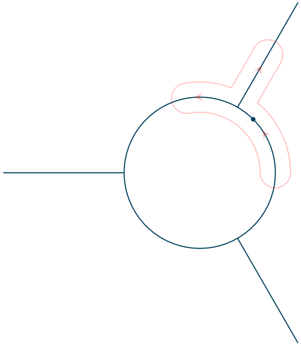


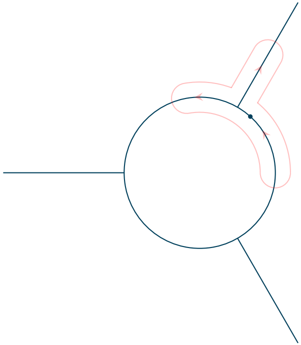


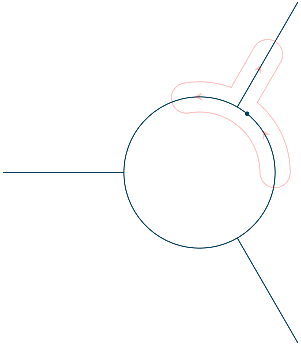


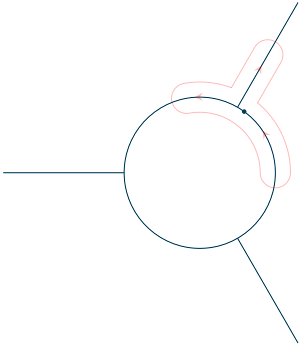


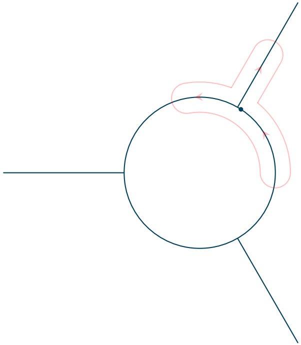


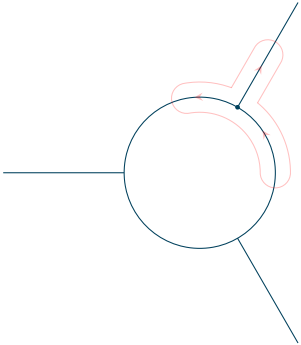


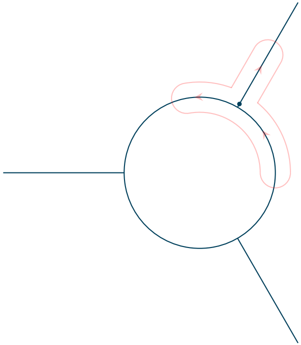


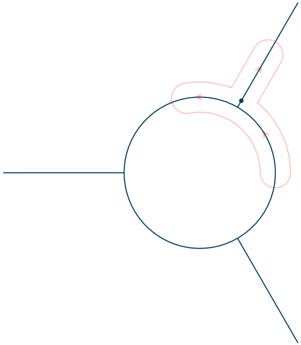


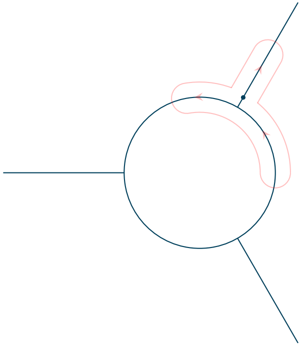


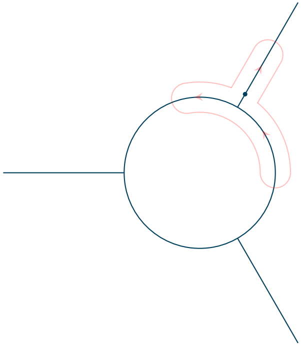


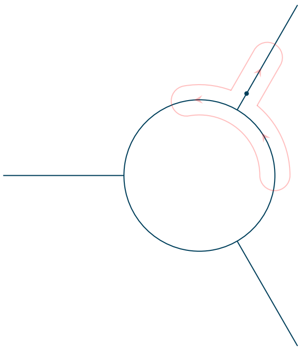


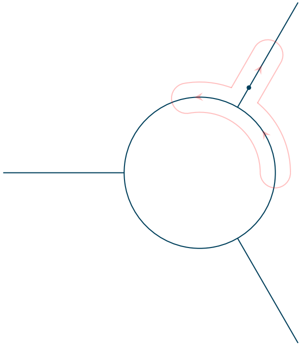


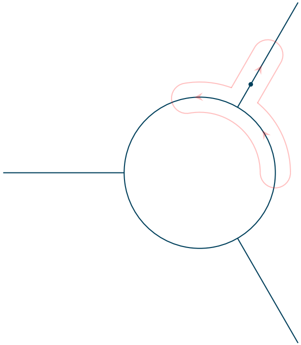


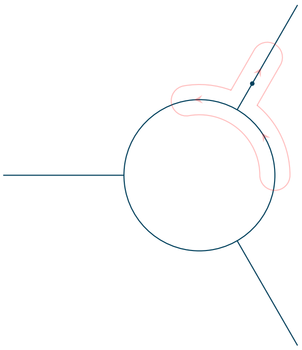


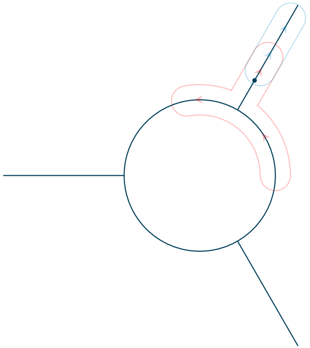


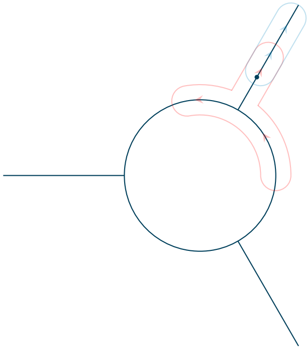


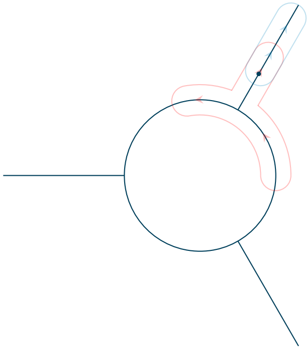


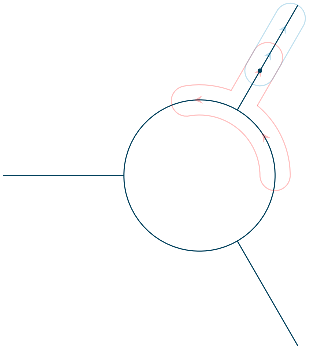


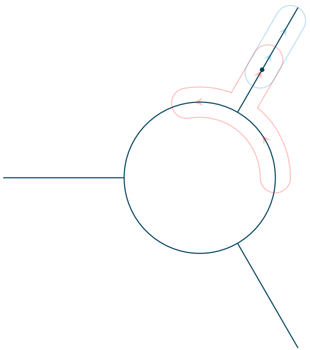


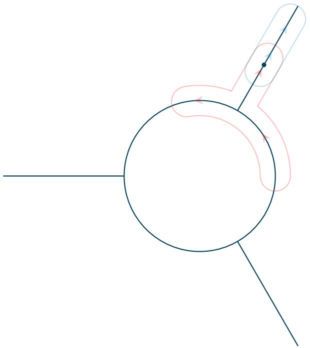


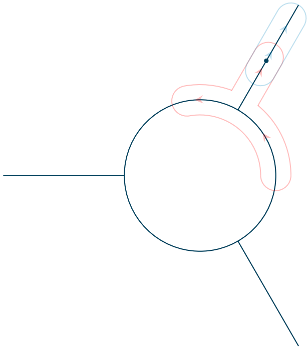


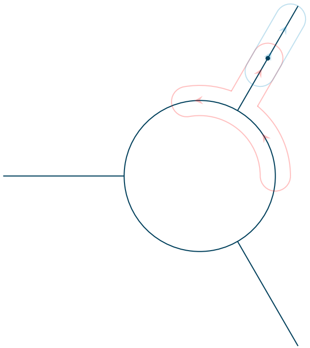


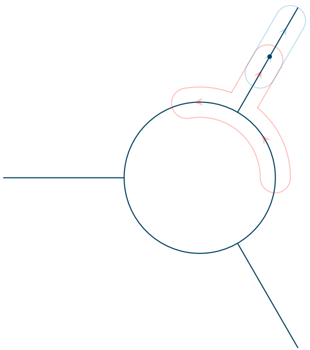


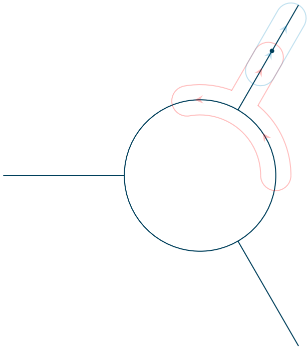


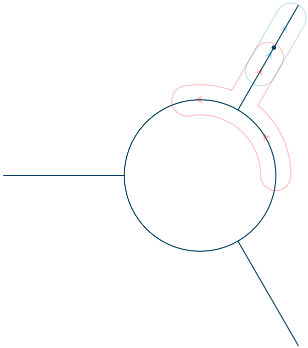


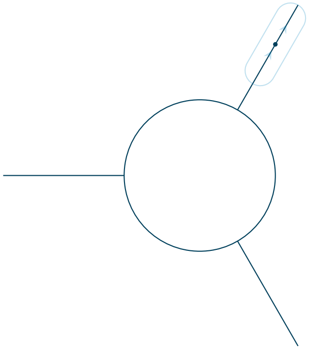


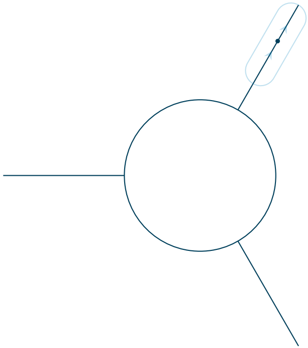


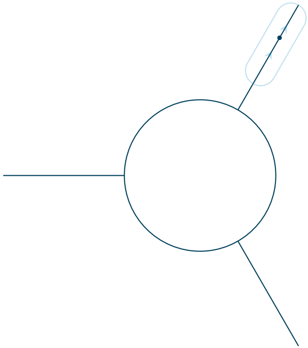


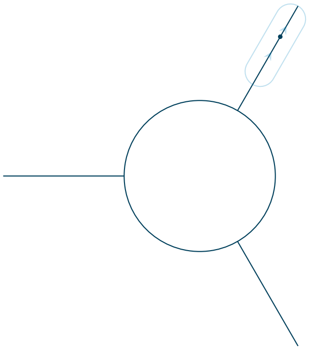


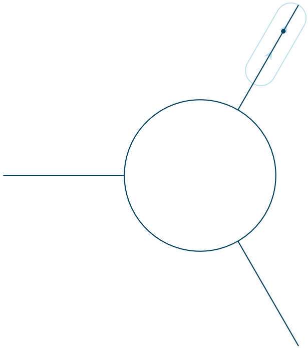


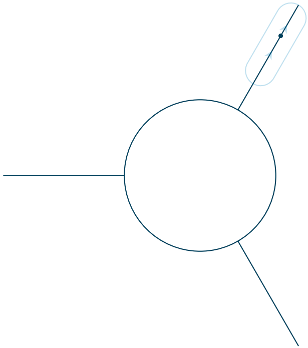


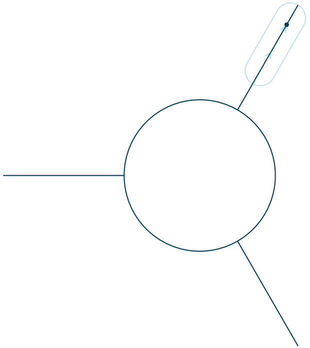


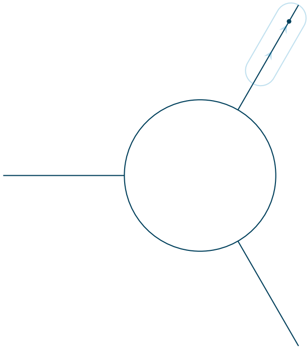


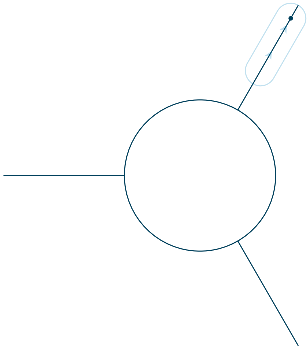


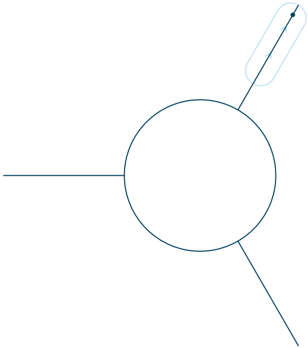


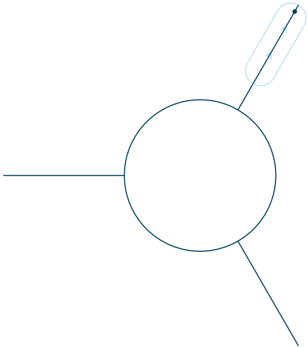


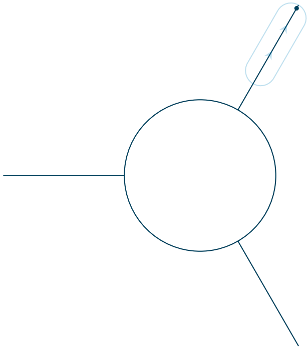


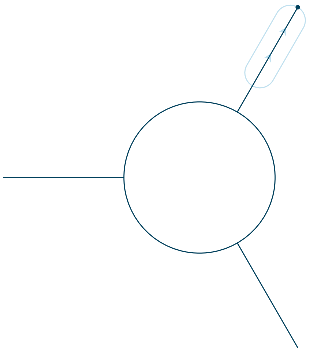












A convenient open covering

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If B, B' are centred at v and v' we have

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Ordered open stars

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The locally ordered metric graph construction is functorial.

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$\{a\} \times]t - \varepsilon, t + \varepsilon[$ if $\varepsilon \leqslant \min(t, 1 - t)$, and
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The partial order $\sqsubseteq$ and the metric d_G on the ball centered at v of radius ε are characterized by the following properties:

$d_G((a,t),v) = 1-t$	$(a,t) \sqsubseteq v$	if $t \in]1-\varepsilon, 1[$
$d_G(v,(a,t)) = t$	$v \sqsubseteq (a,t)$	if $t \in]0,\varepsilon[$
$d_G((a,t),(a,t')) = t'-t$	$(a,t) \sqsubseteq (a,t')$	if $t \leqslant t'$ and $(t,t' \in]0,\varepsilon[$ or $t,t' \in]1-\varepsilon,1[)$
$d_G((a,t),(a,t')) = \min\{t'-t, 1-(t'-t)\}$	$(a,t') \sqsubseteq (a,t)$	if $t \in]0,\varepsilon[$ and $t' \in]1-\varepsilon,1[$
$d_G((a,t),(b,t')) = d_G((a,t),v) + d_G(v,(b,t'))$		if $a \neq b$
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If $\varepsilon \leqslant \frac{1}{4}$ then the ball centered at v of radius ε, say B, is geodesically stable: for all p, $q \in B$, the union of the images of the geodesics from p to q is nonempty and contained in B.

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The *standard ordered base* of G is the collection of ordered open balls of radii $\varepsilon \leqslant \frac{1}{2}$ with their 'canonical' partial order.