Category $\mathcal{C}$

Definition (the "underlying graph" part)

- $\mathrm{Ob}(\mathcal{C})$: collection of objects
- $\mathrm{Mo}(\mathcal{C})$: collection of morphisms
- ∂^-, ∂^+ : mappings source, target as follows

$$\mathrm{Mo}(\mathcal{C}) \underset{\partial^+}{\overset{\partial^-}{\rightrightarrows}} \mathrm{Ob}(\mathcal{C})$$

- We define the homset $\mathcal{C}(x, y) := \left\{\gamma \in \mathrm{Mo}(\mathcal{C}) \;\middle|\; \partial^- \gamma = x \text{ and } \partial^+ \gamma = y\right\}$

Category $\mathcal{C}$

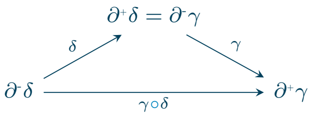
Definition (the "underlying local monoid" part)

- id : provides each object with an identity

$$\mathrm{Mo}(\mathcal{C}) \underset{\partial^{+}}{\overset{\partial^{-}}{\rightrightarrows}} \mathrm{Ob}(\mathcal{C}), \qquad \mathrm{Ob}(\mathcal{C}) \xrightarrow{\ \mathrm{id}\ } \mathrm{Mo}(\mathcal{C})$$

- The binary composition is a partially defined and often denoted by $\circ$

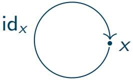
$$\left\{(\gamma,\delta) \;\middle|\; \gamma,\delta \text{ morphisms of } \mathcal{C} \text{ s.t. } \partial^{-}\gamma = \partial^{+}\delta\right\} \xrightarrow{\text{composition}} \mathrm{Mo}(\mathcal{C})$$



Category $\mathcal{C}$

Definition (the axioms)

- The composition law is associative
- For all objects x one has $\partial^- \mathrm{id}_x = x = \partial^+ \mathrm{id}_x$



- For all morphisms γ one has $\mathrm{id}_{\partial^+\gamma} \circ \gamma = \gamma = \gamma \circ \mathrm{id}_{\partial^-\gamma}$

Standard examples

- $\mathcal{Set}$: the category of sets.
- $\mathcal{Mon}$: the category of monoids
- $\mathcal{Cmon}$: the category of commutative monoids
- $\mathcal{Gr}$: the category of groups
- $\mathcal{Pre}$: the category of preordered sets.
- $\mathcal{Pos}$: the category of posets.
- Any preordered set can be seen as a category in which any homset has at most one element.
- Any monoid can be seen as a category with a single object.
- The opposite of a category is obtained by reversing all its arrows (i.e. by swapping the roles of the source and the target)

Some special kinds of morphisms

- f is an isomorphism when there exists g such that both $f \circ g$ and $g \circ f$ are identities.
- Two objects related by an isomorphism are said to be isomorphic.
- A groupoid is a category that only has isomorphisms.
- f is a monomorphism when it is left-cancellative i.e. for all g_1, g_2, $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$.
- f is a epimorphism when it is right-cancellative i.e. for all g_1, g_2, $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$.
- any isomorphism is both monomorphism and an epimorphism, the converse is false in general (e.g. $\mathcal{Pos}$).
- if $r \circ s = \mathrm{id}$ then r is called a retract/split epimorphism and s is called a section/split monomorphism.

The category of graphs ($\mathcal{Grph}$)

The elements of V are the vertices and those of A are the arrows
In particular A and V are sets

Objects

$$\begin{array}{c} A \\ s\downarrow\,\downarrow t \\ V \end{array}$$

Morphisms

$$\begin{array}{ccc} A & \xrightarrow{\phi_1} & A' \\ s\downarrow\,\downarrow t & & s'\downarrow\,\downarrow t' \\ V & \xrightarrow[\phi_0]{} & V' \end{array}$$

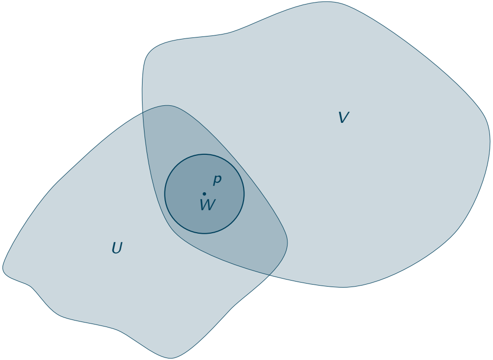
Composition

$$\begin{array}{ccccc} A & \xrightarrow{\phi_1} & A' & \xrightarrow{\psi_1} & A'' \\ s\downarrow\,\downarrow t & & s'\downarrow\,\downarrow t' & & s''\downarrow\,\downarrow t'' \\ V & \xrightarrow[\phi_0]{} & V' & \xrightarrow[\psi_0]{} & V'' \end{array}$$

with $\mathrm{s}'(\phi_1(\alpha)) = \phi_0(\partial^- \alpha)$ and $\mathrm{t}'(\phi_1(\alpha)) = \phi_0(\partial^+ \alpha)$

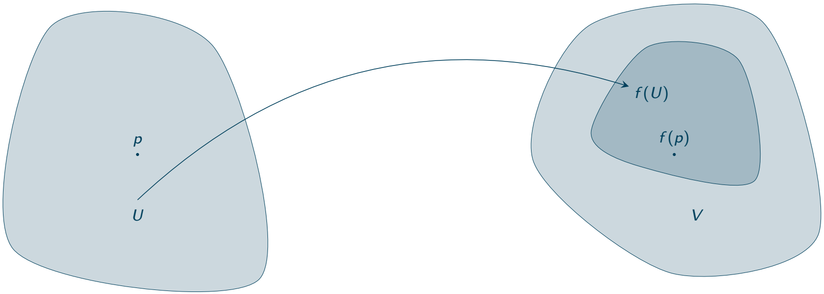
The category of bases of topologies ($\mathscr{B}as$)

A *base of a topology* is a collection of sets $\mathcal{B}$ such that for all $U, V \in \mathcal{B}$, all $p \in U \cap V$, there exists $W \in \mathcal{B}$ such that $p \in W \subseteq U \cap V$.



The category of bases of topologies ($\mathcal{B}as$)

A map $f : \mathcal{B} \to \mathcal{B}'$ is *continuous* when for every point p of $\mathcal{B}$, every $V \in \mathcal{B}'$ with $f(p) \in V$, there exists $U \in \mathcal{B}$ with $p \in U$ such that $f(U) \subseteq V$.



The category of topological spaces ($\mathcal{T}op$)

A topological space is a set X and a collection $\Omega_X \subseteq \mathcal{P}(X)$ s.t.

1) $\emptyset \in \Omega_X$ and $X \in \Omega_X$
2) Ω_X is stable under union
3) Ω_X is stable under finite intersection

Equivalently, a topological space is a base of a topology stable under union.

A continuous map $f : (X, \Omega_X) \to (Y, \Omega_Y)$ is a map $f : X \to Y$ s.t.

$$\forall x \in X \ \forall V \in \Omega_Y \text{ s.t. } f(x) \in V, \ \exists U \in \Omega_X \text{ s.t. } x \in U \text{ and } f(U) \subseteq V$$

or equivalently

$$\forall V \in \Omega_Y \ f^{-1}(V) \in \Omega_X$$

The elements of Ω_X are called the open subsets of X.
The complement of an open subsets is said to be closed.

Functors f from $\mathcal{C}$ to $\mathcal{D}$

Definition (preserving the "underlying graph")

A functor $f : \mathcal{C} \to \mathcal{D}$ is defined by two "mappings" $\mathsf{Ob}(f)$ and $\mathsf{Mo}(f)$ such that

$$
\begin{array}{ccc}
\mathsf{Mo}(\mathcal{C}) & \underset{\partial^{+}}{\overset{\partial^{-}}{\rightrightarrows}} & \mathsf{Ob}(\mathcal{C}) \\
{\scriptstyle \mathsf{Mo}(f)}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{Ob}(f)} \\
\mathsf{Mo}(\mathcal{D}) & \underset{\partial^{+\prime}}{\overset{\partial^{-\prime}}{\rightrightarrows}} & \mathsf{Ob}(\mathcal{D})
\end{array}
$$

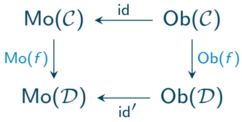
with $\partial^{-\prime}(\mathsf{Mo}(f)(\alpha)) = \mathsf{Ob}(f)(\partial^{-}\alpha)$ and $\partial^{+\prime}(\mathsf{Mo}(f)(\alpha)) = \mathsf{Ob}(f)(\partial^{+}\alpha)$

Hence it is in particular a morphism of graphs.

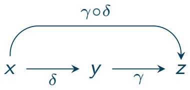
Functors f from $\mathcal{C}$ to $\mathcal{D}$

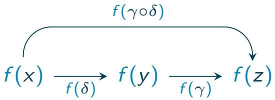
Definition (preserving the "underlying local monoid")

The "mappings" $\mathrm{Ob}(f)$ and $\mathrm{Mo}(f)$ also make the following diagram commute

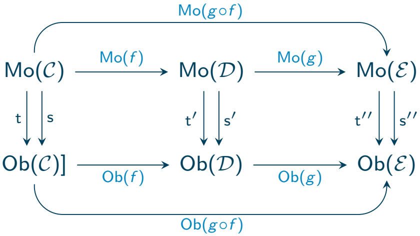


and satisfies $\mathrm{Mo}(f)(\gamma \circ \delta) = \mathrm{Mo}(f)(\gamma) \circ \mathrm{Mo}(f)(\delta)$





Functors compose as morphisms of graphs do



Hence functors should be thought of as morphisms of categories

The small categories and their funtors form a (large) category denoted by $\mathcal{C}at$

Some forgetful functors

$$(M,*,e) \in \mathcal{Mon} \mapsto M \in \mathcal{Set}$$
$$(X,\Omega) \in \mathcal{Top} \mapsto X \in \mathcal{Set}$$
$$(X,\sqsubseteq) \in \mathcal{Pos} \mapsto X \in \mathcal{Set}$$

$$\mathcal{C} \in \mathcal{Cat} \mapsto \mathrm{Ob}(\mathcal{C}) \in \mathcal{Set}$$
$$\mathcal{C} \in \mathcal{Cat} \mapsto \mathrm{Mo}(\mathcal{C}) \in \mathcal{Set}$$

$$\mathcal{C} \in \mathcal{Cat} \mapsto \left(\mathrm{Mo}(\mathcal{C}) \underset{\partial^-}{\overset{\partial^+}{\rightrightarrows}} \mathrm{Ob}(\mathcal{C}) \right) \in \mathcal{Grph}$$

Some small functors

(functor between small categories)

The morphisms of monoids are the functors between small categories with a single object

The morphisms of preordered sets are the functors between small categories whose homsets contain at most one element

The actions of a monoid M over a set X are the functors from M to $\mathcal{Set}$ which sends the only element of M to X

Given a functor $f : \mathcal{C} \to \mathcal{D}$ and two objects x and y we have the mapping

$$
\begin{aligned}
f_{x,y} \quad : \quad \mathcal{C}[x, y] \quad &\to \quad \mathcal{D}[\mathrm{Ob}(f)(x), \mathrm{Ob}(f)(y)] \\
\alpha \quad &\mapsto \quad \mathrm{Mo}(f)(\alpha)
\end{aligned}
$$

- f is faithful when for all objects x and y the mapping $f_{x,y}$ is one-to-one (injective)
- f is full when for all objects x and y the mapping $f_{x,y}$ is onto (surjective)
- f is fully faithful when it is full and faithful
- f is an embedding when it is faithful and $\mathrm{Ob}(f)$ is one-to-one
- f is an equivalence when it is fully faithful and every object of $\mathcal{D}$ is isomorphic to an object of the form $f(C)$ with $C \in \mathcal{C}$.

Some full embeddings in $\mathcal{Cat}$

Remark : The full embeddings compose

$$\mathcal{Pre} \hookrightarrow \mathcal{Cat}$$
$$\mathcal{Mon} \hookrightarrow \mathcal{Cat}$$
$$\mathcal{Pos} \hookrightarrow \mathcal{Pre}$$
$$\mathcal{Gr} \hookrightarrow \mathcal{Mon}$$
$$\mathcal{Cmon} \hookrightarrow \mathcal{Mon}$$
$$\mathcal{Ab} \hookrightarrow \mathcal{Cmon}$$
$$\mathcal{Ab} \hookrightarrow \mathcal{Gr}$$
$$\mathcal{Set} \hookrightarrow \mathcal{Pos}$$

Topological spaces and their bases

Full embedding $I : \mathcal{Top} \to \mathcal{Bas}$.

Space functor $\mathit{Sp} : \mathcal{Bas} \to \mathcal{Top}$ sending $\mathcal{B}$ to $\{\bigcup \mathcal{C} \mid \mathcal{C} \subseteq \mathcal{B}\}$.

Given $\mathcal{B} \in \mathcal{Bas}$, we denote by $U\mathcal{B}$ the underlying set of $\mathcal{B}$, i.e. the union of all the elements of $\mathcal{B}$. E.g.: bases of $\mathbb{R}^2$.

Given $\mathcal{B} \in \mathcal{Bas}$, the identity map on $U\mathcal{B}$ induces an isomorphism from $\mathcal{B}$ to $\mathit{Sp}(\mathcal{B})$ which we denote by $\mathcal{B} \Rrightarrow \mathit{Sp}(\mathcal{B})$; and an isomorphism from $\mathit{Sp}(\mathcal{B})$ to $\mathcal{B}$ which we denote by $Sp(\mathcal{B}) \Rrightarrow \mathcal{B}$. We have $(\mathcal{B} \Rrightarrow \mathit{Sp}(\mathcal{B}))^{-1} = (Sp(\mathcal{B}) \Rrightarrow \mathcal{B})$

The functors I and Sp are equivalences of categories.

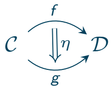
Natural Transformations

morphisms of functors from $f : \mathcal{C} \to \mathcal{D}$ to $g : \mathcal{C} \to \mathcal{D}$

A natural transformation $\eta : f \to g$ is a collection of morphisms $(\eta_x)_{x \in \mathrm{Ob}(\mathcal{C})}$ where $\eta_x \in \mathcal{D}[f(x), g(x)]$ and such that for all $\alpha \in \mathcal{C}[x, y]$ we have $\eta_y \circ f(\alpha) = g(\alpha) \circ \eta_x$ i.e. the following diagram commute

$$x \xrightarrow{\alpha} y \qquad \begin{array}{ccc} f(x) & \xrightarrow{f(\alpha)} & f(y) \\ {\scriptstyle \eta_x}\downarrow & & \downarrow{\scriptstyle \eta_y} \\ g(x) & \xrightarrow[g(\alpha)]{} & g(y) \end{array}$$

This description is summarized by the following diagram



If every η_x is an isomorphism of $\mathcal{D}$, then η is said to be a natural isomorphism, its inverse η^{-1} is $(\eta_x^{-1})_{x \in \mathrm{Ob}(\mathcal{C})}$.

A functor $f : \mathcal{C} \to \mathcal{D}$ is an equivalence iff there exists a functor $g : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\mathrm{id}_{\mathcal{C}} \cong g \circ f$ and $\mathrm{id}_{\mathcal{D}} \cong f \circ g$.

E.g.: we have $\mathrm{id}_{\mathcal{Top}} = I \circ Sp$ and the collection $B \Rrightarrow Sp(B)$ for $B \in \mathcal{Bas}$ is a natural isomorphism from $\mathrm{id}_{\mathcal{Bas}}$ to $Sp \circ I$.

The overall idea of algebraic topology

Every functor preserves the isomorphisms

Problem: prove the topological spaces X and Y are *not* the same
Strategy: find a functor F defined over $\mathcal{Top}$ such that $F(X) \ncong F(Y)$

More topological notions

The interior of a subset A of X is the greatest open subset of X contained in A.

Then closure of a subset A of X is the least closed subset of X containing A.

A neighbourhood of a subset A of X is a subset of X whose interior contains A.

A topological space X is said to be Hausdorff when for all $x, x' \in X$, if $x \neq x'$ then x and x' have disjoint neighbourhoods.

A subset Q of X is said to be saturated when

$$Q \quad = \quad \bigcap \{ U \mid U \text{ open and } Q \subseteq U \}$$

Every subset of a Hausdorff space is saturated.

Compactness and local compactness

Let X be a topological space.

- An open covering of X is a collection of open subsets of X whose union is X.
- X is said to be compact when every open covering of X admit a finite sub-covering.
- X is said to be locally compact when for every $x \in X$, every open neighbourhood U of x contains a saturated compact neighbourhood of x.

A Hausdorff space is locally compact iff each of its points admits a compact neighbourhood.

The connected component functor

1) A topological space X is said to be connected when its only closed-open subsets are $\emptyset$ and X
2) A union of connected subspaces sharing a point is connected
3) The connected components of a topological space induce a partition of its underlying set
4) Any connected subset of X is contained in a connected component of X
5) Any continuous direct image of a connected subset of X is connected

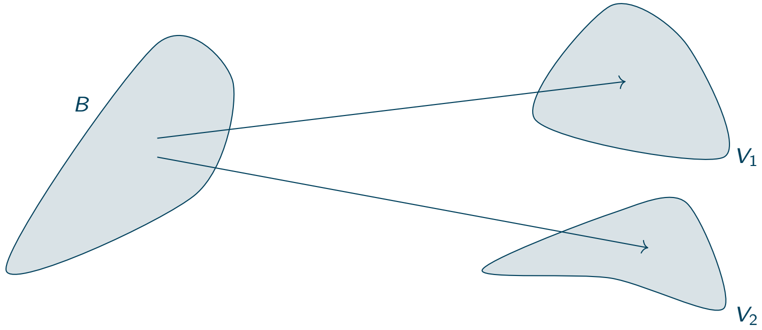
$$\mathcal{Top} \xrightarrow{\pi_0} \mathcal{Set}$$

$$\begin{array}{ccc} X & & \pi_0(X) \\ \downarrow f & \longmapsto & \pi_0(f) \downarrow \\ Y & & \pi_0(Y) \end{array}$$

An application

The continuous image of a connected space is connected

The image of the space B is entirely contained in a connected component of the space V.



This situation is abstracted by classifying continuous maps from B to V according to which connected component (V_1 or V_2) the single connected components of B (namely B itself) is sent to. There are exactly two set theoretic maps from the singleton $\{B\}$ to the pair $\{V_1, V_2\}$ hence there is at most (in fact exactly) two kinds of continuous maps from B to V.

$$\{B\} \rightrightarrows \{V_1, V_2\}$$

In particular B and V are not homeomorphic.

Application

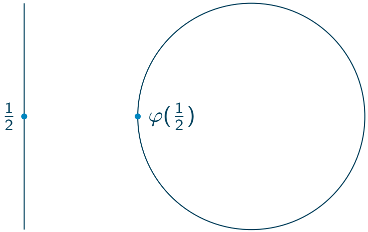
The compact interval and the circle are not homeomorphic

Let $\mathbb{S}^1 := \{z \in \mathbb{C} \mid |z| = 1\}$ be the Euclidean circle and suppose $\varphi : [0,1] \to \mathbb{S}^1$ is a homeomorphism.

Then φ induces a homeomorphism

$$[0, \tfrac{1}{2}[\,\cup\,]\tfrac{1}{2}, 1] \;\to\; \mathbb{S}^1 \setminus \{\varphi(\tfrac{1}{2})\}$$

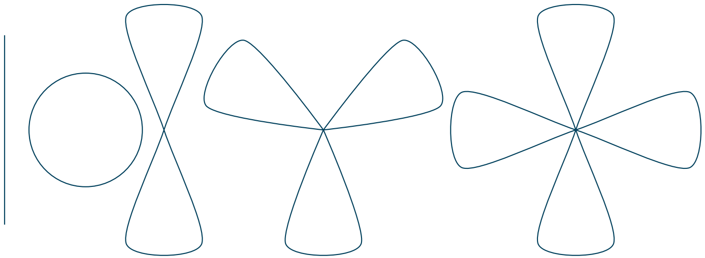
which does not exist!



Generalization

Bouquets of circles

These topological spaces are pairwise not homeomorphic. Why ?



Metric spaces

A metric space is a set X together with a mapping $d : X \times X \to \mathbb{R}_+ \cup \{\infty\}$ such that:

- $d(x, y) \quad = \quad 0 \quad \Leftrightarrow \quad x = y$
- $d(x, y) \quad = \quad d(y, x)$
- $d(x, z) \quad \leqslant \quad d(x, y) + d(y, z)$

The open balls $B(c, r) = \{x \in X \mid d(c, x) < r\}$ with $x \in X$ and $r > 0$ form a base of a topology.

Goal: turn any graph into metric space in a functorial way.

Metric space morphisms

- $\mathcal{Met}_{emb}$ $f : X \to Y$ s.t. $\forall x, x' \in X,\ d_Y(f(x), f(x')) \ = \ d_X(x, x')$
- $\mathcal{Met}_{ctr}$ $f : X \to Y$ s.t. $\forall x, x' \in X,\ d_Y(f(x), f(x')) \ \leqslant \ d_X(x, x')$
- $\mathcal{Met}$ $f : X \to Y$ s.t. $\exists r \in \,]0, \infty[\ \forall x, x' \in X,\ d_Y(f(x), f(x')) \ \leqslant \ r \cdot d_X(x, x')$
- $\mathcal{Met}_{top}$ $f : X \to Y$ s.t. $\forall x \in X\ \forall \varepsilon > 0\ \exists \eta > 0,\ f\big(B(x, \eta)\big) \ \subseteq \ B\big(f(x), \varepsilon\big)$

$$\mathcal{Met}_{emb} \hookrightarrow \mathcal{Met}_{ctr} \hookrightarrow \mathcal{Met} \hookrightarrow \mathcal{Met}_{top} \overset{full}{\hookrightarrow} \mathcal{Top}$$

Length spaces

The length $\ell(\gamma)$ of a path $\gamma : [0, r] \to (X, d)$ is the least upper bound of the collection of sums

$$\sum_{i=0}^{n} d\big(\gamma(t_{i+1}), \gamma(t_i)\big)$$

where $n \in \mathbb{N}$ and $0 = t_0 \leqslant \cdots \leqslant t_n = r$.
The metric space (X, d) is a length space when the distance between two points $x, x' \in X$ is the following greatest lower bound

$$\inf \left\{ \ell(\gamma) \mid \gamma \text{ is a path from } x \text{ to } x' \right\}$$

A path γ from x to x' such that $\ell(\gamma) = d(x, x')$ is said to be geodesic.
The space is said to be geodesic when any two points are related by a geodesic path.

The Hopf-Rinow theorem

Metric Spaces of Non-Positive Curvature, *M. R. Bridson, and A. Haefliger*, 1999

A metric space is said to be complete when all its Cauchy sequences admit a limit.

Let X be a length space.
If X is complete and locally compact, then

- every closed bounded subset of X is compact, and
- X is a geodesic space.

Isometric embedding in $\mathbb{R}^n$

- $\mathbb{R}^n$ is a geodesic space
- $\mathbb{R}^n \setminus \{0\}$ with the distance inherited from $\mathbb{R}^n$ is a length space, not a geodesic one.
- $\mathbb{R}^n \setminus [0,1]^n$ with the distance inherited from $\mathbb{R}^n$ is not a length space.
- Any metric space (X, d) is associated to a length space (X, d_ℓ) with

$$d_\ell(x, x') \quad = \quad \inf \left\{ \ell(\gamma) \mid \gamma \text{ is a path from } x \text{ to } x' \right\}$$

Neighbours

$$G : A \underset{\partial^+}{\overset{\partial^-}{\rightrightarrows}} V$$

- The underlying set of the metric graph is $A \times]0,1[\sqcup V$
- Two points p, p' are said to be neighbours when there is an arrow a such that $p, p' \in \{a\} \times]0,1[\sqcup \{\partial^- a, \partial^+ a\}$

Distance between two neighbours

- If $\partial^- a \neq \partial^+ a$ there is a canonical bijection

$$\phi : \{a\} \times]0,1[\ \sqcup \{\partial^- a, \partial^+ a\} \to [0,1]$$

  In that case $d(p,p') = |t - t'|$ with $t = \phi(p)$ and $t' = \phi(p')$.
- If $\partial^- a = \partial^+ a$ there is a canonical bijection

$$\phi : \{a\} \times]0,1[\ \sqcup \{\partial^- a, \partial^+ a\} \to [0,1[$$

  In that case

$$d(p,p') = \min \left\{ |t - t'|, 1 - |t - t'| \right\}$$

  with $t = \phi(p)$ and $t' = \phi(p')$.

Itinerary

An itinerary on $A\times]0,1[\ \sqcup V$ is a (finite) sequence $p_0,\ldots,p_q$ of points such that p_k and p_{k+1} are neighbours for $k \in \{0,\ldots,q-1\}$.

The length of that itinerary is

$$\ell(p_0,\ldots,p_q) \quad = \quad \sum_{k=0}^{q-1} d(p_k,p_{k+1})$$

The distance between two points p and p' of $A\times]0,1[\ \sqcup V$ is

$$d(p,p') \quad = \quad \inf\left\{\ell(p_0,\ldots,p_q) \mid p_0,\ldots,p_q \text{ is a itinerary from } p \text{ to } p'\right\}$$

The metric graph associated with G is the metric space

$$\left(A\times]0,1[\ \sqcup V \ ,\ d\right)$$

Open balls

The open ball of radius $r < 1$ centered at the vertex v is the set

$$\{v\} \quad \cup \quad \{a \mid \partial^- a = v\} \times]0, r[\quad \cup \quad \{a \mid \partial^+ a = v\} \times]1 - r, 1[$$

For $(a, t) \in \{a\} \times]0, 1[$ the open ball of radius $r \leqslant \min\{t, 1 - t\}$ centered at the vertex (a, t) is the set

$$\{a\} \times]t - r, t + r[$$

That collection of open balls forms a base of open sets.

If $r \leqslant \frac{1}{4}$ then $B(c, r)$ is geodesically stable, i.e. for all p, $q \in B(c, r)$

$$\{p, q\} \quad \subseteq \quad \bigcup \{\mathrm{im}(\gamma) \mid \gamma \text{ geodesic from } p \text{ to } q\} \quad \subseteq \quad B(c, r)\,.$$

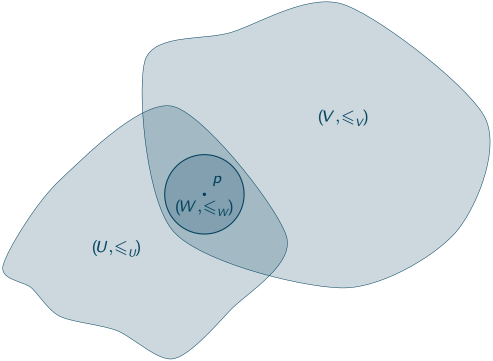
The metric graph construction is functorial from $\mathcal{Grph}$ to $\mathcal{Met}_{ctr}$

Every finite graph with weighted arrows (in $\mathbb{R}_+ \setminus \{0\}$) with can be embedded in $\mathbb{R}^3$.

The category of ordered bases ($\mathcal{OB}$)

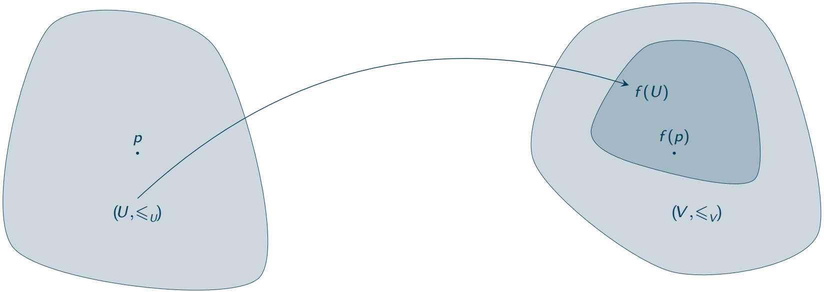
We write that $(X, \leqslant_X)$ is a subposet of $(Y, \leqslant_Y)$, or $(X, \leqslant_X) \hookrightarrow (Y, \leqslant_Y)$, when $X \subseteq Y$ and $a \leqslant_X b \Leftrightarrow a \leqslant_Y b$ for all $a, b \in X$.

An *ordered base* is a collection of posets $\mathcal{B}$ such that for all $(U, \leqslant_U), (V, \leqslant_V) \in \mathcal{B}$, every $p \in U \cap V$, there exists $(W, \leqslant_W) \in \mathcal{B}$ such that $p \in (W, \leqslant_W) \hookrightarrow (U, \leqslant_U), (V, \leqslant_V)$.



The category of ordered bases ($\mathcal{OB}$)

A map $f : \mathcal{U} \to \mathcal{V}$ is *locally order-preserving* when for every point p of $\mathcal{U}$, every $(V, \leqslant_V) \in \mathcal{V}$ with $f(p) \in V$, there exists $(U, \leqslant_U) \in \mathcal{U}$ with $p \in U$ such that $f(U) \subseteq V$ and f is order-preserving from $(U, \leqslant_U)$ to $(V, \leqslant_V)$.



Ordered bases and locally order-preserving maps form the category $\mathcal{OB}$.

The underling topology of an ordered base

If $\mathcal{B}$ is an ordered base, then $U\mathcal{B} = \{UB \mid B \in \mathcal{B}\}$ is a base of a topology (UB denotes the underlying set of the poset B).

If $f : \mathcal{B} \to \mathcal{B}'$ is locally order-preserving, then $Uf : U\mathcal{B} \to U\mathcal{B}'$ is continuous; we have a forgetful functor $\mathcal{OB} \to \mathcal{B}as$.

We have a functor $U : \mathcal{OB} \to \mathcal{S}et$ obtained as the composite $\mathcal{OB} \to \mathcal{B}as \to \mathcal{S}et$.

The underlying space functor $Sp : \mathcal{OB} \to \mathcal{T}op$ is the composite $\mathcal{OB} \to \mathcal{B}as \to \mathcal{T}op$.

We write $\mathcal{B} \sim \mathcal{B}'$ when $Sp(\mathcal{B}) = Sp(\mathcal{B}')$ and $\mathcal{B} \cup \mathcal{B}'$ is still an ordered base; and we say that $\mathcal{B}$ and $\mathcal{B}'$ are equivalent.

The relation $\sim$ is an equivalence relation on the collection of ordered bases over a given set.

If $\mathcal{A} \sim \mathcal{A}'$ and $\mathcal{B} \sim \mathcal{B}'$, then any map $f : U\mathcal{A} \to U\mathcal{B}$ is locally order-preserving from $\mathcal{A}$ to $\mathcal{B}$ iff it is so from $\mathcal{A}'$ to $\mathcal{B}'$.

Locally ordered spaces

An ordered base $\mathcal{B}$ is said to be maximal when for every poset X, if UX is open in $Sp(\mathcal{B})$ and $\mathcal{B} \cup \{X\}$ is still an ordered base, then $X \in \mathcal{B}$.

A locally ordered space is a maximal ordered base.

We denote by $\mathcal{LoSp}$ the full subcategory of $\mathcal{OB}$ whose objects are the locally ordered spaces.

Lemma: Every ordered base is contained in a unique maximal ordered base.

Proposition: the full embedding $\mathcal{LoSp} \to \mathcal{OB}$ is an equivalence of categories whose quasi-inverse is the functor that assigns its locally ordered space to every ordered base.

The locally ordered line

Examples of equivalent ordered bases on $\mathbb{R}$

- $\{(I, \leqslant) \mid I$ open interval of $\mathbb{R}\}$,
- $\{(U, \leqslant) \mid U$ open subset of $\mathbb{R}\}$,
- $\{(U, \sqsubseteq_U) \mid U$ open subset of $\mathbb{R}\}$ where $x \sqsubseteq_U y$ stands for $x \leqslant y$ and $[x, y] \subseteq U$,
- $\{(U, \sqsubseteq'_U) \mid U$ open subset of $\mathbb{R}\}$ where $x \sqsubseteq'_U y$ is any extension of $\sqsubseteq_U$.

Suppose that $[0, 1] \cup [2, 3]$ is a locally ordered subspace of $\mathbb{R}$, the map $t \in [0, 1] \cup [2, 3] \mapsto t + 2 \pmod 4 \in [0, 1] \cup [2, 3]$ is locally order-preserving. A directed path on an ordered base $\mathcal{B}$ is a locally order-preserving map defined over some compact interval equipped with the ordered base inherited from $\mathbb{R}$.

The locally ordered circle

Examples of equivalent ordered bases on $\mathbb{S}^1$

- $\{(A, \leqslant) \mid A \text{ open arc}\}$ where $\leqslant$ is the order induced by $\mathbb{R}$ and the restriction of the exponential map to an open subinterval of $\{t \in \mathbb{R} \mid e^{it} \in A\}$ of length at most 2π,
- $\{(U, \sqsubseteq_U) \mid U \text{ proper open subset of } \mathbb{S}^1\}$ where $x \sqsubseteq_U y$ means that the anticlockwise compact arc from x to y is included in U,
- $\{(U, \sqsubseteq'_U) \mid U \text{ proper open subset of } \mathbb{S}^1\}$ where $\sqsubseteq'_U$ is any extension of the partial order $\sqsubseteq_U$.

Ordered spaces

Topology and Order, L. Nachbin, 1965

An ordered space is a topological space X together with a partial order $\sqsubseteq$ on (the underlying set of) X.
If the relation $\sqsubseteq$ is closed in the sense that

$$\{(a, b) \in X \times X \mid a \sqsubseteq b\}$$

is a closed subset of $X \times X$, then X is said to be a partially ordered space (or pospace).
A ordered space morphism is an order-preserving continuous map.

Ordered spaces and their morphisms form the category $\mathcal{Ord}$.

The underlying space of a pospace is Hausdorff.

Examples

- The real line with standard topology and order.
- Any subset of a pospace with the induced topology and order.
- The collection of compact subsets of a metric space equipped with the Hausdorff distance is a metric space.

$$\begin{aligned} d_H(K, K') &= \sup\left\{d(x, K'), d(x', K) \mid x \in K; x' \in K'\right\} \\ d(x, K) &= \inf\left\{d(x, k) \mid k \in K\right\} \end{aligned}$$

  The induced topological space ordered by inclusion is a pospace.
- Problem: there is no pospace on the circle whose collection of directed paths is

$$\left\{e^{i\theta(t)} \mid \theta : [0, r] \to \mathbb{R} \text{ increasing}\right\}$$

Ordered spaces as locally ordered spaces

Each ordered space $(X, \sqsubseteq)$ can be seen as a locally ordered space

$$\Big(X, \{(U, \sqsubseteq_{|_U}) \mid U \text{ open subset of } X\}\Big)$$

The resulting functor is:

- faithful
- not injective on object (hence not an embedding)
- not full

Directed loops on locally ordered spaces

A locally order-preserving map $\delta : [a, b] \to \mathcal{X}$ whose image is contained in $C \in \mathcal{X}$ induces an order-preserving map from $[a, b]$ to C.

A directed path δ on a local pospace X is constant iff its extremities are equal and there exists $C \in \mathcal{X}$ that contains the image of δ.

A vortex is a point every neighbourhood of which contains a non-constant directed loop.

A local pospace has no vortex.

A convenient open covering

Let $\mathcal{B}$ be the collection of open balls B of $|G|$ such that

- B is centred at a vertex and its radius is $\leqslant \frac{1}{3}$, or
- $B = \{a\} \times U$ for some arrow a and some open interval $U \subseteq \,]0,1[$ of length $\leqslant \frac{1}{3}$.

Given $B, B' \in \mathcal{B}$ if B is of the second kind, then so is $B \cap B'$.

If B, B' are centred at v and v' we have

- $v \neq v' \;\Rightarrow\; B \cap B' = \emptyset$ and
- $v = v' \;\Rightarrow\; B \subseteq B'$ or $B' \subseteq B$

Ordered open stars

An element B of $\mathcal{B}$ centred at v of radius $r \leqslant \frac{1}{3}$ is the disjoint union of $\{v\}$ together with

- $\{a\}\times]0,r[$ for each arrow a such that $\partial^- a = v$
- $\{a\}\times]1-r,1[$ for each arrow a such that $\partial^+ a = v$

The partial order on B is characterized by the following constraints:

- each branch $\{a\}\times]1-r,1[$ and $\{a\}\times]0,r[$ inherits its order from $\mathbb{R}$
- $\{v\} \sqsubseteq \{a\}\times]0,r[$ for each arrow a such that $\partial^- a = v$
- $\{a\}\times]1-r,1[\sqsubseteq \{v\}$ for each arrow a such that $\partial^+ a = v$

We have $B \cap B' \neq \emptyset \ \Rightarrow \ B \cap B' \in \mathcal{B}$ and

$$\sqsubseteq_{B|_{B\cap B'}} \quad = \quad \sqsubseteq_{B\cap B'} \quad = \quad \sqsubseteq_{B'|_{B\cap B'}}$$

The metric graph of $|G|$ thus becomes a local pospace.

The locally ordered metric graph construction is functorial.

Description

There exists a (unique) intrinsic metric d_G on $|G|$ such that the open balls of radii $\varepsilon > 0$ about (a,t) and v are
$\{a\} \times]t-\varepsilon, t+\varepsilon[$ if $\varepsilon \leqslant \min(t, 1-t)$, and
$\{a \in G^{(1)} \mid \mathrm{tgt}(a) = v\} \times]1-\varepsilon, 1[\ \cup \ \{v\} \ \cup \ \{a \in G^{(1)} \mid \mathrm{src}(a) = v\} \times]0, \varepsilon[$ if $\varepsilon \leqslant \frac{1}{2}$.

The partial order $\sqsubseteq$ and the metric d_G on the ball centered at v of radius ε are characterized by the following properties:

$d_G((a,t), v) = 1 - t$	$(a,t) \sqsubseteq v$	if $t \in]1-\varepsilon, 1[$
$d_G(v, (a,t)) = t$	$v \sqsubseteq (a,t)$	if $t \in]0, \varepsilon[$
$d_G((a,t),(a,t')) = t' - t$	$(a,t) \sqsubseteq (a,t')$	if $t \leqslant t'$ and $(t, t' \in]0,\varepsilon[$ or $t, t' \in]1-\varepsilon, 1[)$
$d_G((a,t),(a,t')) = \min\{t'-t, 1-(t'-t)\}$	$(a,t') \sqsubseteq (a,t)$	if $t \in]0,\varepsilon[$ and $t' \in]1-\varepsilon, 1[$
$d_G((a,t),(b,t')) = d_G((a,t),v) + d_G(v,(b,t'))$		if $a \neq b$
	$(a,t) \sqsubseteq (b,t')$	if $t \in]1-\varepsilon, 1[$ and $t' \in]0,\varepsilon[$

If $\varepsilon \leqslant \frac{1}{4}$ then the ball centered at v of radius ε, say B, is geodesically stable: for all p, $q \in B$, the union of the images of the geodesics from p to q is nonempty and contained in B.

The *standard ordered base* of G is the collection of ordered open balls of radii $\varepsilon \leqslant \frac{1}{2}$ with their 'canonical' partial order.