Directed Algebraic Topology and Concurrency

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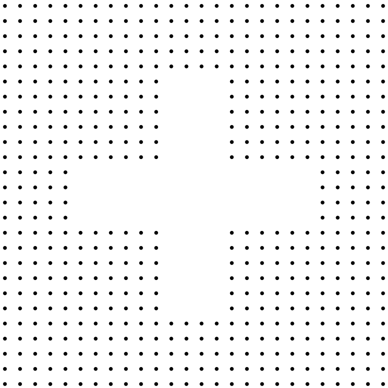
MPRI : Concurrency (2.3)

Thursday, the 28^{th} of January 2010

From discrete to continuous

The discrete semantic of `P(a).P(b).V(b).V(a)|P(b).P(a).V(a).V(b)`



By construction, time is "discrete".

From discrete to continuous

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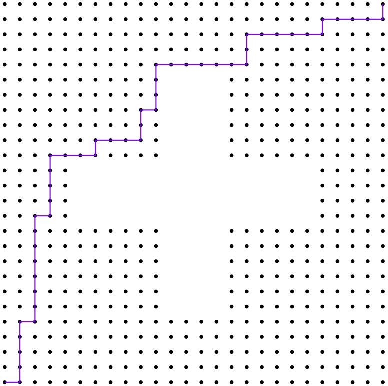

An execution trace consists on an interlacing of “atomic” actions.
This model does not allow “true concurrency”.

From discrete to continuous

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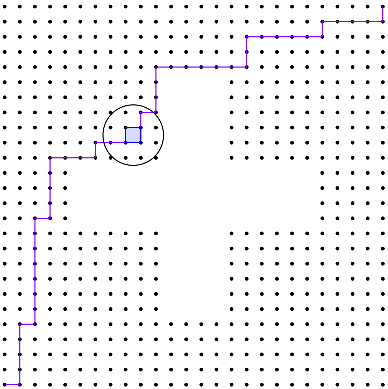

We can locally permute some actions of the given path and thus yield another path which is seen as "equivalent"

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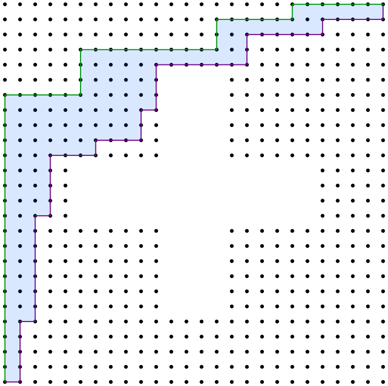

But identifying two paths may require many permutations.
From a combinatorial point of view, this approach is not efficient.

From discrete to continuous

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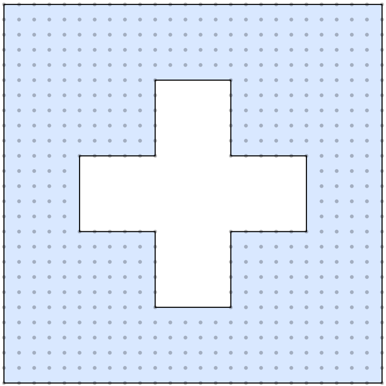

Using topology we define a continuous model

From discrete to continuous

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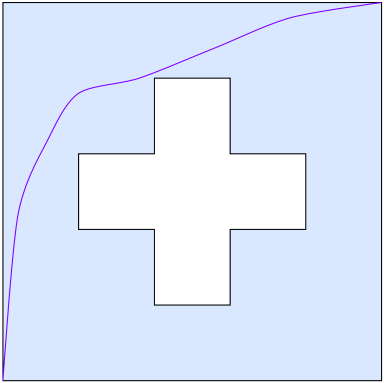

The resulting model allows "true concurrency".
The execution traces are represented by the directed paths.

From discrete to continuous

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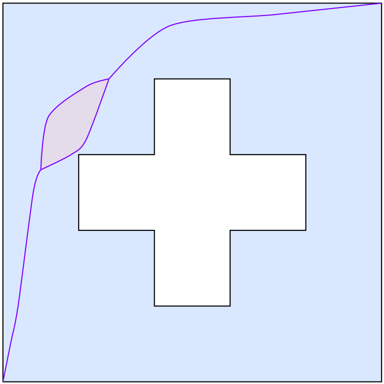

The local permutation of actions are then replaced by (directed) homotopies

From discrete to continuous

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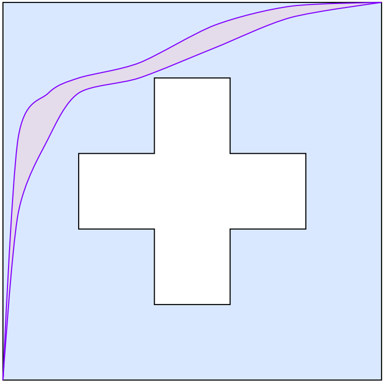

The (directed) homotopies actually allow "global" permutation of actions
so they could be combinatorially more efficient
provided we find a handy representation

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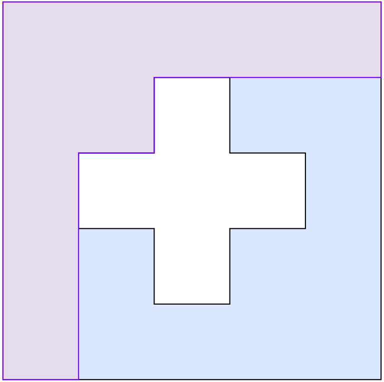


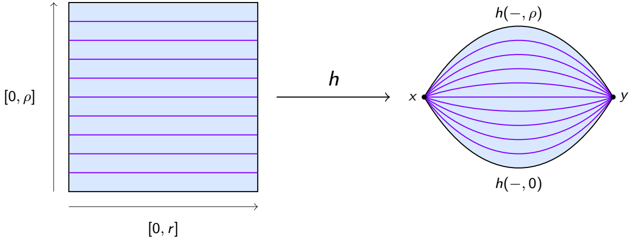
In fact all equivalent paths between two given points
can be easily described as a union of “rectangles”
$[0,1]\times[0,5]\cup[0,3]\times[2,5]\cup[0,5]\times[4,5]$

Let X be a pospace and $r, \rho \in \mathbb{R}_+$

A directed homotopy is a morphism of pospaces $h \in \mathcal{P}o\big[[0,r] \times [0,\rho], X\big]$ such that the mappings

$$h(0,-) : s \in [0,\rho] \mapsto h(0,s) \text{ and } h(r,-) : s \in [0,\rho] \mapsto h(r,s)$$

are constant

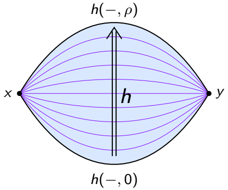


Directed homotopy between directed paths

seen as directed paths

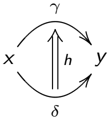
h is also a path on the pospace $X^{[0,r]}$ since

$$h \in \mathcal{P}o\big[[0,r] \times [0,\rho], X\big] \quad \cong \quad \mathcal{P}o\big[[0,\rho], X^{[0,r]}\big]$$



Defining $\gamma := h(-,\rho)$ and $\delta := h(-,0)$, the second point of view leads us

to introduce the following notation



Directed Homotopies and Natural Transformations

The directed homotopies formally have the same properties as
the natural transformations replacing

“category” by “point”

“functor” by “path”

and

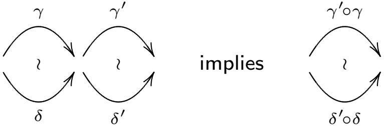
“natural transformation” by “directed homotopy”

Congruence over a small category $\mathcal{C}$

A congruence over $\mathcal{C}$ is an equivalence relation $\sim$ over $\mathrm{Mo}(\mathcal{C})$ such that

1) $\gamma \sim \delta$ implies $\mathsf{s}(\gamma) = \mathsf{s}(\delta)$ and $\mathsf{t}(\gamma) = \mathsf{t}(\delta)$

2) $\gamma \sim \delta$, $\gamma' \sim \delta'$ and $\mathsf{s}(\gamma') = \mathsf{t}(\gamma)$ implies $\gamma' \circ \gamma \sim \delta' \circ \delta$



Then the we can define the quotient category $\mathcal{C}/\!\sim$ defining $[\gamma] \circ [\delta] = [\gamma \circ \delta]$ and we have the quotient functor $q : \mathcal{C} \to \mathcal{C}/\!\sim$ defining $q(\gamma) = [\gamma]$

The underlying preorder of a small category $\mathcal{C}$

$$\begin{array}{ccc}
\mathcal{C}at & \longrightarrow & \mathcal{P}re \\
\mathcal{C} & & (\mathrm{Ob}(\mathcal{C}), \preccurlyeq_{\mathcal{C}}) \\
\Big\downarrow f & \longmapsto & \mathrm{Ob}(f) \Big\downarrow \\
\mathcal{D} & & (\mathrm{Ob}(\mathcal{D}), \preccurlyeq_{\mathcal{D}})
\end{array}$$

with

$$x \preccurlyeq_{\mathcal{C}} y \text{ when } \mathcal{C}[x, y] \neq \emptyset$$

Comparing paths defined on distinct segments

Given $\gamma \in \mathcal{Po}[[0,r],X]$ and $\delta \in \mathcal{Po}[[0,r'],X]$ put $\gamma \preccurlyeq \delta$ when there exist $\theta \in \mathcal{Po}[[0,1],[0,r]]$ and $\theta' \in \mathcal{Po}[[0,1],[0,r']]$ and a directed homotopy from $\gamma \circ \theta$ to $\delta \circ \theta'$.

Loop-free paths and Regular paths

see U. Fahrenberg and M. Raußen paper "Reparametrizations of Continuous Paths"

Let X be a Hausdorff space and $\gamma \in \mathcal{Top}\big[[0,r],X\big]$

- γ is said to be loop-free when $\gamma(t) = \gamma(t') \;\Rightarrow\; \gamma$ is constant on $[t,t']$
- If X Hausdorff and $\gamma \in \mathcal{Top}\big[[0,r],X\big]$ loop-free then $\mathrm{im}(\gamma) \cong [0,1]$ or $\mathrm{im}(\gamma) \cong \{0\}$
- γ is said to be regular when γ constant on $[t,t'] \neq \emptyset$ implies that $t = t'$ or $[t,t'] = [0,r]$
- there exist θ_0, θ_1 s.t. $\gamma \circ \theta_0 = \delta \circ \theta_1$ iff there exist ξ, θ_2, θ_3 such that $\gamma = \xi \circ \theta_2$ and $\delta = \xi \circ \theta_3$
- for all γ there exists a regular path γ' and θ such that $\gamma = \gamma' \circ \theta$
- if $\gamma \circ \theta_0 = \delta \circ \theta_1$ with γ and δ regular, then there exists an φ <u>iso</u> s.t. $\delta = \gamma \circ \varphi$

Reparametrizations and Directed Homotopies

- Let $\gamma \in \mathcal{P}o\big[[0,r],X\big]$ then $h(s,t) = \gamma(t)$ is a directed homotopy
- If $\gamma, \delta \in \mathcal{P}o\big[[0,r],X\big]$, $\mathrm{im}(\gamma) = \mathrm{im}(\delta)$ and $\gamma \sqsubseteq \delta$ then

$$h(t,s) := \varphi\Big(\varphi^{-1}\circ\gamma(t) + s\cdot\big(\varphi^{-1}\circ\delta(t) - \varphi^{-1}\circ\gamma(t)\big)\Big)$$

  is a directed homotopy from γ to δ with $\varphi : \; [0,1] \xrightarrow{\cong} X$
- If $\gamma, \delta \in \mathcal{P}o\big[[0,r],X\big]$, $\mathrm{im}(\gamma) = \mathrm{im}(\delta)$ then we can define the directed path

$$\gamma \vee \delta : t \in [0,r] \mapsto \max(\gamma(t), \delta(t))$$

Comparing paths defined on distinct segments

- The relation $\preccurlyeq$ is a preorder (but it is not so easy to prove)
- We denote by $\sim$ the equivalence relation generated by $\preccurlyeq$ i.e. $\gamma \sim \delta$ iff there is a "zigzag" of directed homotopies

$$\gamma \;\Nearrow\; \Nwarrow \;\cdots\; \Nearrow\; \Nwarrow \;\delta$$

- The relation $\sim$ is actually a congruence over $\overrightarrow{P}(X)$ as a consequence of the "Godement product" construction

The Fundamental Category functor over $\mathcal{P}o$

The preceding construction gives rise to a functor $\overrightarrow{\pi_1}$ from $\mathcal{P}o$ to $\mathcal{C}at$ since for all $f \in \mathcal{P}o[X, Y]$ and all directed homotopies h between paths on X, the composite $f \circ h$ is a directed homotopy between paths on Y.

$$\overrightarrow{\pi_1} : \mathcal{P}o \longrightarrow \mathcal{C}at$$

$$\begin{array}{ccc} X & & \overrightarrow{\pi_1}(X) \\ \downarrow f & \longmapsto & \downarrow \overrightarrow{\pi_1}(f) \\ Y & & \overrightarrow{\pi_1}(Y) \end{array}$$

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with

$$\overrightarrow{\pi_1}(f) : \overrightarrow{\pi_1}(X) \longrightarrow \overrightarrow{\pi_1}(Y)$$

$$\begin{array}{ccc} p & & f(p) \\ \Big\downarrow [\gamma] & \longmapsto & \Big\downarrow [f \circ \gamma] \\ q & & f(q) \end{array}$$

The directed real line and plane

The fundamental category of the directed real line $\overrightarrow{\mathbb{R}}$ is the poset $(\mathbb{R}, \leqslant)$ seen as a small category

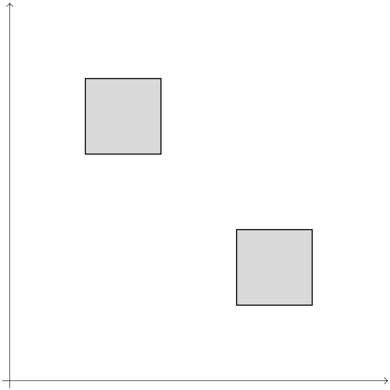
The fundamental category of the directed real plane $\overrightarrow{\mathbb{R}} \times \overrightarrow{\mathbb{R}}$ is the poset $(\mathbb{R}, \leqslant) \times (\mathbb{R}, \leqslant)$ seen as a small category.
Indeed, given γ and δ sharing the same extremities we define $\gamma \vee \delta$ so

$$h(t,s) = (1-s)\cdot\gamma(t) + s\cdot(\gamma\vee\delta)(t) \text{ and } h'(t,s) = (1-s)\cdot\delta(t) + s\cdot(\gamma\vee\delta)(t)$$

are directed homotopies

In general we have $\overrightarrow{\pi_1}(X \times Y) \cong \overrightarrow{\pi_1}(X) \times \overrightarrow{\pi_1}(Y)$

Two squares on the antidiagonal



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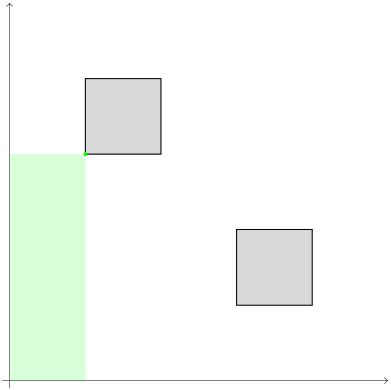

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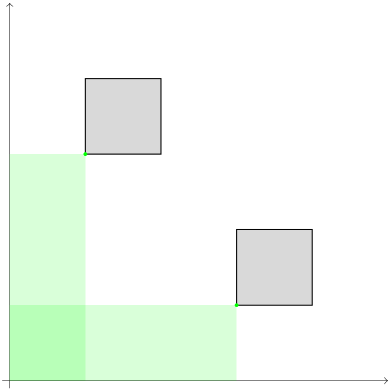

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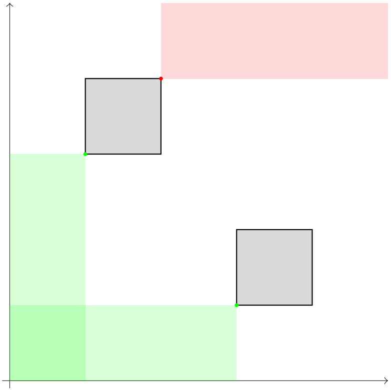

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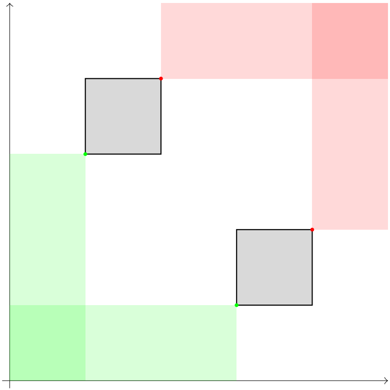

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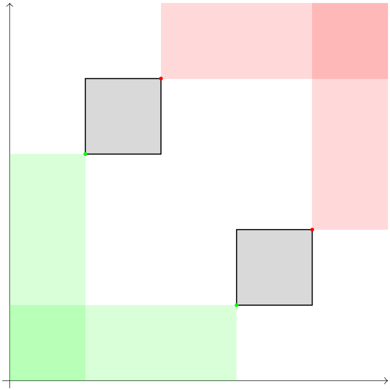
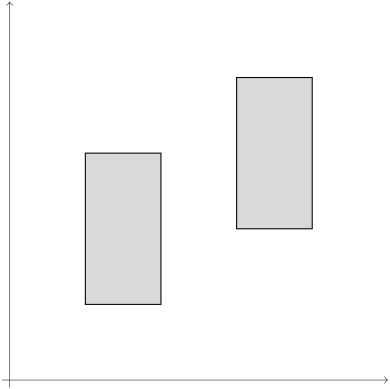


One has 9 “components”

In dimension 2 it suffices to draw the “past cones” from bottom left corners and the “future cones” from upper right ones

Holes cast shadows



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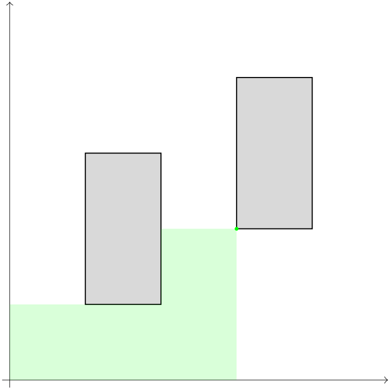

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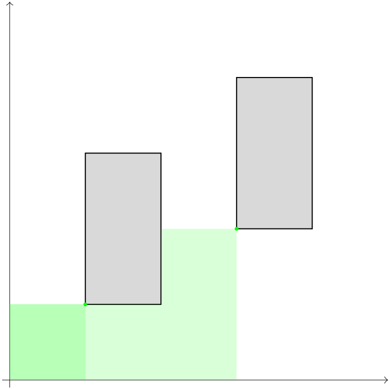

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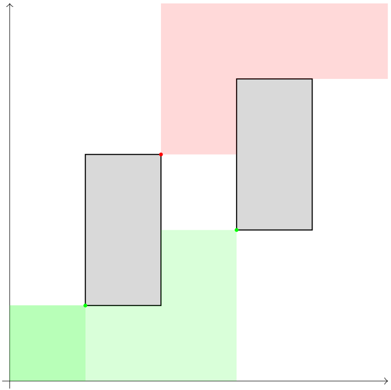

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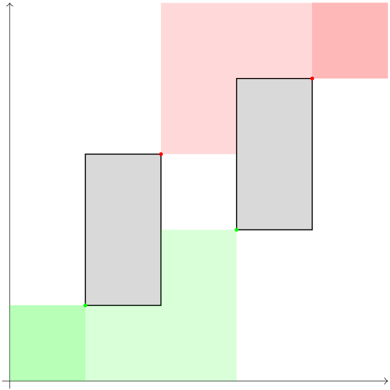

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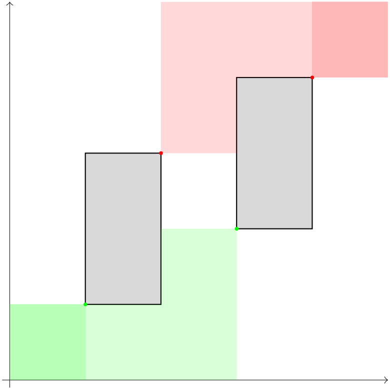
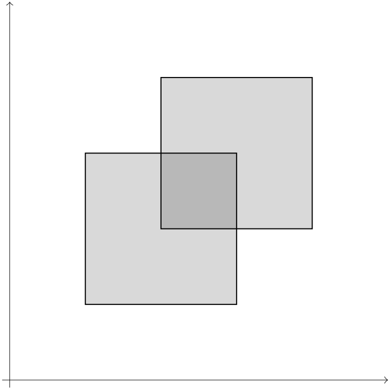


One has 7 “components”

Corners in holes do not shed any light



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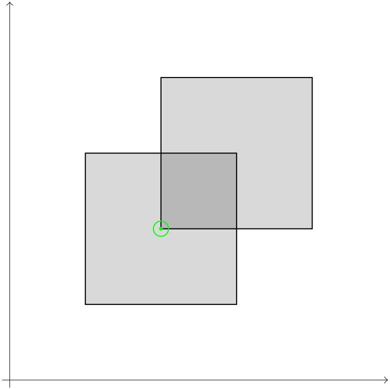

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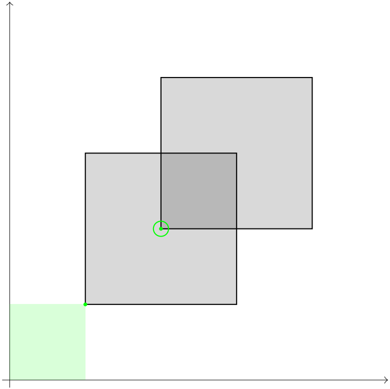

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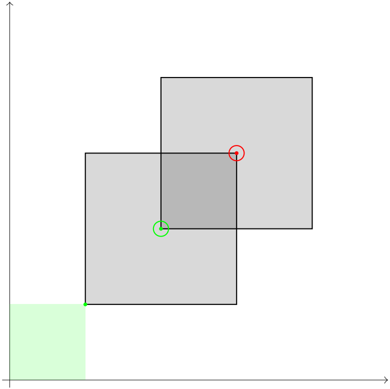

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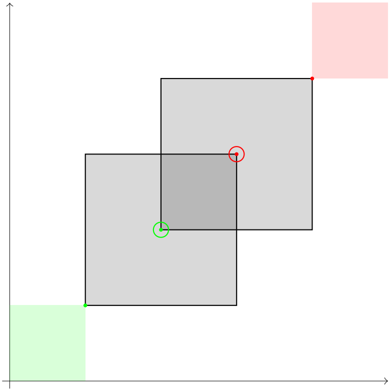

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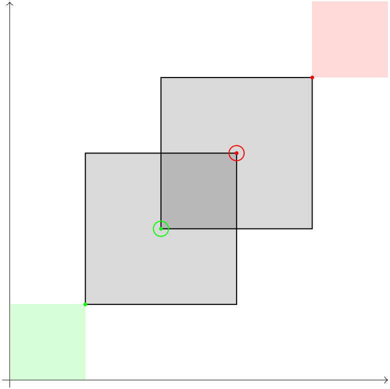
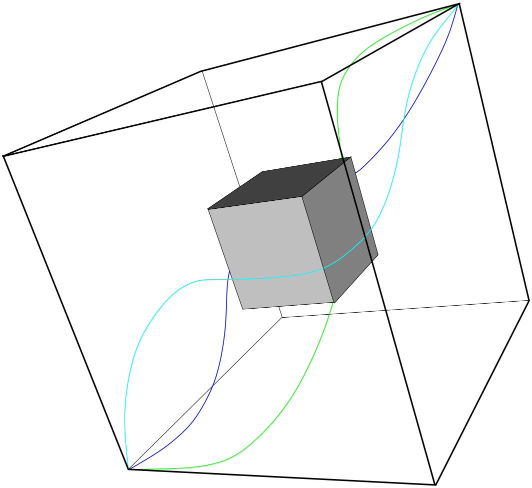


One has 4 “components”

The floating cube



Up to directed homotopy equivalence,
there is a unique directed path from $(0, 0, 0)$ to $(3, 3, 3)$

The floating cube

ss the picture suggests, there are 26 “components”