Directed Algebraic Topology and Concurrency

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MPRI : Concurrency (2.3)

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Functors f from $\mathcal{C}$ to $\mathcal{D}$

Definition (preserving the "underlying graph")

A functor $f : \mathcal{C} \to \mathcal{D}$ is defined by two "mappings" Ob(f) and Mo(f) such that

$$\begin{array}{ccc} \mathrm{Mo}(\mathcal{C}) & \overset{\mathrm{s}}{\underset{\mathrm{t}}{\rightrightarrows}} & \mathrm{Ob}(\mathcal{C}) \\ \mathrm{Mo}(f) \downarrow & & \downarrow \mathrm{Ob}(f) \\ \mathrm{Mo}(\mathcal{D}) & \overset{\mathrm{s'}}{\underset{\mathrm{t'}}{\rightrightarrows}} & \mathrm{Ob}(\mathcal{D}) \end{array}$$

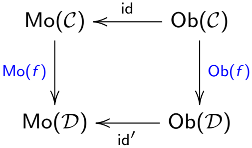
with $\mathrm{s}'(\mathrm{Mo}(f)(\alpha)) = \mathrm{Ob}(f)(\mathrm{s}(\alpha))$ and $\mathrm{t}'(\mathrm{Mo}(f)(\alpha)) = \mathrm{Ob}(f)(\mathrm{t}(\alpha))$

Hence it is in particular a morphism of graphs.

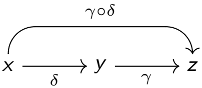
Functors f from $\mathcal{C}$ to $\mathcal{D}$

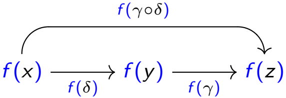
Definition (preserving the "underlying local monoid")

The "mappings" Ob(f) and Mo(f) also make the following diagram commute

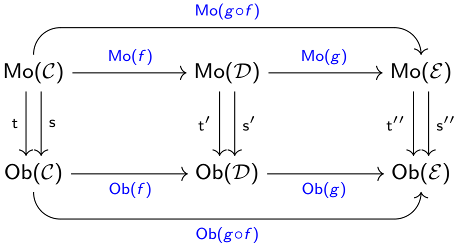


and satisfies $\mathrm{Mo}(f)(\gamma \circ \delta) = \mathrm{Mo}(f)(\gamma) \circ \mathrm{Mo}(f)(\delta)$





Functors compose as the morphisms of graphs do



Hence the functors should be thought of as the morphisms of categories

The small categories and their funtors form a (large) category denoted by $\mathcal{C}at$

Functors terminology

Given a functor $f : \mathcal{C} \to \mathcal{D}$ and two objects x and y we have the mapping

$$
\begin{aligned}
f_{x,y} : \ \mathcal{C}[x,y] &\longrightarrow \mathcal{D}[\mathrm{Ob}(f)(x), \mathrm{Ob}(f)(y)] \\
\alpha &\longmapsto \mathrm{Mo}(f)(\alpha)
\end{aligned}
$$

f is faithful when for all objects x and y the mapping $f_{x,y}$ is one-to-one (injective)
f is full when for all objects x and y the mapping $f_{x,y}$ is onto (surjective)
f is fully faithful when it is full and faithful
f is an embeding when it is faithful and $\mathrm{Ob}(f)$ is one-to-one

Some small functors
(functor between small categories)

The morphisms of monoids are the functors between small categories with a single object

The morphisms of preordered sets are the functors between small categories whose homsets contain at most one element

The actions of a monoid M over a set X are the functors from M to $\mathcal{S}et$ which sends the only element of M to X

Some full embedings in $\mathcal{Cat}$

Remark : The full embedings compose

$$\mathcal{Pre} \hookrightarrow \mathcal{Cat} \qquad \mathcal{Cmon} \hookrightarrow \mathcal{Mon}$$
$$\mathcal{Mon} \hookrightarrow \mathcal{Cat} \qquad \mathcal{Ab} \hookrightarrow \mathcal{Cmon}$$
$$\mathcal{Pos} \hookrightarrow \mathcal{Pre} \qquad \mathcal{Ab} \hookrightarrow \mathcal{Gr}$$
$$\mathcal{Gr} \hookrightarrow \mathcal{Mon} \qquad \mathcal{Set} \hookrightarrow \mathcal{Pos}$$

Some forgetful functors

$$(M, *, e) \in \mathcal{Mon} \mapsto M \in \mathcal{Set}$$
$$(X, \Omega) \in \mathcal{Top} \mapsto X \in \mathcal{Set}$$
$$(X, \sqsubseteq) \in \mathcal{Pos} \mapsto X \in \mathcal{Set}$$
$$(X, \Omega, \sqsubseteq) \in \mathcal{Po} \mapsto (X, \Omega) \in \mathcal{Haus}$$
$$\mathcal{C} \in \mathcal{Cat} \mapsto \mathrm{Ob}(\mathcal{C}) \in \mathcal{Set}$$
$$\mathcal{C} \in \mathcal{Cat} \mapsto \mathrm{Mo}(\mathcal{C}) \in \mathcal{Set}$$

$$\mathcal{C} \in \mathcal{Cat} \mapsto \left(\mathrm{Mo}(\mathcal{C}) \overset{t}{\underset{s}{\rightrightarrows}} \mathrm{Ob}(\mathcal{C}) \right) \in \mathcal{Grph}$$

The homset functors

Let x be an object of a category $\mathcal{C}$

$$\mathcal{C}[-,x] : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{S}et$$

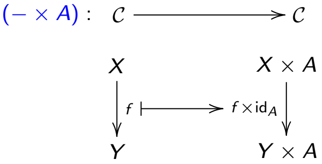
$$\begin{array}{ccc} y & & \mathcal{C}[y,x] \\ \downarrow \delta & \longmapsto & (-\circ\delta) \downarrow \\ z & & \mathcal{C}[z,x] \end{array}$$

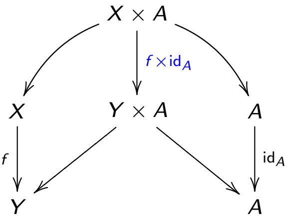
with

$$(-\circ\delta) : \mathcal{C}[y,x] \longrightarrow \mathcal{C}[z,x]$$
$$\gamma \longmapsto \gamma \circ \delta$$

The product functor

Let A be an object of $\mathcal{C}$ such that for all objects X of $\mathcal{C}$ the Cartesian product $X \times A$ exists.





with $f \times \mathrm{id}_A$ defined by right hand side diagram
(the unlabelled arrows being the projection morphism)

Natural Transformations

from f to g (functors)

A natural transformation from $f : \mathcal{C} \longrightarrow \mathcal{D}$ to $g : \mathcal{C} \longrightarrow \mathcal{D}$ is a collection of morphisms $(\eta_x)_{x \in \mathrm{Ob}(\mathcal{C})}$ where $\eta_x \in \mathcal{D}[f(x), g(x)]$ and such that for all $\alpha \in \mathcal{C}[x, y]$ we have $\eta_y \circ f(\alpha) = g(\alpha) \circ \eta_x$ i.e. the following diagram commute

$$x \xrightarrow{\alpha} y \qquad \begin{array}{ccc} f(x) & \xrightarrow{f(\alpha)} & f(y) \\ {\scriptstyle \eta_x}\downarrow & & \downarrow{\scriptstyle \eta_y} \\ g(x) & \xrightarrow[g(\alpha)]{} & g(y) \end{array}$$

Natural Transformations

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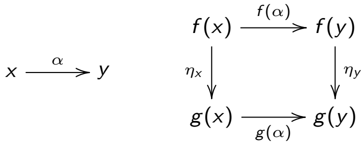
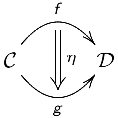


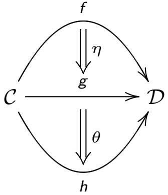
This description is summarized by the following diagram

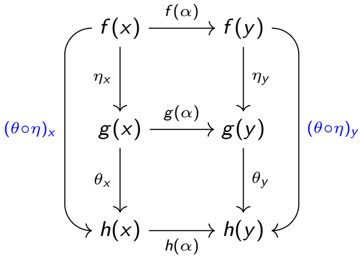


Natural Transformations compose

(a.k.a "vertical composition")

Composition is defined by $(\theta \circ \eta)_x = \theta_x \circ \eta_x$





The functors from $\mathcal{C}$ to $\mathcal{D}$ and the natural transformations between them form the category $\mathrm{Fun}[\mathcal{C}, \mathcal{D}]$ or $\mathcal{D}^{\mathcal{C}}$ (guess the identities)

A functor from $\mathcal{C}$ to $\mathcal{S}et^{\mathcal{C}^{op}}$

involving natural transformations

The category $\mathrm{Fun}[\mathcal{C}^{op}, \mathcal{S}et]$ is often denoted by $\hat{\mathcal{C}}$

A functor from $\mathcal{C}$ to $\mathcal{S}et^{\mathcal{C}^{op}}$

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For all objects x of $\mathcal{C}$ the functor $\mathcal{C}[-, x]$ is an object of $\hat{\mathcal{C}}$

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For all morphisms $\gamma : x \to x'$ of $\mathcal{C}$,
the natural transformation $(\gamma \circ -)$ is a morphism of $\hat{\mathcal{C}}$ defined by

$$(\gamma \circ -) : \; \mathcal{C}[y, x] \longrightarrow \mathcal{C}[y, x']$$

$$\delta \longmapsto \gamma \circ \delta$$

A functor from $\mathcal{C}$ to $\mathcal{S}et^{\mathcal{C}^{op}}$

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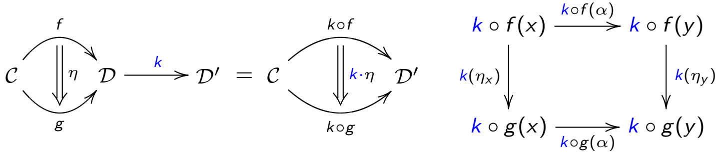
The previous data give rise to a functor because the composition of $\mathcal{C}$ is associative

This functor is refered to as the Yoneda embeding

Natural Transformations admit "scalar" products

on the left and on the right



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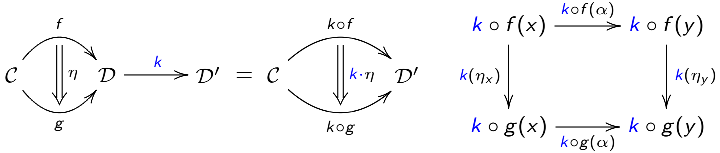
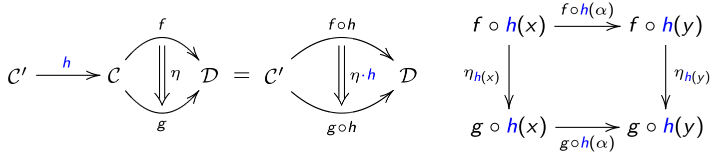
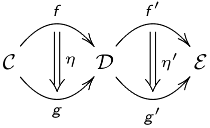




Natural Transformations juxtapose

The "horizontal composition" or Godement product

From the following diagram



Natural Transformations juxtapose

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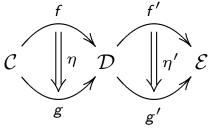
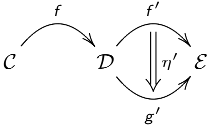
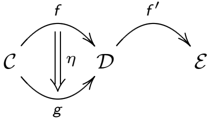


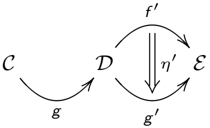
we can deduce
four natural transformations
as shown beside



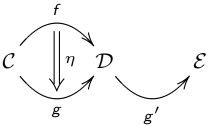
$= \eta' \cdot f$



$= f' \cdot \eta$



$= \eta' \cdot g$



$= g' \cdot \eta$

Natural Transformations juxtapose

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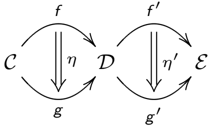

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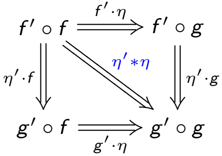
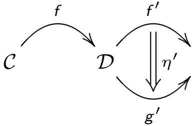
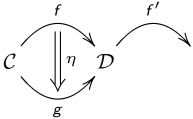


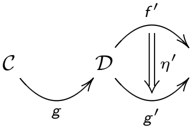
then the outter shape of the above diagram commutes thus defining $\eta' * \eta$



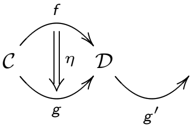
$= \eta' \cdot f$



$= f' \cdot \eta$



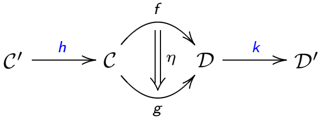
$= \eta' \cdot g$



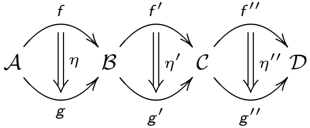
$= g' \cdot \eta$

Algebraic properties

homogeneous associativity



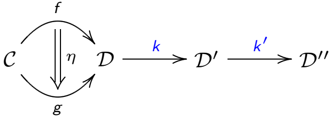
$$(k \cdot \eta) \cdot h = k \cdot (\eta \cdot h)$$



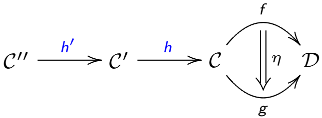
$$(\eta'' * \eta') * \eta = \eta'' * (\eta' * \eta)$$

Algebraic properties

heterogeneous associativity



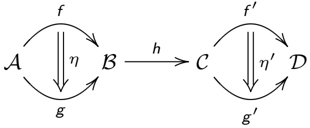
$$(k' \circ k) \cdot \eta = k' \cdot (k \cdot \eta)$$



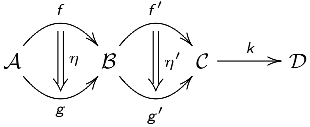
$$(\eta \cdot h) \cdot h' = \eta \cdot (h \circ h')$$

Algebraic properties

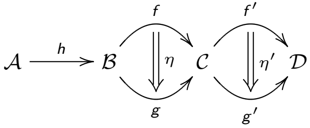
heterogeneous associativity



$$(\eta' \cdot h) * \eta = \eta' * (h \cdot \eta)$$

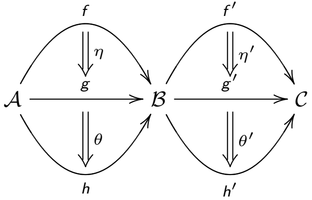


$$k \cdot (\eta' * \eta) = (k \cdot \eta) * \eta'$$



$$(\eta' * \eta) \cdot h = \eta' * (\eta \cdot h)$$

Algebraic properties
Godement exchange law



Algebraic properties

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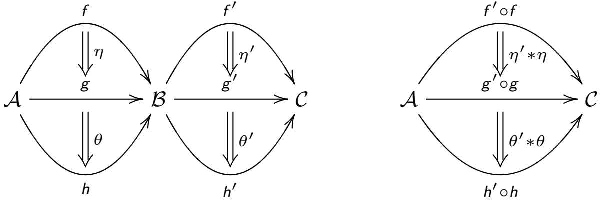

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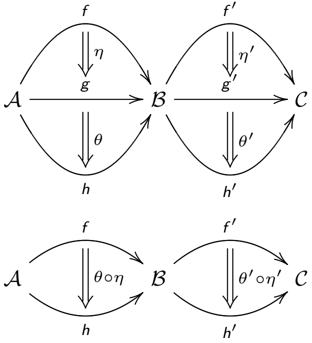

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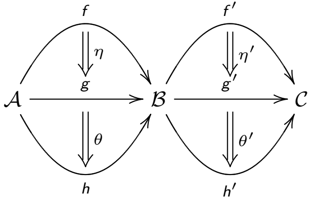
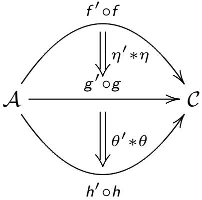
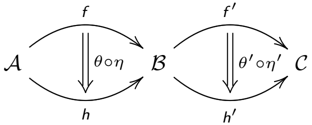






$$(\theta' * \theta) \circ (\eta' * \eta) = (\theta' \circ \eta') * (\theta \circ \eta)$$

Definition
by means of unit and co-unit

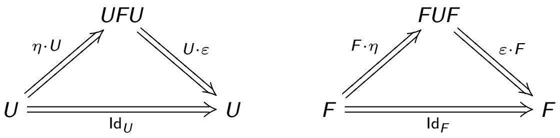
Given two functors $\mathcal{C} \underset{F}{\overset{U}{\rightleftarrows}} \mathcal{D}$

we say that F is left adjoint to U, U is right adjoint to F and we denote by $F \dashv U$ when there exist two natural transformations

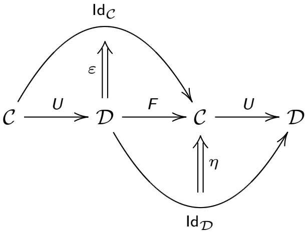
$$\mathrm{Id}_{\mathcal{D}} \overset{\eta}{\Longrightarrow} U \circ F \text{ (unit) and } F \circ U \overset{\varepsilon}{\Longrightarrow} \mathrm{Id}_{\mathcal{C}} \text{ (co-unit)}$$

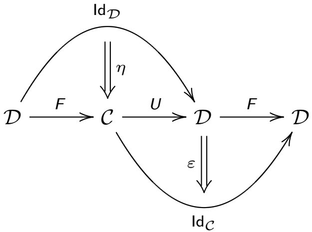
such that

$$(U \cdot \varepsilon) \circ (\eta \cdot U) = \mathrm{Id}_U \text{ and } (\varepsilon \cdot F) \circ (F \cdot \eta) = \mathrm{Id}_F$$



Definition
Diagrams





Definition
by means of unit and homset isomorphism

Given two functors as below

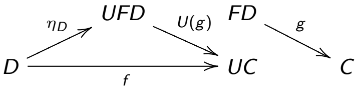
$$\mathcal{C} \underset{F}{\overset{U}{\rightleftarrows}} \mathcal{D}$$

we say that F is left adjoint to U, U is right adjoint to F and we denote by $F \dashv U$ when there exist a natural transformation

$$\mathrm{Id}_{\mathcal{D}} \overset{\eta}{\Longrightarrow} U \circ F \text{ (unit)}$$

such that the following map is a bijection

$$\begin{aligned} \mathcal{C}[F(D), C] &\longrightarrow \mathcal{D}[D, U(C)] \\ g &\longmapsto U(g) \circ \eta_D \end{aligned}$$



Definition

by means of co-unit and homset isomorphism

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we say that F is left adjoint to U, U is right adjoint to F and we denote by $F \dashv U$ when there exist a natural transformation

$$F \circ U \overset{\varepsilon}{\Longrightarrow} \mathrm{Id}_{\mathcal{C}} \text{ (co-unit)}$$

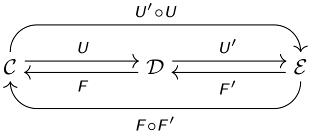
such that the following map is a bijection

$$\begin{array}{rcl} \mathcal{D}[D, U(C)] & \longrightarrow & \mathcal{C}[F(D), C] \\ f & \longmapsto & \varepsilon_C \circ F(f) \end{array}$$

$$\begin{array}{ccccccc} & & FUC & \xleftarrow{F(f)} & & UC & \xleftarrow{f} D \\ & \swarrow^{\varepsilon_C} & & & & & \\ C & & \xleftarrow{\quad g \quad} & & FD & & \end{array}$$

Uniqueness and Composition

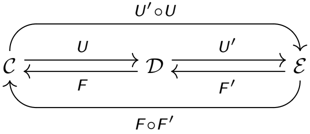
- The left (respectively right) adjoint is unique up to isomorphism
- If $F \dashv U$, $F' \dashv U'$ and $\mathrm{dom}(U') = \mathrm{cod}(U)$ then $F \circ F' \dashv U' \circ U$



What are the unit and the co-unit ?

Uniqueness and Composition

- The left (respectively right) adjoint is unique up to isomorphism
- If $F \dashv U$, $F' \dashv U'$ and $\mathrm{dom}(U') = \mathrm{cod}(U)$ then $F \circ F' \dashv U' \circ U$



What are the unit and the co-unit ?
Respectively $(U' \cdot \eta \cdot F') \circ \eta'$ and $\varepsilon \circ (F \cdot \varepsilon' \cdot U)$

Free ⊣ Underlying

situations where the right adjoint is said to be “forgetful”

- The functor $U : \mathcal{Mon} \longrightarrow \mathcal{Set}$ sends a monoid to its underlying set
- The functor $U : \mathcal{Cmon} \longrightarrow \mathcal{Set}$ sends a commutative monoid to its underlying set
- The functor $U : \mathcal{Cat} \longrightarrow \mathcal{Grph}$ sends a small category to its underlying graph
- The functor $U : \mathcal{Po} \longrightarrow \mathcal{Haus}$ sends a pospace to its underlying topological space

(Find their left adjoints)

- The functor $U : \mathcal{Top} \longrightarrow \mathcal{Set}$ sends a topological space to its underlying set. It has both a left and a right adjoint.

Inclusion $\dashv$ Reflection

situations where the right adjoint is called the "reflector"

- All the embedings given on slide 6 admit a left adjoint
- The left adjoint of $(\{\text{intervals of } \mathbb{R}\}, \subseteq) \hookrightarrow (\{\text{subsets of } \mathbb{R}\}, \subseteq)$ is provided by the convex hull
- In general, every Galois connection is an adjunction.

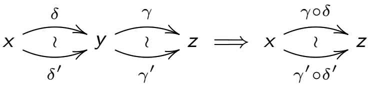
The reflector of $\mathcal{Pre} \hookrightarrow \mathcal{Cat}$

Congruences

A congruence on a small category $\mathcal{C}$ is an equivalence relation $\sim$ over $\mathrm{Mo}(\mathcal{C})$ such that

1) $\gamma \sim \gamma'$ implies $\mathrm{s}(\gamma) = \mathrm{s}(\gamma')$ and $\mathrm{s}(\delta) = \mathrm{s}(\delta')$
2) $\gamma \sim \gamma'$, $\delta \sim \delta'$ and $\mathrm{s}(\gamma) = \mathrm{t}(\delta)$ implies $\gamma \circ \delta \sim \gamma' \circ \delta'$

In diagrams we have



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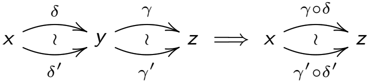

Hence the $\sim$-equivalence class of $\gamma \circ \delta$ does not depend on the $\sim$-equivalence classes of γ and δ and we have a quotient category $\mathcal{C}/\sim$ in which the composition is given by

$$[\gamma] \circ [\delta] = [\gamma \circ \delta]$$

Moreover the set-theoretic quotient map $q : \gamma \in \mathrm{Mo}(\mathcal{C}) \mapsto [\gamma] \in \mathrm{Mo}(\mathcal{C})/\sim$ induces a functor $q : \mathcal{C} \to \mathcal{C}/\sim$

The left adjoint of $\mathcal{Pre} \hookrightarrow \mathcal{Cat}$

Congruences

Reminder : A preorder on X can be seen as a small category whose set of objects is X and such that there is at most one morphism from an object to another.

The relation $\delta \sim \delta'$ defined by $\mathrm{s}(\delta) = \mathrm{s}(\delta')$ and $\mathrm{t}(\delta) = \mathrm{t}(\delta')$ is a congruence.

The left adjoint of $\mathcal{Pre} \hookrightarrow \mathcal{Cat}$ sends a small category $\mathcal{C}$ to the quotient category $\mathcal{C}/\sim$ which is actually a preorder

The associated quotient functors $q : \mathcal{C} \to \mathcal{C}/\sim$ for $\mathcal{C}$ running through the collection of all small categories provide the unit of the adjunction

Exponentiable object

We consider a category $\mathcal{C}$

An object E is said to be exponentiable when the functor $(E \times -)$ is well-defined and admits a right adjoint which is then denoted by $(-)^E$.

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The topological space $[0,1]$ is exponentiable (in $\mathcal{Top}$) by equiping the set $\mathcal{Top}\big[[0,1], X\big]$ with the compact-open topology

The pospace $[0,1]$ is exponentiable (in $\mathcal{Po}$) by equiping the topological space set $X^{[0,1]}$ with the pointwise order i.e. $\gamma \sqsubseteq \delta$ iff $\forall t \in [0,1], \gamma(t) \sqsubseteq_X \delta(t)$

The Moore category functor over $\mathcal{P}o$

Let $\overrightarrow{X}$ be a pospace

Reminder: for any real number $r \geqslant 0$ the compact segment $[0, r]$ with its standard topology and its standard order is a pospace

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The objects of $\overrightarrow{\pi_1}(X)$ are the points of X and the homsets (whose elements are called the directed paths) are given by

$$(\overrightarrow{\pi_1}(X))[x, x'] = \bigcup_{r \geqslant 0} \left\{ \delta \in \mathcal{P}o\big[[0, r], UX\big] \;\middle|\; \delta(0) = x \text{ and } \delta(r) = x' \right\}$$

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The composition is given by the concatenation, suppose $\delta \in \mathcal{P}o\big[[0, r], UX\big]$ and $\gamma \in \mathcal{P}o\big[[0, r'], UX\big]$ satisfying $\delta(r) = \gamma(0)$ then we have

$$\begin{array}{ccc} [0, r + r'] & \longrightarrow & UX \\ t & \longmapsto & \begin{cases} \delta(t) & \text{if } 0 \leqslant t \leqslant r \\ \gamma(t - r) & \text{if } r \leqslant t \leqslant r + r' \end{cases} \end{array}$$

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$$\begin{array}{ccl} [0, r+r'] & \longrightarrow & UX \\ t & \longmapsto & \begin{cases} \delta(t) & \text{if } 0 \leqslant t \leqslant r \\ \gamma(t-r) & \text{if } r \leqslant t \leqslant r+r' \end{cases} \end{array}$$

The identities are the directed paths defined over the degenerated segment $\{0\}$

The Moore category functor over $\mathcal{Po}$
Functoriality

The preceding construction gives rise to a functor $\overrightarrow{P}$ from $\mathcal{Po}$ to $\mathcal{Cat}$ since for all $f \in \mathcal{Po}[X, Y]$ and all directed path γ on X, the composite $f \circ \gamma$ is a directed path on Y.

$$\overrightarrow{P} : \mathcal{Po} \longrightarrow \mathcal{Cat}$$

$$\begin{array}{ccc} X & & \overrightarrow{P}(X) \\ \Big\downarrow f & \longmapsto & \Big\downarrow \overrightarrow{P}(f) \\ Y & & \overrightarrow{P}(Y) \end{array}$$

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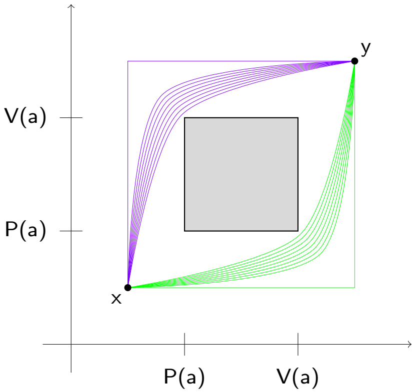
with

$$\overrightarrow{P}(f) : \overrightarrow{P}(X) \longrightarrow \overrightarrow{P}(Y)$$

$$\begin{array}{ccc} p & & f(p) \\ \downarrow & \gamma \longmapsto f\circ\gamma & \downarrow \\ q & & f(q) \end{array}$$

The Moore category of $[\![Q]\!]$

where Q is the PV program `P(a).V(a) | P(a).V(a)`



There are infinitely many paths from x to y.

We would like to classifying them according to whether they run under or above the square.

Directed homotopy between directed paths
Formal definition

Let γ and δ be two directed paths on X defined over the segment $[0, r]$

A directed homotopy from γ to δ is $h \in \mathcal{P}o\big[[0,r] \times [0,\rho], X\big]$ such that

1) The mappings $h(0,-) : s \in [0,\rho] \mapsto h(0,s)$ and $h(r,-) : s \in [0,\rho] \mapsto h(r,s)$ are constant

2) The mappings $h(-,0) : t \in [0,r] \mapsto h(t,0)$ and $h(-,\rho) : s \in [0,r] \mapsto h(t,\rho)$ are γ and δ

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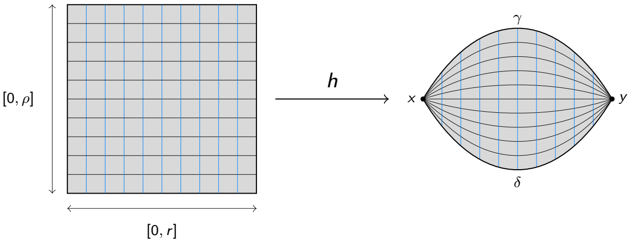
As a consequence we have $\gamma(0) = \delta(0)$ and $\gamma(r) = \delta(r)$. Writting $\gamma \sqsubseteq \delta$ when there exists a directed homotopy from γ to δ we define a partial order on the collection of directed paths on X defined over $[0, r]$.

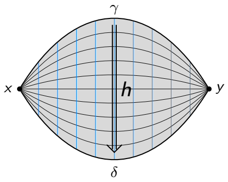
The two faces of directed homotopies

h can be seen as a morphism from $[0,r] \times [0,\rho]$ to X i.e. $h \in \mathcal{P}o\big[[0,r] \times [0,\rho], X\big]$



The two faces of directed homotopies

but also as a path from γ to δ in the pospace $X^{[0,r]}$ i.e. $h \in \mathcal{Po}\big[[0,\rho], X^{[0,r]}\big]$



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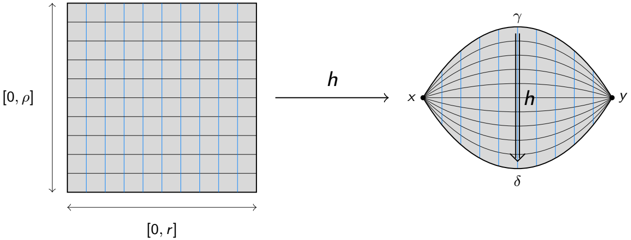
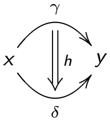


The second point of view leads us to introduce the following notation



Directed Homotopies and Natural Transformations

The directed homotopies formally have the same properties as the natural transformations replacing

"category" by "point"

"functor" by "path"

and

"natural transformation" by "directed homotopy"

(See slides 17-27)

Comparing two paths γ and δ

defined over $[0, r]$ and $[0, r']$ with possibly $r \neq r'$

Write $\gamma \rtimes \delta$ when there exists two continuous increasing maps θ and θ' from $[0, 1]$ onto (surjective) $[0, r]$ and $[0, r']$ such that there exists a directed homotopy from $\gamma \circ \theta$ to $\delta \circ \theta'$

If $h : [0, r'] \times [0, \rho] \to X$ is a directed homotopy from α to β and θ continuous increasing from $[0, r]$ onto $[0, r']$ then $h \circ (\theta \times \mathrm{id}_{[0,\rho]})$ is a directed homotopy from $\alpha \circ \theta$ to $\beta \circ \theta$

Then denote by $\sim$ the congruence over $\overrightarrow{P}(X)$ generated by the relation $\rtimes$ the fundamental category of X is denoted by $\overrightarrow{\pi_1}(X)$ and defined as the quotient

$$\overrightarrow{P}(X)/\sim$$

The Fundamental Category functor over $\mathcal{Po}$

The preceding construction gives rise to a functor $\overrightarrow{\pi_1}$ from $\mathcal{Po}$ to $\mathcal{Cat}$ since for all $f \in \mathcal{Po}[X, Y]$ and all directed homotopies h between paths on X, the composite $f \circ h$ is a directed homotopy between paths on Y.

$$\overrightarrow{\pi_1} : \mathcal{Po} \longrightarrow \mathcal{Cat}$$

$$\begin{array}{ccc} X & & \overrightarrow{\pi_1}(X) \\ \downarrow f & \longmapsto & \downarrow \overrightarrow{\pi_1}(f) \\ Y & & \overrightarrow{\pi_1}(Y) \end{array}$$

The Fundamental Category functor over $\mathcal{P}o$

The preceding construction gives rise to a functor $\overrightarrow{\pi_1}$ from $\mathcal{P}o$ to $\mathcal{C}at$ since for all $f \in \mathcal{P}o[X, Y]$ and all directed homotopies h between paths on X, the composite $f \circ h$ is a directed homotopy between paths on Y.

$$\overrightarrow{\pi_1} : \mathcal{P}o \longrightarrow \mathcal{C}at$$

$$\begin{array}{ccc} X & & \overrightarrow{\pi_1}(X) \\ \downarrow f & \longmapsto & \overrightarrow{\pi_1}(f) \downarrow \\ Y & & \overrightarrow{\pi_1}(Y) \end{array}$$

with

$$\overrightarrow{\pi_1}(f) : \overrightarrow{\pi_1}(X) \longrightarrow \overrightarrow{\pi_1}(Y)$$

$$\begin{array}{ccc} p & & f(p) \\ \downarrow [\gamma] & \longmapsto & [f \circ \gamma] \downarrow \\ q & & f(q) \end{array}$$