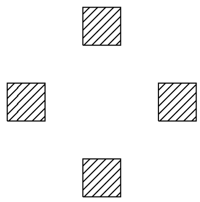
2.3.1: Concurrency

friday, the 3$^{\text{rd}}$ of march 2023
duration: 3h

Exercise 1: Denote the following (hashed) isothetic region by X:



1) Write a program whose forbidden region is X.

2a) What are the maximal blocks of the forbidden region?

2b) How many maximal blocks are there in the *complement* of forbidden region?

3) Give the prime factorization of the geometric model
(i.e. the complement of the hashed region).

4) Draw the category of components of the geometric model.

Exercise 2: Consider the program Π

```
var x, y, z = 0;
proc p = x:=1; y:=1; z:=1
proc q = x:=x*y*z; y:=x*y*z; z:=x*y*z
init p q
```

1) Draw the geometric model of Π.

2) Draw the category of components of the geometric model of Π.

3) How many directed paths are there from the initial point of the program to the 'final point' (i.e. any point after which there is no more instruction to execute) ?

4a) What are all the possible outcomes of the programme Π (i.e. all the possible values of the 3-tuple (x, y, z) after an execution of the program Π) ?

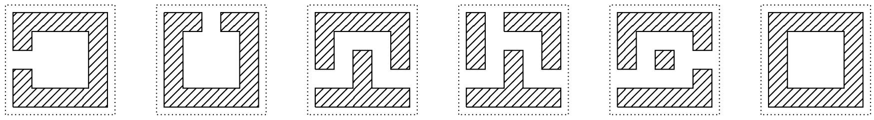
4b) Same question for the k-tuple $(\mathtt{x}_1,\mathtt{x}_2,\ldots,\mathtt{x}_k)$ (with $k > 3$) if the program under consideration is

```
var x_1, x_2, ..., x_k = 0;
proc p = x_1:=1; x_2:=1; ...; x_k:=1
proc q = x_1:=x_1*x_2*···*x_k;
         x_2:=x_1*x_2*···*x_k;
                  ⋮
         x_k:=x_1*x_2*···*x_k
init p q
```

Exercise 3: Let $F \subseteq \mathbb{R}^2$ be *bounded*, i.e. there exists a rectangle R such that $F \subseteq R$. The deadlock attractor of the *complement* of F (i.e. $\mathbb{R}^2 \setminus F$) is the set of points $p \in \mathbb{R}^2 \setminus F$ such that every directed path which starts at p (i.e. $\gamma(0) = p$) and leaves the rectangle R (i.e. $\exists t'\, \gamma(t') \notin R$) visits F (i.e. $\exists t\, \gamma(t) \in F$).

1) Given a directed path $\gamma : [0, T] \to \mathbb{R}^2$, prove that if $\gamma(0), \gamma(t) \in F$, $\gamma(t') \notin R$, and $t' \leqslant t''$ then $t < t'$ and $\gamma(t'') \notin R$.

2) Draw the deadlock attractor of $\mathbb{R}^2 \setminus F$ for every $F \subseteq \mathbb{R}^2$ represented below as an hashed region (the dotted rectangle represents R):



For any subset X of a locally ordered space, the *future cone* and the *past cone* of X are denoted by $\mathrm{cone}^{\mathrm{f}}(X)$ and $\mathrm{cone}^{\mathrm{p}}(X)$ respectively. Given a subset A_0 of an isothetic region, define $A_{k+1} = \mathrm{cone}^{\mathrm{f}}(\mathrm{cone}^{\mathrm{p}}(A_k))$ for $k \in \mathbb{N} \setminus \{0\}$.

3) Prove that for all $k \in \mathbb{N}$, $A_k \subseteq A_{k+1}$.

4) For every $X = \mathbb{R}^2 \setminus F$ with F one of the bounded sets described in 2):

4a) Compute $\bigcup_{k \in \mathbb{N}} A_k$ with A_0 the deadlock attractor of $\mathbb{R}^2 \setminus F$.

4b) What do you observe about the sequence $\{A_k \mid k \in \mathbb{N}\}$?

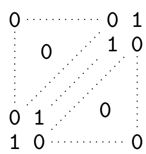
5) Make a conjecture about the sequence $\{A_k \mid k \in \mathbb{N}\}$ when A_0 is any subset of an isothetic region ? (Explain)

Exercise 4: The free commutative monoid of homogeneous languages over Σ is denoted by $\mathcal{H}(\Sigma)$.

1) Decompose the first two homogeneous languages in $\mathcal{H}(\{\mathtt{0,1,2}\})$ and *prove* that the third one is prime.

011	221001	011
001	220001	001
101	011122	101
000	020102	000
100	210021	100
110	021102	110
010	010122	010
111	211021	

2a) Prove that the following homogeneous language is prime in $\mathcal{H}(\{\mathtt{0,1}\})$ (the square matrix contains '1' on the diagonal, '0' anywhere else).



2b) Find a morphism of monoids $\phi : (\mathbb{N} \setminus \{0\}, \times, 1) \to \mathcal{H}(\{\mathtt{0,1}\})$ that induces an *isomorphism* on its image, and such that $\forall n \in \mathbb{N}$, $\mathrm{card}(\phi(n)) = n$. (Explain)

Exercise 5: Describe the fundamental category and the category of components of the following ordered subspaces of $\mathbb{R}^2$ (provide some explanations):

1) $\{(x, y) \in \mathbb{R}^2 \mid x + y \leqslant 0\}$.

2) $(\mathbb{N} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{N})$.

3) $(\mathbb{Q} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Q})$.