# Concurrency

and

Directed Algebraic Topology

- MPRI -

thursday the 10$^{th}$ of march 2015

duration: 3h

Exercise 1:

According to the virtual machine of the PAML language give all the possible outputs $(\mathtt{x}, \mathtt{y})$ of the following program.

```
#variable x = 3
#process p1 = x:=1
#process p2 = x:=2
#process q = (y:=x)+[x=1]+(y:=x)+[x<>1]+(y:=x)
#init p1 p2 q
```

Exercise 2:

a) Provides the following process with a conservative control flow graph
`#process p = P(a)+[`*e*` = 0]+P(b)` with *e* begin some expression.

b) Prove the following process has no *finite* conservative control flow graph:
`#process p = P(a).C(p)`

Exercise 3:

Give the potential function of the following program

```
#mutex a
#process p = P(a).V(a)
#init 2p
```

Exercise 4:

Extend the PAML language and its virtual machine so the instruction `F(`*name*`)`, *name* being an identifier, allows dynamic creation of processes. By convention, if *name* was not associated to a *body of instructions* by some line

`#process` *name* `=` *body of instructions*

then its is considered as the empty process.

Exercise 5:

Give two precubical sets $C$ and $M$ whose realization in the category of topological spaces are respectively the cylinder and the Möbius band (pictures with some explanations suffice).

Exercise 6:

Let $P$ be the following program

```
#mutex a b c
#process x = Pb.Pa.Va.Pc.Vc.Vb
#process y = Pa.Pc.Pb.Vb.Va.Vc
#init x y
```

a) Draw the geometric model (i.e. actually draw the forbidden region in grey) of $P$ (make a rather large picture)

b) Draw the deadlock attractor of $P$

c) Draw the category of components of $P$

Observe that the forbidden region of $P$ is actually contained in some square $[0,r]^2$ with $r > 0$. A dipath is then said to be maximal when its image is contained in $[0,r]^2$ and it cannot be extended (by nonconstant dipaths). On a new picture:

d) Draw one representative of each dihomotopy class of the maximal dipaths on the model of $P$

e) Give a finite collection $H_1, \ldots, H_n$ of subregions of the model such that:
- the image of a maximal dipath is contained in a unique $H_k$
- two maximal dipaths are dihomotopic iff their image are contained in the same $H_k$. The subregions may overlap so it is recommended to draw several pictures. Also note that a maximal dipath may not start at the origin.

Exercise 7:

Consider the following PAML program $P$

```
#synchronization 2 a
#process p = Wa
#init 2p
```

a) Draw $[\![P]\!]$ the geometric model of $P$

b) Compute the fundamental category $\overrightarrow{\pi_1}[\![P]\!]$

c) Compute the category of components $\overrightarrow{\pi_0}\big(\overrightarrow{\pi_1}[\![P]\!]\big)$

Exercise 8:

Consider the following PAML program $P$

```
#mutex a b
#semaphore c
#process p = Pa.Pc.Vc.Va
#process q = Pb.Pc.Vc.Vb
#init 2p 2q
```

a) Compute the forbidden region generated by each resource.

b) Give the maximal cubes of the forbidden region.

c) Compute a decomposition of the state space.

d) Denote by $X$ the geometric model of $P$ and write a PAML program whose geometric model is isomorphic with $X$.

Exercise 9: An $n$-grid, for $n \in \mathbb{N}$ is a subset $S \subseteq \mathbb{Z}^n$. A path of length $n \in \mathbb{N}$ on the grid $S$ is a finite sequence of points $p_0, \ldots, p_n$ of $S$ such that for all $k \in \{1, \ldots, n\}$, $p_{k+1} - p_k$ is a vector whose unique nonzero coordinate is 1.

a) We want to define a category $P(S)$ whose morphisms are the paths on the grid $S$. Describe the sources, the targets, the compositions, and the identities.

b) Prove that $P(S)$ is freely generated by some graph $G(S)$ (describe this graph).

Two paths of length 2 with the same source and the same target are declared to be equivalent so we have a congruence $\sim$ over $P(S)$. Define $F(S)$ as the quotient $P(S)/\sim$.

c) Prove that $F(S)$ is loop-free.

Let $S$ be $\{0, 1, 2\}^3 \setminus \{(1, 1, 1)\}$.

d) Make a picture of the graph $G(S)$.

e) What is the collection of arrows of $G(S)$ that preserve both future cones and past cones in the category $F(S)$ ?

f) Which square of the graph $G(S)$ are not pushouts of $F(S)$ ? Same question with pullbacks.

g) What is the greatest system of weak isomorphisms of $F(S)$ ? What is the category of components of $F(S)$ ?

Exercise 10:

Recall that $\mathcal{cSet} = \mathcal{Set}^{\Box^{+\mathrm{op}}}$ is the category of precubical sets. For any $n \in \mathbb{N}$ and any precubical set $K$, define $\mathrm{trunc}_n(K)$ as the precubical sets obtained by *discarding* all the cubes of dimension greater or equal than $n$. For $n \in \mathbb{N}$, define $\mathcal{cSet}_n$ as the full subcategory of $\mathcal{cSet}$ whose objects are the precubical sets of dimension $n$ i.e. $K_d = \emptyset$ for $d > n$. By convention $\mathcal{cSet}_{-1}$ is the category with only one morphism (and therefore only one object). Let $I_n$ be the inclusion functor of $\mathcal{cSet}_n$ into $\mathcal{cSet}$.

a) Explain why $\mathrm{trunc}_n$ actually extends to a functor $\mathcal{cSet} \to \mathcal{cSet}_{n-1}$.

b) Prove that $\mathrm{trunc}_n$ is right adjoint to $I_{n-1}$.

The standard $n$-cube is denoted by $\Box_n^+$. In particular $\Box_1^+$ is the graph with one arrow between two vertex, $\Box_2^+$ is the square, $\Box_3^+$ is the cube. In general $\Box_n^+$ is the $n$-fold tensor product of $\Box_1^+$.

b) What is the geometric realization (i.e. in $\mathcal{Top}$) of $\mathrm{trunc}_n(\Box_n^+)$ ?

Let $\rceil\_\lfloor : \mathcal{cSet} \to d\mathcal{Top}$ denotes the realization in $d\mathcal{Top}$ and $\overrightarrow{\pi_1} : d\mathcal{Top} \to \mathcal{Cat}$ be the fundamental category functor. Then for all precubical sets $K$, define $F(K)$ as the full subcategory of $\overrightarrow{\pi_1}(\rceil K \lfloor)$ whose set of objects is $K_0$.

c) What is $F(K)$ when $K$ is a graph (i.e. a 1-dimensional precubical set)

d) Compute $F(\square_3^+)$ and $F(\mathrm{trunc}_3(\square_3^+))$

e) Given a precubcial set $K$, what is $F(\mathrm{trunc}_3(K))$ (all the cubes of dimension greater of equal than 3 are dropped)

f) Explain why $F$ actually extends to a functor from $\mathcal{cSet}$ to $\mathcal{Cat}$

g) Prove that $F(K \otimes K') \cong F(K) \times F(K')$

h) Give a direct description of $F$ (i.e. without topological arguments)