Concurrency

and

Directed Algebraic Topology

- MPRI -

thursday the 14^{th} of march 2013

duration: 1h30

Exercise 1: Paths and Classes

Consider the following PV program where `a`, `b`, `c` and `d` are mutex i.e. they cannot be held by more that one process at the time.

```
#mtx a b c d
procs:
processus_1 = P(a).P(b).V(b).V(a).P(c).P(d).V(d).V(c)
processus_2 = P(d).P(c).V(c).V(d).P(b).P(a).V(a).V(b)
init:
processus_1 | processus_2
```

1 - Draw a picture of the geometric model X of the program (make a large and clear picture for the remaining questions will require more drawings on it. You can also make several drawings :)

2 - Draw a directed path γ corresponding to the following execution trace. Two instructions on the same row means they are executed simultaneously.

proc 1	`P(a)`	`P(b)`	`V(b)`	-	`V(a)`	`P(c)`	-	-	`P(d)`	`V(d)`	`V(c)`
proc 2	`P(d)`	-	`P(c)`	`V(c)`	-	`V(d)`	`P(b)`	`P(a)`	`V(a)`	-	`V(b)`

3 - Draw the subset $Y \subseteq X$ such that for all directed paths δ, $\mathsf{img}(\delta) \subseteq Y$ if and only if the execution trace corresponding to δ is the one described in 2.

4 - Draw the subset $E_\gamma \subseteq X$ such that for all directed paths δ, $\mathsf{img}(\delta) \subseteq E_\gamma$ if and only if $\delta \sim \gamma$.

A *deadlock* d is a point of X such that any directed path on X starting at d is constant. The *deadlock area* is the collection of points $x \in X$ such that any directed path δ starting at x admits an extension in the future that finishes at some deadlock.

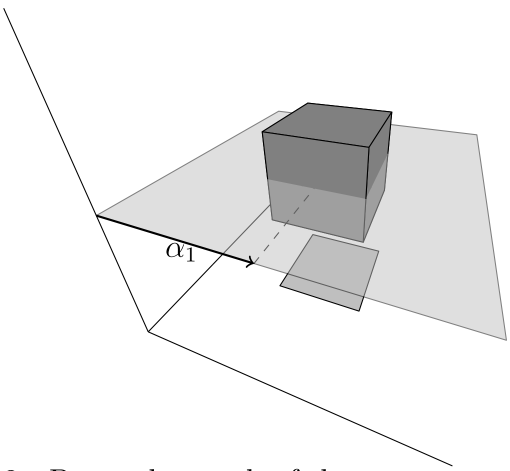
5 - Draw the deadlocks and the deadlock area of X (make a new picture).

Reminder: There exists a partition $C_0 \sqcup ... \sqcup C_n = X$ and a mapping that

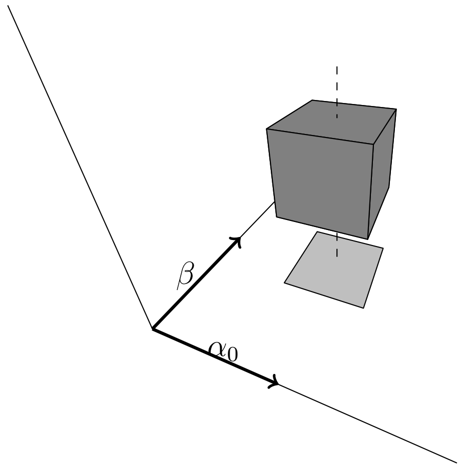
sends $\gamma \in \{\text{directed paths on } X\} \mapsto E_\gamma \subseteq \{C_0, ..., C_n\}$ such that for all directed paths γ and δ, $\gamma \sim \delta$ iff $E_\gamma = E_\delta$. The elements of the least partition satisfying the preceding property are called the components.
6 - Draw the components of X (make a new picture).

<u>Exercise 2</u>: Components of the Cube
Let Σ be some Yoneda system of the (fundamental category of the) "floating cube". You can illustrate your answers drawing on the picture.

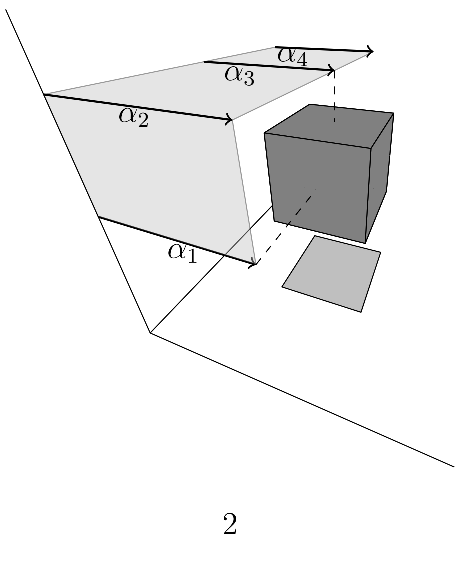
1 - Prove α_1 is not a Yoneda morphism and $\alpha_1 \notin \Sigma$.



2 - Prove the pushout of α_0 along β does not exists and $\alpha_0 \notin \Sigma$.



3 - Prove that each of the gray square is a pullback and α_2, α_3 and $\alpha_4 \notin \Sigma$.



A category $\mathcal{C}$ is said to be loop-free when for all objects x and y, if $\mathcal{C}[x, y] \neq \emptyset$ and $\mathcal{C}[y, x] \neq \emptyset$ then $x = y$ and $\mathcal{C}[x, x] = \{\mathsf{id}_x\}$. It is said to be connected when for all objects x and y there exists a finite sequence of objects $x_0, ..., x_n$ such that $x_0 = x$, $x_n = y$ and for all $i = 0, ..., n-1$ $\mathcal{C}[x_i, x_{i+1}] \neq \emptyset$ or $\mathcal{C}[x_{i+1}, x_i] \neq \emptyset$. In the sequel, the acronym *nflcc* stands for *nonempty finite loopfree connected category*. We denote by M the free commutative monoid of all nonempty finite loopfree connected categories together with the cartesian product as composition law and $\mathbb{1}$ (the category with a single object and a single morphism) as neutral element.

Exercise 3:

A collection of nonidentity morphisms of a category $\mathcal{C}$ is said to be generating when any nonidentity morphism of $\mathcal{C}$ can be written as a composite of elements of G. A category $\mathcal{C}$ is said to be free when it admits a generating collection of morphisms G such that $\gamma_n \circ ... \circ \gamma_0 = \delta_m \circ ... \circ \delta_0$ with $\gamma_k, \delta_k \in G$ implies $n = m$ and for all $k = 1, ..., n$ $\gamma_k = \delta_k$.

1a - Find a finite loop-free category that is not free.

1b - Find a loop-free category with no least generating family.

1c - Prove that any finite loop-free category admits a least (with respect to inclusion $\subseteq$) generating family.

2a - Find a prime element of M that is not free.

2b - Given two connected categories $\mathcal{A} \neq \mathbb{1}$ and $\mathcal{B} \neq \mathbb{1}$, prove the cartesian product $\mathcal{A} \times \mathcal{B}$ is not free.

2c - Prove that any element of M that is free (as a category) is a prime element of M.

We drop the connectedness hypothesis and denote by M' the commutative monoid of nonempty finite loopfree categories.

3a - Prove that any element of M' can be written as a product of irreducible elements of M'.

3b - Find an irreducible element of M' that is not prime and conclude about M' (adapt the example of $\mathbb{N}[X]$).