Concurrency

and

Directed Algebraic Topology

- MPRI -

thursday the 10^{th} of march 2011

duration: 1h30

Exercise 1: Maximal Subcubes

An n-*cube* is a subset of $\mathbb{R}_+^n$ of the form $I_1 \times \cdots \times I_n$ where each I_k is an interval. An n-*cubical area* is a finite union of n-cubes i.e.

$$X = C_1 \cup \cdots \cup C_p \quad \text{where } p \in \mathbb{N}\backslash\{0\} \text{ and each } C_k \text{ is an } n\text{-cube}$$

An n-cube C such that $C \subseteq X$ is called a *subcube* of X. Moreover if for all subcubes C' of X we have

$$C \subseteq C' \;\Rightarrow\; C = C'$$

then C is called a *maximal subcube* of X. Given a cubical area X, we put

$$M(X) := \mathsf{Card}\big\{\text{maximal subcubes of } X\big\}$$

thus defining a morphism of commutative monoids in particular for all cubical areas X and Y we have

$$M(X \times Y) = M(X) \times M(Y)$$

1) Given an n-cube C compute $M(C)$ (give a short explanation).

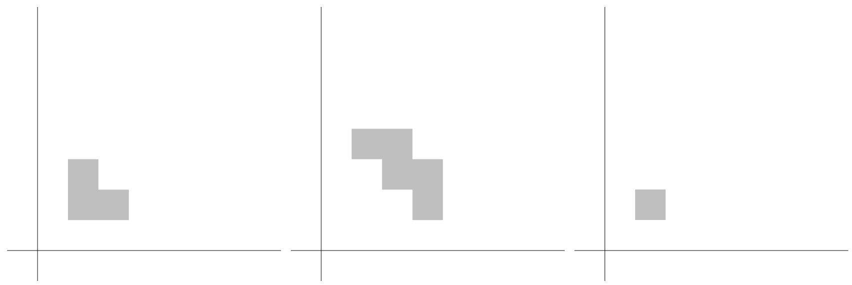
Reminder: any subcube of a cubical area X is included in some *maximal* subcube of the cubical area X.

2) Given an n-cubical area, prove if $M(X) = 1$ then X is an n-cube (give a short explanation).

3) Prove if $M(X)$ is a prime number then $X = C \times X'$ where C is a cube and X' is prime and not an interval (hint: any cubical area has a unique decomposition in prime cubical areas).

The result provided by the 3^{rd} question of the exercise 1 can be used in the rest of the exam.

Exercise 2: Staircases, PV programs and Categories of Components

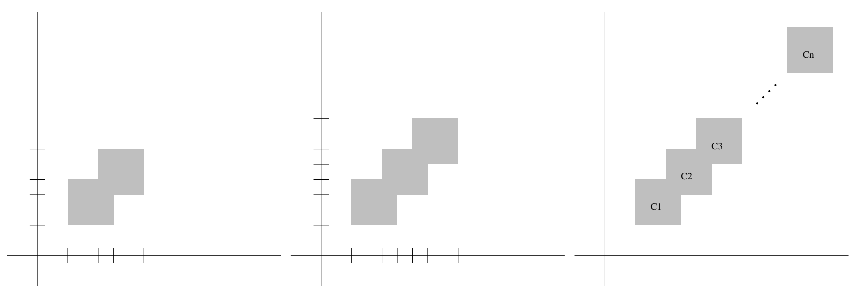


X Y S_1

1) For each cubical area X, Y and S_1 (the *complement* of the grey cubical area) count the number of maximal subcubes of and explain why X, Y and S_1 are prime.

A point a of a cubical area A is called a *deadlock* when any path on A starting at a is *constant*. The *deadlock attractor* of a cubical area A is the set of points $x \in A$ such that for all path γ starting at x there *exists* a path δ such that $\delta(0) = \gamma(1)$ and $\delta(1)$ is a deadlock.

2) What are the deadlocks and the deadlock attractors of X and Y ?



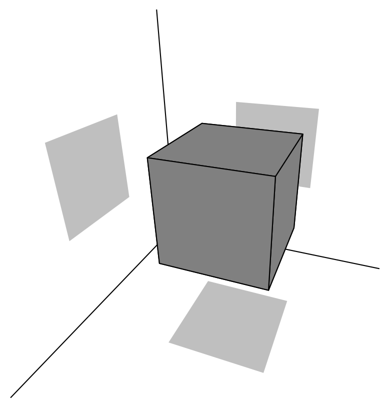
S_2 S_3 S_n

3) Give 2 PV programs whose geometric models are S_2 and S_3
4) What are the categories of components of S_2 and S_3 ?

In general S_n is the union of n squares along the diagonal, formally
$S_n := [1, 4] \times [1, 4] \cup [3, 6] \times [3, 6] \cup \cdots \cup [2n - 1, 2n + 2] \times [2n - 1, 2n + 2]$

5) By a simple geometric argument, prove the number of maximal subcubes of S_n is *even* (i.e. can be divided by 2)
6) Give the exact number of maximal subcubes of S_n
7) What is the category of components of S_n ?

Exercise 3: Dimension 3



1) Find a PV program whose geometric model is X, the (complement of the) gray cube.

2a) Given two points a and b of a pospace $\overrightarrow{X} = (X, \sqsubseteq)$ consider the subpospace $\overrightarrow{A}$ with $A := \{x \in X \mid a \sqsubseteq x \sqsubseteq b\}$ then compare $\overrightarrow{\pi_1}(\overrightarrow{A})[a,b]$ and $\overrightarrow{\pi_1}(\overrightarrow{X})[a,b]$ i.e. the collection of paths from a to b on $\overrightarrow{A}$ and the collection of paths from a to b on $\overrightarrow{X}$.
2b) Find $a, b \in X$ such that $\mathsf{Card}\big(\overrightarrow{\pi_1}(\overrightarrow{X})[a,b]\big) = 2$

In the sequel, the acronym *nflcc* stands for "nonempty finite loopfree connected category".

Exercise 4: nflcc!
Reminder: any nflcc can be written as a product of prime nflcc's in a unique way (up to permutation of the terms) and for all nflcc's $\mathcal{A}$ and $\mathcal{B}$ be a nflcc we have
$Ob(\mathcal{A} \times \mathcal{B}) = Ob(\mathcal{A}) \times Ob(\mathcal{B})$ and $Mo(\mathcal{A} \times \mathcal{B}) = Mo(\mathcal{A}) \times Mo(\mathcal{B})$

1) prove $\mathcal{C}$ prime $\Rightarrow$ $\mathsf{Card}(Ob(\mathcal{C})) \geqslant 2$ and $\mathsf{Card}(Mo(\mathcal{C})) \geqslant 3$

2a) prove $\mathsf{Card}(Ob(\mathcal{C}))$ prime (number) $\Rightarrow$ $\mathcal{C}$ prime (nflcc)
2b) prove $\mathsf{Card}(Mo(\mathcal{C}))$ prime (number) $\Rightarrow$ $\mathcal{C}$ prime (nflcc)

3) Find an infinite family $(\mathcal{C}_n)_{n \in \mathbb{N}}$ of prime nflcc's such that for all $n \in \mathbb{N}$, $\mathsf{Card}(Ob(\mathcal{C}_n)) < \mathsf{Card}(Ob(\mathcal{C}_{n+1}))$

4) Find a *prime* nflcc $\mathcal{C}$ such that $\mathsf{Card}(Ob(\mathcal{C}))$ and $\mathsf{Card}(Mo(\mathcal{C}))$ are not prime (numbers) and $\mathcal{C}$ is not free.