Concurrency

and

Directed Algebraic Topology

MPRI

Exam 2010 : Answers

Exercice 1

Question 1:

The forbidden area is $([1,2[\cup[3,4[) \times ([1,2[\cup[3,4[)$ (the bounds are not so important) so it is obviously factorized as a Cartesian product.

Question 2:

Warning : The model of the program is the complement of the forbidden area. In this example, the forbidden area can be factorized, but its complement cannot.

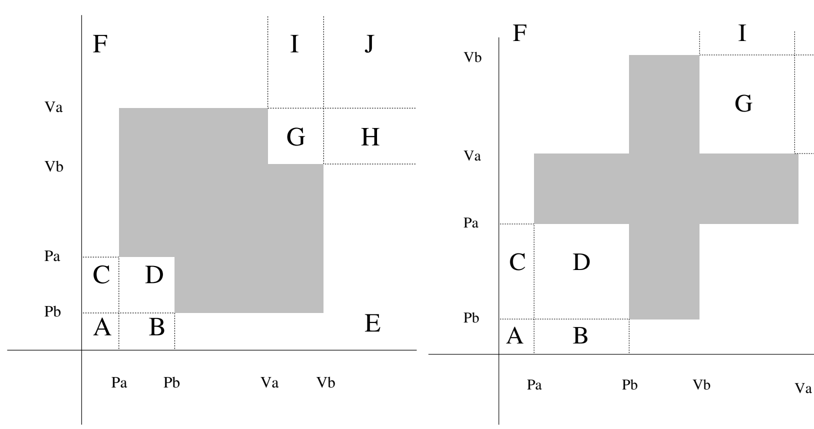
The two processes do not run independently since the model of the program is

$[0,1[\times[0,+\infty[\quad \cup \quad [2,3[\times[0,+\infty[\quad \cup \quad [4,+\infty[\times[0,+\infty[$

$[0,+\infty[\times[0,1[\quad \cup \quad [0,+\infty[\times[2,3[\quad \cup \quad [0,+\infty[\times[4,+\infty[$

which cannot be factorized. Of course one can easily guess the result since the two processes request the ressource `a` which cannot be shared.

Question 3:

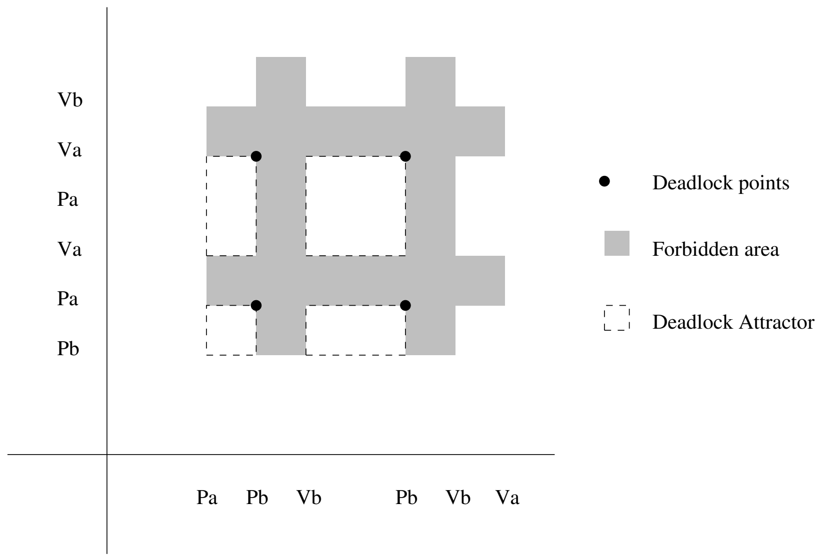


Question 4:

Their fundamental categories are isomorphic since they are determined from the components.

Exercice 2

Questions 1, 2 and 3:



The fundamental category of the model is not connected.

Question 4:

`P(a).V(a).P(a).V(a)|P(a).V(a).P(a).V(a)|P(a).V(a).P(a).V(a)`

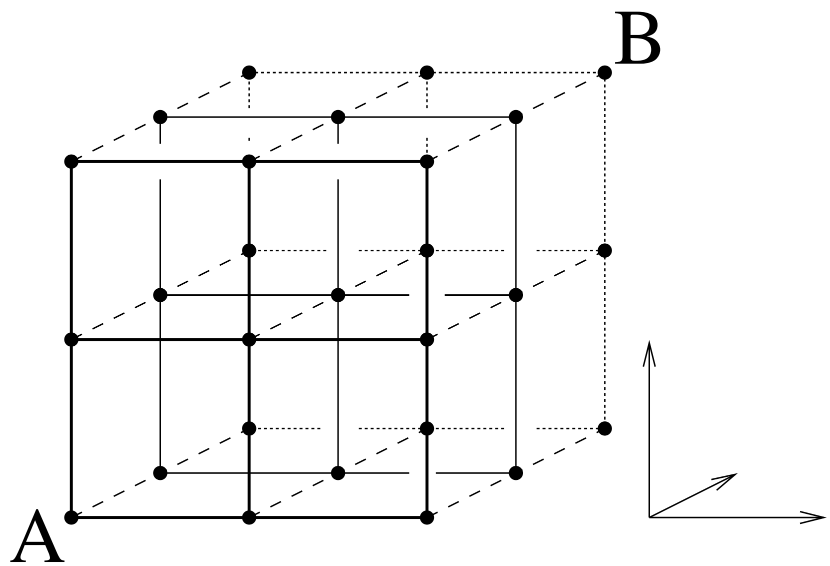
avec `a` smaphore d'arit 2.

Question 5:

The maximum occurs if we take, for example, $x := (0,0,0)$ and $y = (5,5,5)$. Combinatorially, it can be calculated as follow : remark that the pair of actions `P(a).V(a)` is in some sense "atomic" since once a process holds the mutex `a`, it is not more available until its owner releases it. So you have 3 processes sequentially trying to perform 2 atomic actions. Therefore the execution traces, up to dihomotopy, are in bijection with the number of anagrams of the word 112233, which corresponds to the case where the first process performs its 2 actions first, then the second one and finally the third one.

$$\frac{6!}{(2!)^3} = \frac{720}{8} = 90$$

A more "geometric" approach consists on counting the number of directed paths from A to B on the following directed graph.



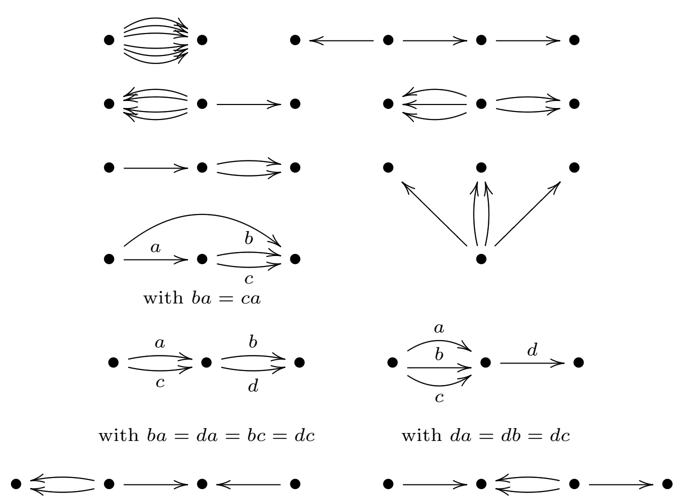
Exercice 3

Question 1:

Warning : a category of size 8 has exactly 8 morphisms (the identities have to be counted among the morphisms). One also has to count the compositions, for example the following category has 6 morphisms : 3 identities, 2 morphisms represented by the arrows and the last one being given by their composite.

$$\bullet \longrightarrow \bullet \longrightarrow \bullet$$

There are 11 connected loop-free categories of size 8. The two last ones are not free, in other words there are relations between the paths over their underlying graph.



Question 2:
Given two small categories $\mathcal{C}$ and $\mathcal{D}$, the set of objects and the set of morphisms of the Cartesian product $\mathcal{C} \times \mathcal{D}$ are respectively $\mathsf{Ob}(\mathcal{C}) \times \mathsf{Ob}(\mathcal{D})$ and $\mathsf{Mo}(\mathcal{C}) \times \mathsf{Mo}(\mathcal{D})$. Therefore the mapping $\mathcal{C} \mapsto \mathsf{card}(\mathsf{Ob}(\mathcal{C}))$ and $\mathcal{C} \mapsto \mathsf{card}(\mathsf{Mo}(\mathcal{C}))$ provide two examples of such mappings. Of course there is also the "trivial" mapping $\mathcal{C} \mapsto 1$, yet it is not so interesting.

Question 3a:
The category $\mathbf{1}$ is the only category with a single object and a single morphism, hence $\Phi(m, 1) = 1$. If $m = 0$ then $\Phi(m, 1) = 0$ since a category with at least 1 object has at least 1 identity. If $m \geqslant 2$ then $\Phi(m, 1) = 0$ because in a loop-free category, there is no morphism from an object x to itself but its identity id_{x}.
If $m \leqslant 2$ then $\Phi(m, 2) = 0$ since a category with 2 objects has at least 2 identities, hence a category with 2 objects and at most 2 morphisms cannot be connected.
If $m \geqslant 3$ then $\Phi(m, 2) = 1$. Let us call a and b the only objects of some connected loop-free category with at least 3 morphsims. Since the category is connected we can suppose there is a morphism from a to b. Moreover $\mathcal{C}$ is loop-free hence there is no morphism from b to a. as a consequence we have

$$\mathsf{card}(\mathcal{C}[a, b]) = m - 2 \qquad \begin{tikzcd} a \arrow[r, bend left, "\alpha_1"] \arrow[r, bend right, "\alpha_{m-2}"'] \arrow[r, phantom, "\vdots"] & b \end{tikzcd}$$

Question 3b:
Given a small category, there is at least as many objects as morphisms i.e. $m < x$ implies $\Phi(m, x) = 0$, the result easily follows.
Actually we have a more accurate result derived from a classical fact of combinatorial graph theory : suppose a bridge can relate 2 islands and you want to connect $n \geqslant 1$ islands. Then you need to build at least $n - 1$ bridges. The optimal configuration being provided by any tree, in particular the "linear order of length n".

`1 — 2 — ··· — (n-1) — n`

Then we have $m < 2x - 1$ implies $\Phi(m, x) = 0$.

Question 4a:
Reflexivity : $x \preccurlyeq x$ because $\mathsf{id}_{\mathsf{x}} \in \mathcal{C}[\mathsf{x}, \mathsf{x}]$.
Transitivity : $x \preccurlyeq y$ and $y \preccurlyeq z$ means that we have some $\delta \in \mathcal{C}[x, y] \neq \emptyset$ and $\gamma \in \mathcal{C}[y, z] \neq \emptyset$. It follows that $\gamma \circ \delta \in \mathcal{C}[x, z] \neq \emptyset$.
Antisymetry : $x \preccurlyeq y$ and $y \preccurlyeq x$ means that $\mathcal{C}[x, y] \neq \emptyset$ and $\mathcal{C}[y, x] \neq \emptyset$. Since $\mathcal{C}$ is loop-free we have $x = y$.

Question 4b:
The object part of the morphism R is provided by the mapping $\mathcal{C} \mapsto (\mathsf{Ob}(\mathcal{C}), \preccurlyeq)$.
Suppose f is a functor from $\mathcal{C}$ to $\mathcal{D}$ and let x and y be 2 objects of $\mathcal{C}$ such that

$x \preccurlyeq y$. In other words we have some morphism δ of $\mathcal{C}$ from x to y. Since f is a functor, $\mathsf{Mo}(f)(\delta)$ is a morphism of $\mathcal{D}$ from $\mathsf{Ob}(f)(x)$ to $\mathsf{Ob}(f)(y)$ and we have $\mathsf{Ob}(f)(x) \preccurlyeq \mathsf{Ob}(f)(y)$. Hence the morphism part of the functor R is provided by the mapping $(f : \mathcal{C} \to \mathcal{D}) \mapsto (\mathsf{Ob}(f) : \mathsf{Ob}(\mathcal{C}) \to \mathsf{Ob}(\mathcal{D}))$.

Question 4c:
Let $(X \preccurlyeq)$ be a poset. The set of objects of the associated category is X, and the set of morphisms is the "graph" of the binary relation $\preccurlyeq$ i.e. the collection of ordered pairs

$$\big\{(x, y) \in X \times X \;\big|\; x \preccurlyeq y\big\}$$

The source and the target of (x, y) are respectively x and y. The composition is given by

$$(y, z) \circ (x, y) := (x, z)$$

which is sound since $\preccurlyeq$ is transitive. By reflexivity, the identities are the ordered pairs (x, x) for x ranging in X. This category is loop-free since $\preccurlyeq$ is antisymetric. We have defined the object part of the functor.
We define the morphism part of the functor R. Given a morphism of poset f, $R(f)$ has to be a functor between the corresponding loop-free categories. The object part of $R(f)$ is just f while its morphism part is given by the mapping $(x, y) \mapsto \big(f(x), f(y)\big)$

Question 4d:
Given a loop-free category $\mathcal{C}$, the category $IR(\mathcal{C})$ is as follows : its objects are the objects of $\mathcal{C}$ and its morphisms are the ordered pairs (x, y) such that $\mathcal{C}[x, y] \neq \emptyset$. Then the following mappings

$$\mathsf{id}_{\mathsf{Ob}(\mathcal{C})} : \mathsf{x} \in \mathsf{Ob}(\mathcal{C}) \mapsto \mathsf{x} \in \mathsf{Ob}(\mathcal{C}) \text{ and } \delta \in \mathsf{Mo}(\mathcal{C}) \mapsto \big(\mathsf{s}(\delta), \mathsf{t}(\delta)\big) \in \mathsf{Mo}(IR(\mathcal{C}))$$

induce a functor from $\mathcal{C}$ to $IR(\mathcal{C})$ i.e. a morphism of Cat. The collection of functors thus defined is actually a natural transformation from $\mathsf{id}_{\mathsf{Cat}}$ to $I \circ R$.

Actually the category of posets is isomorphic with the full subcategory of Cat whose objects are loop-free categories whose homsets have at most 1 element.

Question 5a:
Warning : "generating set" usually means "which generates all non-identity morphisms", but the question was ambiguous about that so two answers are possible.
Given a poset $(X, \preccurlyeq)$ say that the ordered pair $(x, y) \in X \times X$ covers x if for all $z \in X$, $x \preccurlyeq z \preccurlyeq y$ implies $x = z$ or $z = y$. Let $\mathcal{C}$ be a small category, the smallest generating can be described as the colelction of morphisms δ such that $(\mathsf{s}(\delta), \mathsf{t}(\delta))$ covers $\mathsf{s}(\delta)$ in the poset $R(\mathcal{C})$ (see Question 4).
In a less abstract way, the elements of the smallest generating set are the morphisms α such that $\alpha = \gamma \circ \delta$ implies either γ or δ is an identity.
A simple remark proves this collection is generating, given a composable se-

quence $(\delta_n, \ldots, \delta_1)$ of morphisms of a loop-free category $\mathcal{C}$, if $p < q$ and $\delta_p = \delta_q$ then for all $p \leqslant n \leqslant q$ we have $\delta_p = \delta_n = \mathsf{id}$. So if we suppose $\mathcal{C}$ has finitely many morphisms, the length of the composable sequences which do not contain any identity is bounded by the number of morphisms of $\mathcal{C}$.

Question 5b:
If the identities are admited in the generating sets, the answer is $A \times B$, otherwise the generating set of the Cartesian product is the collection of morphisms $\mathsf{id}_\mathsf{a} \times \beta$ and $\alpha \times \mathsf{id}_\mathsf{b}$ for $\alpha \in A$, $\beta \in B$, a object of $\mathcal{A}$ and b object of $\mathcal{B}$.
Actually I wanted to mean "without identities". The notion of least generating set of a finite loop-free category is the first step to the proof of the theorem of decomposition of non-empty finite connected loop-free categories.

Question 6:
Consider the polynomial

$$1 + X + X^2 + X^3 + X^4 + X^5 = (1 + X^3)(1 + X + X^2) = (1 + X)(1 + X^2 + X^4)$$

and "replace" the variable X by the loop-free category $\bullet \longrightarrow \bullet$ (actually any prime loop-free category would work). In order to make this idea sound, one interprets the product of polynomials as the Cartesian product of categories and the sum of polynomials as the disjoint union of categories. We provide some details. Given two sets A and B, one might have $A \cap B \neq \emptyset$, in order to obtain two disjoint copies of A and B we consider $A \times \{0\} \cong A$ and $B \times \{1\} \cong B$ and we define the disjoint union $A \sqcup B$ as the usual union $A \times \{0\} \cup B \times \{1\}$. Now given two categories $\mathcal{C}$ and $\mathcal{D}$, we define the disjoint union of $\mathcal{C}$ and $\mathcal{D}$ by considering the disjoint union of their objects $\mathsf{Ob}(\mathcal{C}) \sqcup \mathsf{Ob}(\mathcal{D})$ and the disjoint union of their morphisms $\mathsf{Mo}(\mathcal{C}) \sqcup \mathsf{Mo}(\mathcal{D})$. Remark that if $\mathcal{C}$ and $\mathcal{D}$ are non empty, there disjoint union is disconnected. Moreover, any decomposition in $\mathbb{N}[X]$ induces a decomposition in $\mathbb{M}_f$ <u>because</u> the Cartesian product <u>distributes</u> over the disjoint union. Then we have two decompositions of

$$1 \sqcup \mathcal{C} \sqcup \mathcal{C}^2 \sqcup \mathcal{C}^3 \sqcup \mathcal{C}^4 \sqcup \mathcal{C}^5$$

and $1 + \mathcal{C}$ is irreducible since it has 3 (prime number) objects. It remains to check that $1 + \mathcal{C}$ is not prime, for example by proving that it does not divide $1 + \mathcal{C}^3$ nor $1 + \mathcal{C} + \mathcal{C}^2$. To do so remark that the number of connected components of an element of M_f induces a morphism of monoids from $\mathbb{M}_f^*$ to $(\mathbb{N}, \times, 1)$. Then $1 + \mathcal{C}^3$ and $1 + \mathcal{C} + \mathcal{C}^2$ have respectively 2 and 3 (two prime numbers) connected components therefore they cannot be factorized.

The construction described above is inspired from a very well-known other one. Indeed the set of polynomials $\mathbb{Z}[X]$ has a ring structure and given any $\alpha \in \mathbb{Z}$, the association $X \mapsto \alpha$ defines a unique ring morphism from $\mathbb{Z}[X]$ to $\mathbb{Z}$ known as the "evaluation morphism at α".

In we endow $\mathbb{M}_f$ with Cartesian product, disjoint union, $\mathbf{1}$ and the empty category as product, sum, unit and zero, we almost have a ring (the additive structure is just a monoid, not a group). The same way $\mathbb{N}[X]$ almost has a ring structure. Then for all finite catgories $\mathcal{C}$ one has an "evaluation at $\mathcal{C}$" mapping from $\mathbb{N}[X]$ to $\mathbb{M}_f$ which preserves sum, product, unit and zero and which is entirely determined by $X \mapsto \mathcal{C}$.