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CATEGORIES OF COMPONENTS AND LOOP-FREE CATEGORIES

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ABSTRACT. Given a groupoid $\mathcal{G}$ one has, in addition to the equivalence of categories E from $\mathcal{G}$ to its skeleton, a fibration F (Definition 1.11) from $\mathcal{G}$ to its set of connected components (seen as a discrete category). From the observation that E and F differ unless $\mathcal{G}[x,x] = \{\mathrm{id}_x\}$ for every object x of $\mathcal{G}$, we prove there is a fibered equivalence (Definition 1.12) from $\mathcal{C}[\Sigma^{-1}]$ (Proposition 1.1) to $\mathcal{C}/_\Sigma$ (Proposition 1.8) when Σ is a *Yoneda*-system (Definition 2.5) of a loop-free category $\mathcal{C}$ (Definition 3.2). In fact, all the equivalences from $\mathcal{C}[\Sigma^{-1}]$ to $\mathcal{C}/_\Sigma$ are fibered (Corollary 4.5). Furthermore, since the quotient $\mathcal{C}/_\Sigma$ shrinks as Σ grows, we define the component category of a loop-free category as $\mathcal{C}/_{\overline{\Sigma}}$ where $\overline{\Sigma}$ is the greatest *Yoneda*-system of $\mathcal{C}$ (Proposition 3.7).

1. Introduction and purpose

Although loop-free categories (Definition 3.2) have been introduced by *André Haefliger* (as *small categories without loops* or *scwols*) for a very different purpose [6, 18, 19], our interest in them comes from the algebraic study of *partially ordered spaces* or *pospaces* [26, 33].

The source of our motivation for studying pospaces lies on the notion of *progress graph* which has been proposed by *Edsger W. Dijkstra* as a natural model for concurrency[1] [8]: by their very definition, progress graphs are special instances of pospaces, however, the framework they offer is too restricted from a theoretical point of view. Thus, in practice, any PV program[2] gives rise to a pospace (more precisely a progress graph) we would like to abstract in the same fashion as topological spaces are abstracted by groupoids.

To this aim, we notice that loop-free categories arise in the context of pospaces as groupoids do in the context of spaces, precisely, we define the fundamental (loop-free) categories of pospaces (Section 5) as one defines the fundamental groupoids of spaces [24]. Formally speaking, the properties of the fundamental category functor $\overrightarrow{\pi_1}$ (from the category of pospaces PoTop to the category of loop-free categories LfCat) are similar to those of the usual fundamental groupoid functor (from the category of topological spaces Top to the category of groupoids Grd). In particular, both of them preserve push-out squares under (almost) the same hypothesis (*van Kampen* Theorems) and the proof of this result, given by *Eric Goubault* [14] (for pospaces) and *Marco Grandis* [16] (for d-

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[1] A branch of theoretical computer science.
[2] The PV language is one of the simplest one among those which allow parallel programming.

spaces), follows the classical one [24]. Yet, both PoTop and Top are complete, cocomplete and admit a cartesian closed reflective subcategory that have all the "reasonable" objects of the category in which they are included [23], still both LfCat and Grd are epireflective subcategories of the category of small categories Cat. Up to now, there is a strong and straightforward analogy between the algebraic approach of partially ordered spaces and the one of topological spaces. Applying the same pattern, one can even adapt the notion of fundamental groupoid to several other contexts that extend the one of pospaces [13, 16, 31]. Things would not be more complicated if we were not entrapped by the need for finite representation, which is an ubiquitous problem when one intends to make concrete calculation using computers. A natural idea to solve it consists on defining a notion which plays, in the context of pospaces or other, the role that arcwise connected components play in the framework of spaces [11, 17]. Indeed, the information contained in the fundamental groupoid of an arcwise connected topological space is already contained in its *fundamental group*; in fact, the fundamental groupoid of a space is entirely determined by the fundamental groups of its arcwise connected components. This property comes from two more general and algebraic facts (compare to Theorem 4.1):

1. the skeleton of any groupoid $\mathcal{G}$ is obtained as the coproduct *in* Grd of the family of groups $\mathcal{G}[x,x]$ for x ranging over the set C which contains exactly one object taken from each connected component of $\mathcal{G}$ and

2. there is a fibration (Definition 1.11) from $\mathcal{G}$ to its set of connected components (seen as a discrete category) given by the quotient functor generated by the collection of morphisms of $\mathcal{G}$ (Proposition 1.8).

The two previous properties suggest that any groupoid $\mathcal{G}$ has a some sort of "finite presentation" as soon as its collection of connected components is finite and the groups $\mathcal{G}[x,x]$ are finitely generated. For instance, the fundamental groupoid of the euclidean circle has a single component and its fundamental group is $\mathbb{Z}$ though its collection of objects is uncountable.

On the contrary, for any object x of any loop-free category $\mathcal{C}$, the monoid $\mathcal{C}[x,x]$ is reduced to $\{\mathrm{id}_x\}$ and in general, the quotient turning all the morphisms of a loop-free category into identities is not a fibration, therefore, recovering any piece of information about $\mathcal{C}$ from its set of connected components and the structure of the monoids $\mathcal{C}[x,x]$ is hopeless. However, considering properties 1 and 2 as a guideline, we aim at giving a suitable notion of component (based on the study of loop-free categories) in the context of pospaces: this task will be completed when we have proved Theorem 4.1. Furthermore, we expect that our construction produces finite results when it is applied to progress graphs, this fact is illustrated in Section 5 though it has not been formally proved yet. Still, some results by *Lisbeth Fajstrup* take a step in this direction [9]. Besides, the construction we are about to describe actually holds for any small category (Theorem 2.6), nevertheless it is way more fruitfully applied to loop-free ones (Theorem 3.8 and 4.1).

The method presented here is an improvement of an earlier one given in [11], indeed, it has the "good" theoretical properties the original one lacks of (Theorem 2.6, 3.8, 4.1 as well as a *Van Kampen* like Theorem [22, 23]). *Marco Grandis* has an alternative approach [17], however, even on basic examples (free of any "pathology"), the results he obtains differ from ours.

Let us now specify some pieces of notation: $\mathcal{C}$ is always put for a small category, for any morphism f of $\mathcal{C}$, we respectively denote the source and the target of f by $s(f)$ and $t(f)$, we put $e(f)$ for the pair $\{s(f), t(f)\}$, $Iso(\mathcal{C})$ for the set of isomorphisms of $\mathcal{C}$ and given x and y, two objects of $\mathcal{C}$, the homset of morphisms $\mathcal{C}$ whose source and target are respectively x and y is denoted by $\mathcal{C}[x,y]$. In general, for any collection Σ of morphisms of $\mathcal{C}$, we put $\Sigma[x,y] := \mathcal{C}[x,y] \cap \Sigma$. The group of autofunctors of $\mathcal{C}$ is denoted by $\mathsf{Aut}(\mathcal{C})$.

The category of fractions of $\mathcal{C}$ over Σ is described in [3, 12], in regard of its importance in the rest of this paper we give a reminder about it, forewarning the reader that the problem of smallness of homsets will be ignored because this construction will only be used for small categories.

1.1. Proposition. *Given a category $\mathcal{C}$ and a collection Σ of morphisms of $\mathcal{C}$, there exists a category, unique up to isomorphism, denoted by $\mathcal{C}[\Sigma^{-1}]$ and called* **category of fractions** *(of $\mathcal{C}$ over Σ), as well as a unique functor I_Σ from $\mathcal{C}$ to $\mathcal{C}[\Sigma^{-1}]$ such that:*

1. *the functor I_Σ sends any morphism of Σ to an isomorphism of $\mathcal{C}[\Sigma^{-1}]$ and*
2. *for any functor F from $\mathcal{C}$ to $\mathcal{D}$ sending any morphism of Σ to an isomorphism of $\mathcal{D}$, there is a unique functor G from $\mathcal{C}[\Sigma^{-1}]$ to $\mathcal{D}$ such that $F = G \circ I_\Sigma$.*

In addition, we can choose $\mathcal{C}[\Sigma^{-1}]$ and I_Σ so that for all objects x of $\mathcal{C}$ we have $I_\Sigma(x) = x$.

The category of groupoids is a reflective subcategory of the category of small categories whose left adjoint is given by $\mathcal{C}[\Sigma^{-1}]$ where Σ is the collection of all morphisms of $\mathcal{C}$.

1.2. Definition. [Calculus of fractions [3, 12]] *Let $\mathcal{C}$ be a category and Σ be a collection of morphisms of $\mathcal{C}$, we say that Σ admits a* **right calculus of fractions** *over $\mathcal{C}$ when the following properties are satisfied:*

1. *all the identities of $\mathcal{C}$ are in the collection Σ,*
2. *the collection Σ is stable under composition,*
3. *for all morphisms γ and σ respectively taken from $\mathcal{C}[x,y]$ and $\Sigma[y',y]$, there are two morphisms γ' and σ' respectively in $\mathcal{C}[x',y']$ and $\Sigma[x',x]$ such that the diagram 1.1 commutes,*
4. *for all morphisms γ and δ of $\mathcal{C}[x,y]$ and all morphisms σ of $\Sigma[y,y']$ such that $\sigma \circ \gamma = \sigma \circ \delta$, there exists a morphism σ' in $\Sigma[x',x]$ such that $\gamma \circ \sigma' = \delta \circ \sigma'$: see diagram 1.2.*

$$\begin{tikzcd} & y & \\ x \arrow[ur, "\gamma"] & & y' \arrow[ul, "\sigma\in\Sigma"'] \\ & x' \arrow[ul, dotted, "\sigma'\in\Sigma"] \arrow[ur, dotted, "\gamma'"'] & \end{tikzcd}$$

Diagram 1.1

$$\begin{tikzcd} x' \arrow[r, dotted, "\sigma'"] & x \arrow[r, bend left, "\gamma"] \arrow[r, bend right, "\delta"'] & y \arrow[r, "\sigma"] & y' \end{tikzcd}$$

Diagram 1.2

The statement 3 is called the **right extension property** *of Σ in $\mathcal{C}$. Reversing all the arrows in the previous definition, we obtain the notion of* **left calculus of fractions**. *When Σ both satisfies axioms of left calculus of fractions and right calculus of fractions, we say that Σ admits a* **left and right calculus of fractions** *over $\mathcal{C}$.*

1.3. Proposition. *Given a category $\mathcal{C}$ and a collection Σ of morphisms admitting a right calculus of fractions, the category of fractions $\mathcal{C}[\Sigma^{-1}]$ can be described in the following way:*

objects *the collection of objects of $\mathcal{C}[\Sigma^{-1}]$ is the collection of objects of $\mathcal{C}$,*

morphisms *the homset $\big(\mathcal{C}[\Sigma^{-1}]\big)[x,y]$ is given by*

$$\Big\{(\gamma,\sigma)\big|\ \sigma\in\Sigma,\ t(\sigma)=x,\ t(\gamma)=y,\ s(\sigma)=s(\gamma)\Big\}\Big/\sim_{x,y}\quad ,$$

where the ordered pair $\big((\gamma,\sigma),(\gamma',\sigma')\big)$ belongs to the equivalence relation $\sim_{x,y}$ when, by definition, there are two morphisms τ and τ' in Σ such that the diagram 1.3 commutes.

The composite of the $\sim_{x,y}$-equivalence class of (γ,σ) followed by the $\sim_{y,z}$-equivalence class of (δ,τ) is the $\sim_{x,z}$-equivalence class of $(\delta\circ\gamma',\sigma\circ\tau')$ where the morphisms γ' and τ' come from the right extension property of Σ in $\mathcal{C}$ and make diagram 1.4 commute.

$$\begin{tikzcd}
 & y & & \\
u \arrow[ur, "\gamma"] \arrow[dr, "\sigma"'] & v \arrow[l, dotted, "\tau"'] \arrow[r, dotted, "\tau'"] & u' \arrow[ul, "\gamma'"'] \arrow[dl, "\sigma'"] \\
 & x &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & v \arrow[r, "\delta"] \arrow[dr, "\tau\in\Sigma"] & z \\
w \arrow[ur, dotted, "\gamma'"] \arrow[dr, dotted, "\tau'\in\Sigma"'] & & y \\
 & u \arrow[ur, "\gamma"] \arrow[r, "\sigma\in\Sigma"'] & x
\end{tikzcd}$$

Diagram 1.3 *Diagram 1.4*

1.4. Definition. [Σ-zigzag] *Two objects x and y of $\mathcal{C}$ are said to be Σ-***zigzag connected** *when $x=y$ or there is a finite sequence $(z_0,\dots,z_{n+1})$ $(n\in\mathbb{N})$ of objects of $\mathcal{C}$ such that $\{z_0,z_{n+1}\}=\{x,y\}$ and for all k in $\{0,\dots,n\}$, one of the sets $\Sigma[z_k,z_{k+1}]$ and $\Sigma[z_{k+1},z_k]$ is not empty; thus we define an equivalence relation over the objects of $\mathcal{C}$ whose equivalence classes are called the Σ-***components** *of $\mathcal{C}$. A Σ-***zigzag** *between x and y is a sequence $(\sigma_n,\dots,\sigma_0)$ whose entries belong to Σ and such that for all k in $\{0,\dots,n\}$, $e(\sigma_k)=\{z_k,z_{k+1}\}$. If Σ contains all the identities of $\mathcal{C}$, then the condition $x=y$ can be omitted.*

A finite sequence $(\sigma_n,\dots,\sigma_0)$ $(n\in\mathbb{N})$ of morphisms of $\mathcal{C}$ is said to be **composable** when for all k in $\{0,\dots,n-1\}$, the target of σ_k is the source of σ_{k+1}, the sequence is said to be Σ-**composable** when for all k in $\{0,\dots,n-1\}$, the target of σ_k is in the same Σ-component as the source of σ_{k+1}, in a yet more general way, for any equivalence relation $\sim$ over the set of objects of $\mathcal{C}$, the sequence $(\sigma_n,\dots,\sigma_0)$ is said to be $\sim$-**composable** when for all k in $\{0,\dots,n-1\}$, the target of σ_k is in the same $\sim$-equivalence class as the source of σ_{k+1}.

Let us define the generalized congruences [1] which are at the core of the construction that we will describe.

1.5. Definition. [Generalized congruences] *Given a small category $\mathcal{C}$, a* **generalized congruence** *over $\mathcal{C}$ is an ordered pair of equivalence relations, denoted by $(\sim_o, \sim_m)$, respectively over the set of objects of $\mathcal{C}$ and over the set of* non empty $\sim_o$*-composable sequences of $\mathcal{C}$. Furthermore, these relations have to satisfy the following properties, which are given both in a formal manner (on the left hand side) and a graphical one (on the right hand side):*

1. if $x \sim_o y$, then $(\mathrm{id}_x) \sim_m (\mathrm{id}_y)$,
$$\begin{tikzcd} x \arrow[d, squiggly, no head, "o"] \\ y \end{tikzcd} \Longrightarrow \begin{tikzcd} x \arrow[r, bend left, "\mathrm{id}_x"] \arrow[dr, phantom, "\wr\, m"] & x \\ y \arrow[r, bend right, "\mathrm{id}_y"'] & y \end{tikzcd}$$

2. if $(\delta_n, \ldots, \delta_0) \sim_m (\gamma_p, \ldots, \gamma_0)$, then $t(\delta_n) \sim_o t(\gamma_p)$ and $s(\delta_0) \sim_o s(\gamma_0)$,
$$\begin{tikzcd} x \arrow[r, bend left, "(\delta_n,\ldots,\delta_0)"] \arrow[dr, phantom, "\wr\, m"] & y \\ x' \arrow[r, bend right, "(\gamma_p,\ldots,\gamma_0)"'] & y' \end{tikzcd} \Longrightarrow \begin{tikzcd} x \arrow[r, bend left, "(\delta_n,\ldots,\delta_0)"] \arrow[d, squiggly, no head, "o"'] & y \arrow[d, squiggly, no head, "o"] \\ x' \arrow[r, bend right, "(\gamma_p,\ldots,\gamma_0)"'] & y' \end{tikzcd}$$

3. if $s(\gamma) = t(\delta)$, then $(\gamma, \delta) \sim_m (\gamma \circ \delta)$,
$$\begin{tikzcd} & y \arrow[dr, "\gamma"] \arrow[d, phantom, "="] & \\ x \arrow[ur, "\delta"] \arrow[rr, "\gamma\circ\delta"'] & {} & z \end{tikzcd} \Longrightarrow \begin{tikzcd} x \arrow[r, bend left=60, "(\gamma,\delta)"] \arrow[r, "(\gamma\circ\delta)"'] \arrow[r, phantom, bend left=30, "\wr\, m"] & z \end{tikzcd}$$

4. if $(\delta_n, \ldots, \delta_0) \sim_m (\delta'_{n'}, \ldots, \delta'_0)$, $(\gamma_p, \ldots, \gamma_0) \sim_m (\gamma'_{p'}, \ldots, \gamma'_0)$ and $t(\delta_n) \sim_o s(\gamma_0)$, then $(\gamma_p, \ldots, \gamma_0, \delta_n, \ldots, \delta_0) \sim_m (\gamma'_{p'}, \ldots, \gamma'_0, \delta'_{n'}, \ldots, \delta'_0)$.

$$\begin{tikzcd} x \arrow[r, bend left, "(\delta_n,\ldots,\delta_0)"] \arrow[dr, phantom, "m\,\wr"] & y \arrow[r, squiggly, no head, "o"] & u \arrow[r, bend left, "(\gamma_p,\ldots,\gamma_0)"] \arrow[dr, phantom, "\wr\, m"] & v \\ x' \arrow[r, bend right, "(\delta'_{n'},\ldots,\delta'_0)"'] & y' & u' \arrow[r, bend right, "(\gamma'_{p'},\ldots,\gamma'_0)"'] & v' \end{tikzcd} \Longrightarrow \begin{tikzcd} x \arrow[r, bend left, "(\gamma_p,\ldots,\gamma_0,\delta_n,\ldots,\delta_0)"] \arrow[dr, phantom, "\wr\, m"] & v \\ x' \arrow[r, bend right, "(\gamma'_{p'},\ldots,\gamma'_0,\delta'_{n'},\ldots,\delta'_0)"'] & v' \end{tikzcd}$$

In Axiom 4, the sequence $(\gamma'_{p'}, \ldots, \gamma'_0, \delta'_{n'}, \ldots, \delta'_0)$ is $\sim_o$-composable (in virtue of Axiom 2 and transitivity of $\sim_o$). Any usual congruence over a category [29] can be seen as a special instance of generalized congruence in which the relation $\sim_o$ is the equality over the collection of objects. Moreover, for any ordered pair of binary relations (R_o, R_m) respectively over the set of objects of $\mathcal{C}$ and over the set of finite non empty sequences of morphisms of $\mathcal{C}$, there is a unique generalized congruence $(\sim_o, \sim_m)$ over $\mathcal{C}$ such that $R_o \subseteq \sim_o$, $R_m \subseteq \sim_m$ and which is minimum in the sense where for any other generalized congruence $(\sim'_o, \sim'_m)$ such that $R_o \subseteq \sim'_o$ and $R_m \subseteq \sim'_m$, we have $\sim_o \subseteq \sim'_o$ and $\sim_m \subseteq \sim'_m$. In this case, the generalized congruence $(\sim_o, \sim_m)$ is said to be **generated** by (R_o, R_m).

1.6. PROPOSITION. [Universal property of a generalized congruence] *Let* $(\sim_o, \sim_m)$ *be a generalized congruence over a category* $\mathcal{C}$ *and* $\mathbb{F}$ *be the collection of functors* F *whose range is* $\mathcal{C}$ *and satisfy the two following properties:*

1. *for all objects* x *and* y *of* $\mathcal{C}$*, if* $x \sim_o y$*, then* $Fx = Fy$*,*
2. *for all* $\sim_o$*-composable sequences* $(\gamma_n, \ldots, \gamma_0)$ *and* $(\delta_p, \ldots, \delta_0)$*,* *if* $(\gamma_n, \ldots, \gamma_0) \sim_m (\delta_p, \ldots, \delta_0)$*, then* $F(\gamma_n) \circ \cdots \circ F(\gamma_0) = F(\delta_p) \circ \cdots \circ F(\delta_0)$*.*

There exists a category, unique *up to isomorphism and denoted by* $\mathcal{C}/\sim$*, as well as a* unique *functor* $Q\sim$ *from* $\mathcal{C}$ *to* $\mathcal{C}/\sim$ *in* $\mathbb{F}$ *such that for any functor* F *in* $\mathbb{F}$*, there is a* unique *functor* G *from* $\mathcal{C}/\sim$ *to* $\mathcal{A}$ *such that* $F = G \circ Q\sim$*. Furthermore,* $Q\sim$ *is an epimorphism of categories and for all objects* y *of* $\mathcal{C}/\sim$*, there is an object* x *of* $\mathcal{C}$ *such that* $Q\sim(x) = y$*.*

1.7. DEFINITION. *The category* $\mathcal{C}/\sim$ *is called the* **quotient** *of* $\mathcal{C}$ *by* $\sim$ *while* $Q\sim$ *is the* **quotient** *functor.*

In the rest of the paper, we will deal with sequences of Σ-composable sequences, so, in order to avoid double indices, we will write $\overrightarrow{\gamma}$ to designate a sequence of morphisms $(\gamma_n, \ldots, \gamma_0)$ with γ_i as a generic element for i in $\{0, \ldots, n\}$. By extension, we will denote by $(\overrightarrow{\gamma_N}, \ldots, \overrightarrow{\gamma_0})$ a sequence of sequences of morphisms using uppercases as indices. We say that a non empty Σ-composable sequence is **normalized** when none of its elements are in Σ.

1.8. PROPOSITION. [Generalized congruence generated by Σ] *Given a category* $\mathcal{C}$ *and a collection* Σ *of morphisms of* $\mathcal{C}$*, the generalized congruence generated by the relations*

$$R_o := \emptyset \quad \textit{and} \quad R_m := \Big\{ \big((\sigma), (\mathrm{id}_{s(\sigma)})\big), \big((\sigma), (\mathrm{id}_{t(\sigma)})\big) \Big| \sigma \in \Sigma \Big\}$$

is analytically described as in the present statement: the relation $\sim_o$ *on objects of* $\mathcal{C}$ *is given by Definition 1.4 and the relation* $\sim_m$ *over the set of non empty* $\sim_o$*-composable sequences*[3] *is the* reflexive, symmetric *and* transitive *closure of the relation* $\sim_m^1$ *defined below:*

1. *for all morphisms* σ *in* $\Sigma[a, b]$*,*

$$\begin{cases} (\gamma_n, \ldots, \gamma_{k+1}, \sigma, \gamma_{k-1}, \ldots, \gamma_0) \sim_m^1 (\gamma_n, \ldots, \gamma_{k+1}, \mathrm{id}_a, \gamma_{k-1}, \ldots, \gamma_0) \\ (\gamma_n, \ldots, \gamma_{k+1}, \sigma, \gamma_{k-1}, \ldots, \gamma_0) \sim_m^1 (\gamma_n, \ldots, \gamma_{k+1}, \mathrm{id}_b, \gamma_{k-1}, \ldots, \gamma_0) \end{cases}$$

2. *for all* k *in* $\{0, \ldots, n-1\}$ *such that* $s(\gamma_{k+1}) = t(\gamma_k)$*,*

$$(\gamma_n, \ldots, \gamma_{k+1}, \gamma_k, \ldots, \gamma_0) \sim_m^1 (\gamma_n, \ldots, \gamma_{k+1} \circ \gamma_k, \ldots, \gamma_0).$$

In other words, for all $\sim_o$*-composable sequences* $\overrightarrow{\gamma}$ *and* $\overrightarrow{\delta}$ *, we write* $\overrightarrow{\gamma} \sim_m \overrightarrow{\delta}$ *when there is a finite sequence of* $\sim_o$*-composable sequences* $\overrightarrow{\alpha_0}, \ldots, \overrightarrow{\alpha_N}$*, where* $N \in \mathbb{N}$*, such that:*

[3] In this case, we also write Σ-composable instead of $\sim_o$-composable.

1. $\overrightarrow{\alpha_0} = \overrightarrow{\gamma}$ *and* $\overrightarrow{\alpha_N} = \overrightarrow{\delta}$,

2. for all K *in* $\{0, \ldots, N-1\}$, $\overrightarrow{\alpha_K} \sim^1_m \overrightarrow{\alpha_{K+1}}$.

Such a sequence $\overrightarrow{\alpha_0}, \ldots, \overrightarrow{\alpha_N}$ *is called a sequence of* $\sim^1_m$*-transformations. Then, the ordered pair* $(\sim_o, \sim_m)$ *is a generalized congruence over* $\mathcal{C}$*. In this case, the quotient category and the quotient functor are respectively denoted by* $\mathcal{C}/_\Sigma$ *and* $Q/_\Sigma$*, moreover, the last one is caracterized by the following universal property: for all functor* F *from* $\mathcal{C}$ *to* $\mathcal{D}$ *sending any morphism of* Σ *to an identity of* $\mathcal{D}$*, there is a unique functor* G *from* $\mathcal{C}/_\Sigma$ *to* $\mathcal{D}$ *such that* $F = G \circ Q_\Sigma$.

In particular, in virtue of Propositions 1.1 and 1.8, there is a unique functor P_Σ from $\mathcal{C}[\Sigma^{-1}]$ to $\mathcal{C}/_\Sigma$ such that $Q_\Sigma = P_\Sigma \circ I_\Sigma$.

The notion of fibration is implicitly contained in the statement of Theorem 4.1.

1.9. Definition. [Fiber over an object [4]] *Given a functor* $F : \mathcal{F} \longrightarrow \mathcal{B}$*, for any object* I *of* $\mathcal{B}$*, the* **fiber of** F **over** I *is the subcategory of* $\mathcal{F}$ *whose objects are those* X *such that* $F(X) = I$ *and whose morphisms are those* f *such that* $F(f) = \mathrm{id}_I$*. The fiber of* F *over* I *is denoted by* $\mathcal{F}_I$.

1.10. Definition. [Cartesian morphism [4]] *Given a functor* $F : \mathcal{F} \longrightarrow \mathcal{B}$ *and a morphism* α *in* $\mathcal{B}[J, I]$*, a morphism* f *of* $\mathcal{F}[Y, X]$ *is said to be* **cartesian** *over* α *when*

1. $F(f) = \alpha$ *and*

2. for all morphisms g *of* $\mathcal{F}[Z, X]$ *such that* $F(g)$ *can be factorized as shown in the diagram 1.5, there exists a* unique *morphism* h *in* $\mathcal{F}[Z, Y]$ *satisfying* $F(h) = \beta$ *and making the diagram 1.6 commute.*

Diagram 1.5

$$\begin{tikzcd} & F(Y) \arrow[dr, "\alpha"] & \\ F(Z) \arrow[ur, "\beta"] \arrow[rr, "F(g)"'] & & F(X) \end{tikzcd}$$

Diagram 1.6

$$\begin{tikzcd} & Y \arrow[dr, "f"] & \\ Z \arrow[ur, "h"] \arrow[rr, "g"'] & & X \end{tikzcd}$$

1.11. Definition. [Fibration [4]] *A functor* $F : \mathcal{F} \longrightarrow \mathcal{B}$ *is a* **fibration** *of* **base** $\mathcal{B}$ *when for all objects* I *and* J *of* $\mathcal{B}$*, for all morphisms* α *in* $\mathcal{B}[J, I]$ *and for all objects* X *of* $\mathcal{F}_I$*, there exists a morphism* f *which is cartesian over* α *and whose target is* X.

The Proposition 1.13 justify the terminology introduced in the next definition:

1.12. Definition. *A functor* F *from* $\mathcal{F}$ *to* $\mathcal{B}$ *is a* **fibered equivalence** *when it is full, faithful and for all objects* y *of* $\mathcal{B}$*, there is an object* x *of* $\mathcal{F}$ *such that* $F(x) = y$.

Any fibered equivalence is obviously an equivalence of category and is also called, in the terminology of *Saunders Mac Lane* [29], a **left adjoint-left inverse**.

1.13. Proposition. *Any fibered equivalence* $F : \mathcal{F} \longrightarrow \mathcal{B}$ *is a* fibration.

PROOF. For each object I of $\mathcal{B}$, we *choose* an object $G(I)$ of $\mathcal{F}$ such that $F(G(I)) = I$. As the functor F is full and faithful, the map 1.1 is a bijection which enables us to define for all morphisms β in $\mathcal{B}[J, I]$ the image of β by G as

the unique element α of $\mathcal{F}[G(J), G(I)]$ such that $F(\alpha) = \beta$. The functor G we have defined

is the right adjoint to F, satisfies $F \circ G = Id_{\mathcal{B}}$ and the co-unit of the adjunction $F \dashv G$ is the identity.

Any fibered equivalence is clearly an equivalence of categories so the unit η of this adjunction is an isomorphism from $Id_{\mathcal{F}}$ to $G \circ F$.

$$\begin{array}{ccc} \mathcal{F}[G(J), G(I)] & \longrightarrow & \mathcal{B}[J, I] \\ \alpha & \longmapsto & F(\alpha) \end{array}$$

Map 1.1

Moreover, for any object X of $\mathcal{F}$, η_X is the unique morphism of $\mathcal{F}[X, GFX]$ such that $F(\eta_X) = \mathrm{id}_{FX}$ since the co-unit of the adjunction $F \dashv G$ is an identity. Now we can prove that F is a fibration: let I be an object of $\mathcal{B}$, X be an object of $\mathcal{F}_I$ and α be a morphism of $\mathcal{B}[J, I]$, we put $f := \eta_X^{-1} \circ G(\alpha)$. Referring to what we have proved in the preamble, we know that f belongs to $\mathcal{F}[Y, X]$, where $Y := G(J)$, and also that $F(f) = F(\eta_X^{-1}) \circ F(G(\alpha)) = \alpha$. Now we check that f is cartesian over α. Let g and β be two morphisms respectively taken from $\mathcal{F}[Z, X]$ and $\mathcal{B}[F(Z), F(Y)]$ and that make the diagram 1.7 commute. Let h be $G(\beta) \circ \eta_Z$, it comes $F(h) = F(G(\beta)) \circ F(\eta_Z) = \beta$.

In addition, since η is an isomorphism from $Id_{\mathcal{F}}$ to $G \circ F$, the diagram 1.8 commutes and provides the equality $g = f \circ h$.

$$\begin{tikzcd} & F(Y) = J \arrow[dr, "F(f)=\alpha"] \arrow[d, phantom, "="] & \\ F(Z) \arrow[ur, "\beta"] \arrow[rr, "F(g)"'] & {} & F(X) = I \end{tikzcd}$$

Diagram 1.7

$$\begin{tikzcd} & GJ = Y \arrow[dr, "G(\alpha)"] & \\ GFZ \arrow[ur, "G(\beta)"] \arrow[rr, "GFg"] & & GI = GFX \arrow[d, dotted, shift right, "\eta_X^{-1}"'] \\ Z \arrow[u, "\eta_Z"] \arrow[rr, "g"'] \arrow[uur, dotted, bend left, "h=G(\beta)\circ\eta_Z"] & & X \arrow[u, shift right, "\eta_X"'] \arrow[uul, dotted, bend right, "f=\eta_X^{-1}\circ G(\alpha)"'] \end{tikzcd}$$

Diagram 1.8

We still have to check that such a morphism h is unique. Let h_1 and h_2 be two morphisms on $\mathcal{F}[Z, Y]$ satisfying $F(h_1) = F(h_2) = \beta$, for F is faithful (as an equivalence of categories) we have $h_1 = h_2$.

■

2. *Yoneda*-systems and categories of components

2.1. DEFINITION. [*Yoneda morphisms*] *Let $\mathcal{C}$ be a category, x and y be two objects of $\mathcal{C}$ and σ be a morphism in $\mathcal{C}[x, y]$, we say that σ is* **inversible**[4] **in the sense of *Yoneda*** *when it satisfies the following conditions:*

[4] A neologism which means "consistent with having an inverse".

preservation of the future cone: *for all objects y' of $\mathcal{C}$,* if $\mathcal{C}[y,y'] \neq \emptyset$, *the map $\big(\mathbb{Y}_{\mathcal{C}}(y')\big)(\sigma)$ is a bijection and*

preservation of the past cone: *for all objects x' of $\mathcal{C}$,* if $\mathcal{C}[x',x] \neq \emptyset$, *the map $\big(\mathbb{Y}_{\mathcal{C}}(\sigma)\big)(x')$ is a bijection.*

$$\big(\mathbb{Y}_{\mathcal{C}}(y')\big)(\sigma) \ : \ \begin{array}{ccc} \mathcal{C}[y,y'] & \longrightarrow & \mathcal{C}[x,y'] \\ \gamma & \longmapsto & \gamma \circ \sigma \end{array} \qquad \big(\mathbb{Y}_{\mathcal{C}}(\sigma)\big)(x') \ : \ \begin{array}{ccc} \mathcal{C}[x',x] & \longrightarrow & \mathcal{C}[x',y] \\ \delta & \longmapsto & \sigma \circ \delta \end{array}$$

We also write, for short, that σ is a ***Yoneda*-morphism**. The terminology as well as the notations $\big(\mathbb{Y}_{\mathcal{C}}(y')\big)(\sigma)$ and $\big(\mathbb{Y}_{\mathcal{C}}(\sigma)\big)(x')$ refer to the *Yoneda* embedding of the category $\mathcal{C}$ in its category of presheaves $\mathsf{Set}^{\mathcal{C}^{op}}$. Moreover, the collection of all the *Yoneda*-morphisms of $\mathcal{C}$ is denoted by *Yoneda*$(\mathcal{C})$.

2.2. Proposition. *The* Yoneda-*morphisms compose.*

Proof. This is a straightforward consequence of the fact that bijections compose. ∎

2.3. Proposition. *Given a category $\mathcal{C}$, two objects x and y of $\mathcal{C}$ and a morphism σ in $\mathcal{C}[x,y]$; the morphism σ is an isomorphism of $\mathcal{C}$ if and only if σ is a* Yoneda-*morphism and $\mathcal{C}[y,x] \neq \emptyset$.*

Proof. Suppose that σ is a *Yoneda*-morphism such that $\mathcal{C}[y,x] \neq \emptyset$, applying Definition 2.1 with $y' = x$ and $x' = y$, there exists a unique morphism γ in $\mathcal{C}[y,x]$ such that $\gamma \circ \sigma = \mathrm{id}_x$ and a unique morphism δ in $\mathcal{C}[y,x]$ such that $\sigma \circ \delta = \mathrm{id}_y$, it follows that σ is an isomorphism. The converse is straightforward. ∎

2.4. Proposition. *Any* Yoneda-*morphism is both a monomorphism and an epimorphism.*

Proof. Immediately comes from the injectivity of maps $\big(\mathbb{Y}_{\mathcal{C}}(y')\big)(\sigma)$ and $\big(\mathbb{Y}_{\mathcal{C}}(\sigma)\big)(x')$ described in Definition 2.1. ∎

2.5. Definition. [*Yoneda-systems*] *Let Σ be a collection of morphisms of a category $\mathcal{C}$ and suppose the variables x, y, x' and y' range over the set of objects of $\mathcal{C}$. The collection Σ is called a* ***Yoneda*-system** *of $\mathcal{C}$ when:*

1. $Iso(\mathcal{C}) \subseteq \Sigma \subseteq Yoneda(\mathcal{C})$,

2. (a) *for all morphisms γ of $\mathcal{C}[x,y]$, for all morphisms σ of $\Sigma[y',y]$, there exist a morphism γ' of $\mathcal{C}[x',y']$ and a morphism σ' of $\Sigma[x',x]$ such that the diagram 2.1 is a* pull-back *square in $\mathcal{C}$,*

   (b) *for all morphisms γ of $\mathcal{C}[x,y]$, for all morphisms σ of $\Sigma[x,x']$, there exist a morphism γ' of $\mathcal{C}[x',y']$ and a morphism σ' of $\Sigma[y,y']$ such that the diagram 2.2 is a* push-out *square in $\mathcal{C}$*

$$\begin{tikzcd}[column sep=small] & y & \\ x \arrow[ur, "\gamma"] & \arrow[phantom, ""]{}\text{\scriptsize\it pull back in } \mathcal{C} & y' \arrow[ul, "\sigma\in\Sigma"'] \\ & x' \arrow[ul, dotted, "\sigma'\in\Sigma"] \arrow[ur, dotted, "\gamma'"'] & \end{tikzcd}$$

Diagram 2.1

$$\begin{tikzcd}[column sep=small] & y' & \\ x' \arrow[ur, dotted, "\gamma'"] & \text{\scriptsize\it push out in } \mathcal{C} & y \arrow[ul, dotted, "\sigma'\in\Sigma"'] \\ & x \arrow[ul, "\sigma\in\Sigma"] \arrow[ur, "\gamma"'] & \end{tikzcd}$$

Diagram 2.2

$$\begin{tikzcd} {} \arrow[r, "\sigma'_1"'] & {} \arrow[r, "\sigma'_2"'] & {} \\ {} \arrow[u] \arrow[r, "\sigma_1"] & {} \arrow[u] \arrow[r, "\sigma_2"] & {} \arrow[u] \end{tikzcd}$$

Diagram 2.3

3. and the collection Σ is stable under composition.

The collection Σ is said to be stable under **change** (respectively **co-change**) **of base** when Axiom (2a) (respectively (2b)) of Definition 2.5 is satisfied.

Before coming to the next theorem, we recall that a **complete lattice** [2, 32] is a partially ordered set[5] $(\mathcal{T}, \sqsubseteq)$ whose subsets[6] have a least upper bound and a greatest lower bound. We also introduce the following notation: if $\left(\Sigma_i\right)_{i\in I}$ is a family of collections of morphisms of a given category $\mathcal{C}$, we denote by $\biguplus_{i\in I}\Sigma_i$ the (collection of morphisms of the) subcategory of $\mathcal{C}$ generated by the set-theoretical union $\bigcup_{i\in I}\Sigma_i$.

2.6. Theorem. [Complete lattice of *Yoneda*-systems of $\mathcal{C}$] *Let $\mathcal{C}$ be a category, the collection of all* Yoneda*-systems of $\mathcal{C}$, denoted by $\mathcal{T}_{\mathcal{C}}$ and equipped with inclusion order $\subseteq$, is a* complete lattice *in which*

1. *the greatest lower bound is given by the set-theoretical intersection $\bigcap$,*
2. *the least upper bound is given by $\biguplus$ and*
3. *the least element is the collection of all the isomorphisms of $\mathcal{C}$.*

Moreover, the greatest element of $\mathcal{T}_{\mathcal{C}}$, denoted by $\overline{\Sigma}$, is the set-theoretical intersection of the elements of the family S_α, indexed by the ordinals, whose members are sets of morphisms of $\mathcal{C}$ and which is transfinitely defined [7, 25, 27, 28] as follows:

initial case *S_0 is the set of all* Yoneda*-morphisms of $\mathcal{C}$.*
Given an ordinal λ such that for all ordinal $\alpha < \lambda$, the set S_α is already defined, we construct the set S_λ in the following way:

successor case *if $\lambda = \alpha + 1$ is the successor of some ordinal α, then $S_{\alpha+1}$ is the set of morphisms σ in S_α which satisfies:*

- *for all morphisms γ satisfying $t(\gamma) = t(\sigma)$, there exist a morphism σ' which* belongs *to S_α and a morphism γ' of $\mathcal{C}$ such that $t(\sigma') = s(\gamma)$, $t(\gamma') = s(\sigma)$, $s(\sigma') = s(\gamma')$ and the square formed by σ, γ, σ' and γ' is a pull-back in $\mathcal{C}$; dually*

[5]Partially ordered sets are also called posets for short.
[6]Even the empty one, so complete lattices cannot be empty.

- *for all morphisms γ satisfying $s(\gamma) = s(\sigma)$, there exist a morphism σ'' which* belongs *to S_α and a morphism γ'' of $\mathcal{C}$ such that $s(\sigma'') = t(\gamma)$, $s(\gamma'') = t(\sigma)$, $t(\sigma'') = t(\gamma'')$ and the square formed by σ'', γ, σ'' and γ'' is a push-out in $\mathcal{C}$.*

limit case *if λ is a limit, which means that it is* not *a successor, the set S_λ is the set-theoretical intersection of the members of the family $(S_\alpha)_{\alpha<\lambda}$.*

PROOF. A routine verification proves that the collection of isomorphisms of a category is a *Yoneda*-system of this category and it is obviously the least one with respect to inclusion.

Let I be a non empty set and $(\Sigma_i)_{i\in I}$ be a family of *Yoneda*-systems of $\mathcal{C}$, by construction, the collection of morphisms of $\mathcal{C}$ described below is stable under composition and satisfies the first point of definition 2.5 since *Yoneda*-morphisms compose (Proposition 2.2).

$$\biguplus_{i\in I}\Sigma_i := \left\{\sigma_n\circ\cdots\circ\sigma_1 \;\middle|\; n\in\mathbb{N}\backslash\{0\},\ \{i_1,\ldots,i_n\}\subseteq I \text{ and } \forall k\in\{1,\ldots,n\}\ \sigma_k\in\Sigma_{i_k}\right\}$$

Given an element $\sigma_n\circ\cdots\circ\sigma_1$ of $\biguplus_{i\in I}\Sigma_i$, where $n\in\mathbb{N}\backslash\{0\}$, we have a finite subset $\{i_1,\ldots,i_n\}$ of I such that for all k in $\{1,\ldots,n\}$, the morphism σ_k is in Σ_{i_k}. Let γ be a morphism of $\mathcal{C}$ sharing the same source as σ_1: the situation is depicted by diagram 2.4. From a finite induction (apply consecutively Definition 2.5 (2*b*) to $\Sigma_{i_1},\ldots,\Sigma_{i_n}$), we obtain a finite sequence of push-out squares (see diagram 2.5) which gives, once pasted, the expected push-out square (see diagram 2.6).

$$\begin{tikzcd} {} \\ {} \arrow[u, "\gamma"] \arrow[r, "\sigma_1\in\Sigma_{i_1}"'] & {} \arrow[r, dotted, no head] & {} \arrow[r, "\sigma_n\in\Sigma_{i_n}"'] & {} \end{tikzcd}$$

Diagram 2.4

$$\begin{tikzcd} {} \arrow[r, dotted, "\sigma'_1\in\Sigma_{i_1}"] & {} & {} \arrow[r, dotted, "\sigma'_n\in\Sigma_{i_n}"] & {} \\ {} \arrow[u, "\gamma"] \arrow[r, "\sigma_1\in\Sigma_{i_1}"'] \arrow[ur, phantom, "\text{push-out}"] & {} \arrow[u, dotted, "\gamma_1"'] \arrow[r, dotted, no head] & {} \arrow[u, dotted, "\gamma_{n-1}"] \arrow[r, "\sigma_n\in\Sigma_{i_n}"'] \arrow[ur, phantom, "\text{push-out}"] & {} \arrow[u, dotted, "\gamma_n"'] \end{tikzcd}$$

Diagram 2.5

$$\begin{tikzcd} {} \arrow[r, dotted, "\sigma'_n\circ\cdots\circ\sigma'_1\in\biguplus_{i\in I}\Sigma_i"] & {} \\ {} \arrow[u, "\gamma"] \arrow[r, "\sigma_n\circ\cdots\circ\sigma_1\in\biguplus_{i\in I}\Sigma_i"'] \arrow[ur, phantom, "\text{push-out}"] & {} \arrow[u, dotted, "\gamma_n"'] \end{tikzcd}$$

Diagram 2.6

Up to duality, the same proof holds for the pull-back squares, using (2*a*) instead of (2*b*).

The collection $\bigcap_{i\in I}\Sigma_i$ is obviously stable under composition and no less clearly satisfies the first point of Definition 2.5. The stability under change (cochange) of base of $\bigcap_{i\in I}\Sigma_i$ is inherited from the stability under change (cochange) of base of Σ_i for each $i\in I$ since push-outs and pull-backs are uniquely defined up to isomorphism.

The operators $\bigcap$ and $\biguplus$ are obviously *associative* over the collection of *Yoneda*-systems of $\mathcal{C}$, this result holds whether the families of morphisms considered are *Yoneda*-systems or not. Now we prove that the description of $\overline{\Sigma}$ is valid.

Clearly, if λ_1 and λ_2 are ordinals such that $\lambda_1\leq\lambda_2$ then $S_{\lambda_2}\subseteq S_{\lambda_1}$, so let $\overline{\Sigma}$ be the set-theoretical intersection of the family (indexed by ordinals) of sets S_α. It comes immediately that all the elements of $\overline{\Sigma}$ are *Yoneda*-morphisms. We prove by transfinite induction that $Iso(\mathcal{C})\subseteq S_\alpha$ for any ordinal α. It is true for S_0 by Proposition 2.3. If it is true for S_α, then so is it for $S_{\alpha+1}$ since the push-out, as well as the pull-back, of any isomorphism along any morphism is still an isomorphism. The case where λ is a limit ordinal is trivial. Still by transfinite induction, we verify that all the sets S_α are stable under composition: the case of S_0 is given by Proposition 2.2. Now suppose S_α is stable under composition and let σ_1 and σ_2 be two elements of $S_{\alpha+1}$ such that $\sigma_2 \circ \sigma_1$ exists, if both inner squares of diagram 2.3 are push-out squares, then so is the outer shape; by construction of $S_{\alpha+1}$, we can suppose that σ_1' and σ_2' belong to S_α which, by hypothesis of induction, is stable under composition and thus contains $\sigma_2' \circ \sigma_1'$. Once again, the same proof holds, up to duality, for pull-back squares and we have proved that $S_{\alpha+1}$ is stable under composition. The case where λ is a limit ordinal is trivial. Because $\mathcal{C}$ is a *small* category, the collection S_α, which is *indexed by the ordinals* and decreasing with respect to inclusion, has to be stationary beyond some ordinal λ. It follows that S_λ is stable under change and co-change of base and contains only *Yoneda*-morphisms. Then, together with what has been proved, S_λ is stable under composition and contains all the isomorphisms of $\mathcal{C}$. It remains to prove, still by transfinite induction, that any set S_α contains all the *Yoneda*-systems of $\mathcal{C}$. It is obvious for S_0. Let Σ be a *Yoneda*-system included in S_α, given σ in Σ and a morphism γ of $\mathcal{C}$ sharing the same target as σ, by Definition 2.5 (diagram 2.1) we can choose σ' in Σ and thus, by hypothesis of induction, in S_α. The same proof holds in the case where γ and σ shares the same source, referring to the diagram 2.2 to prove that we can pick σ'' in S_α. Thus, by construction of $S_{\alpha+1}$, the morphism σ belongs to $S_{\alpha+1}$ and finally $\Sigma \subseteq S_{\alpha+1}$. The case where λ is a limit ordinal is obvious. ■

Given any category $\mathcal{C}$, a consequence of Theorem 2.6 is the existence of the greatest (with respect to inclusion) *Yoneda*-system of $\mathcal{C}$: it is denoted by $\overline{\Sigma}$. Then we define the **category of components** of $\mathcal{C}$ as the quotient of $\mathcal{C}$ by $\overline{\Sigma}$, in other words $\mathcal{C}/_{\overline{\Sigma}}$ with the notation of Proposition 1.8.

2.7. Corollary. [Action of autofunctors on $\overline{\Sigma}$] *The greatest* Yoneda-*system* $\overline{\Sigma}$ *of a small category* $\mathcal{C}$ *and its complementary in the set of morphisms of* $\mathcal{C}$ *are* stable *by the direct image of any* autofunctor *of* $\mathcal{C}$*, in other words* $\overline{\Sigma}$ *and its complementary are stable under the (right)* action *of* $\mathsf{Aut}(\mathcal{C})$ *over the set of morphisms of* $\mathcal{C}$*.*

Proof. Let Φ be an autofunctor of $\mathcal{C}$, then $\Phi(\overline{\Sigma})$ is a *Yoneda*-system of $\mathcal{C}$ so $\Phi(\overline{\Sigma}) \subseteq \overline{\Sigma}$. ■

Given an autofunctor Φ of $\mathcal{C}$, it comes from Corollary 2.7 that the functor $Q_{\overline{\Sigma}} \circ \Phi$ sends any element of $\overline{\Sigma}$ to an identity of $\mathcal{C}/_{\overline{\Sigma}}$ hence, by Proposition 1.8, there exists a *unique endofunctor* $\Phi/_{\overline{\Sigma}}$ of $\mathcal{C}/_{\overline{\Sigma}}$ such that $Q_{\overline{\Sigma}} \circ \Phi = \Phi/_{\overline{\Sigma}} \circ Q_{\overline{\Sigma}}$.

2.8. Proposition. *The map sending any autofunctor* Φ *of* $\mathcal{C}$ *to* $\Phi/_{\overline{\Sigma}}$ *is a morphism of group from* $\mathsf{Aut}(\mathcal{C})$ *to* $\mathsf{Aut}(\mathcal{C}/_{\overline{\Sigma}})$*.*

Proof. Given Φ_1 and Φ_2 two autofunctors of $\mathcal{C}$, we have $Q_{\overline{\Sigma}} \circ \Phi_2 \circ \Phi_1 = \Phi_2/_{\overline{\Sigma}} \circ \Phi_1/_{\overline{\Sigma}} \circ Q_{\overline{\Sigma}}$. It follows, since $(\Phi_2 \circ \Phi_1)/_{\overline{\Sigma}}$ is unique, that $\Phi_2/_{\overline{\Sigma}} \circ \Phi_1/_{\overline{\Sigma}} = (\Phi_2 \circ \Phi_1)/_{\overline{\Sigma}}$. In particular, considering $\Phi_2 = \Phi_1^{-1}$, we see that Φ_1 is an autofunctor of $\mathcal{C}/_{\overline{\Sigma}}$. ■

2.9. Remark. If for all objects x and y of $\mathcal{C}$, we have $\mathcal{C}[x,y]=\emptyset$ if and only if $\mathcal{C}[y,x]=\emptyset$, then Proposition 2.3 implies that $\overline{\Sigma}$ is the set of isomorphisms of $\mathcal{C}$ and thus $S_0=\overline{\Sigma}$. In all "concrete" cases[7], the induction described in Theorem 2.6 stops after only finitely many iterations. Note that Corollary 2.7 is not satisfied for any *Yoneda*-system. In the poset $(\mathbb{R},\leq)$ seen as a small category, the morphisms are the ordered pairs of real numbers (x,y) such that $x\leq y$, we have a *Yoneda*-system of $(\mathbb{R},\leq)$ taking $\Sigma := \{(x,y)\in\mathbb{R}\times\mathbb{R} \mid x\geq 0 \text{ or } y<0\}$. Clearly, given any strict "translation" τ of $(\mathbb{R},\leq)$, which means that τ is an autofunctor of $(\mathbb{R},\leq)$ defined by $x\longmapsto x+t$ where $t\in\mathbb{R}\backslash\{0\}$, the direct image of Σ by τ is not included in Σ.

In regard with the abstract, let us consider a groupoid $\mathcal{G}$, its greatest *Yoneda*-system is the set of all its morphisms and the relation $\sim$ over the objects of $\mathcal{G}$ defined by $x\sim y$ when $\mathcal{G}[x,y]$ is not empty, is an equivalence relation whose classes are called the connected components of $\mathcal{G}$. Consequently, its category of components is the discrete category having exactly one object for each connected component of $\mathcal{G}$. As expected, the notion of $\overline{\Sigma}$-component of a small category *extends* the notion of connected component of a groupoid [24]. Furthermore, given a groupoid $\mathcal{G}$, one has the following equivalent statements:

1. the functor $P_{\overline{\Sigma}}$ is a fibered equivalence,
2. for all objects x and y of $\mathcal{G}$, the set $\mathcal{G}[x,y]$ is either empty or a singleton.

In particular, if $\mathcal{G}$ is the fundamental groupoid of a topological space X [24], each of the preceding statements is also equivalent to the assertion that X is simply connected in the sense where for any base point b of X, the fundamental group $\pi_1(X,b)$ is trivial. Furthermore, the set of arcwise connected components of a topological space X (usually denoted by $\pi_0(X)$ and seen as a small discrete category) is isomorphic to the category of components of the fundamental groupoid of X.

2.10. Proposition. *Any* Yoneda-*system Σ of a category $\mathcal{C}$ is a left and right calculus of fractions over $\mathcal{C}$.*

Proof. We treat the case of the right calculus of fractions. The only point of Definition 1.2 which is not obviously satisfied by Σ is the fourth one: suppose that $\sigma\circ\gamma=\sigma\circ\delta$ with the notation of Definition 1.2, thus σ is a *Yoneda*-morphism hence, by Proposition 2.4, we know that $\gamma=\delta$, then it suffices to take $\sigma' := \mathrm{id}_x$. ∎

2.11. Remark. Let Σ be a *Yoneda*-system of a category $\mathcal{C}$ and consider the diagram 2.7, from point (2a) of Definition 2.5 we obtain, since σ is in Σ, a representative of the pull-back as in diagram 2.8 where σ' is in Σ.

Yet, invoking the fact that γ is an element of Σ, we obtain another representative of the same pull-back, as shown by diagram 2.9, in which the morphism γ'' is in Σ. As a consequence, we have an isomorphism ξ of $\mathcal{C}$ such that $\gamma'=\gamma''\circ\xi$, it follows that γ' is

[7]We refer here to a branch of theoretical computer science, the static analysis of concurrent programs, where the categories of components are used to reduce the size of the models to analyze [15].

also an element of Σ. We can go further: let Σ and Σ' be two *Yoneda*-systems of $\mathcal{C}$ and consider the diagram 2.10, for all pull-back squares as in diagram 2.11, we have $\sigma_* \in \Sigma$ and $\sigma'_* \in \Sigma'$. The same holds for push-out squares.

$$\begin{tikzcd} & x & \\ y \arrow[ur, "\sigma\in\Sigma"] & & z \arrow[ul, "\gamma\in\Sigma"'] \end{tikzcd}$$

Diagram 2.7

$$\begin{tikzcd} & x & & & x & \\ y \arrow[ur, "\sigma"] & & z \arrow[ul, "\gamma"'] & y \arrow[ur, "\sigma"] & & z \arrow[ul, "\gamma"'] \\ & x' \arrow[ul, dotted, "\gamma'"] \arrow[ur, dotted, "\sigma'\in\Sigma"'] \arrow[rrr, dashed, "\xi"'] & & & x'' \arrow[ul, dotted, "\gamma''\in\Sigma"] \arrow[ur, dotted, "\sigma''"'] & \end{tikzcd}$$

Diagram 2.8 *Diagram 2.9*

$$\begin{tikzcd} {} \arrow[r, "\sigma\in\Sigma"] & {} & {} \arrow[l, "\sigma'\in\Sigma'"'] \end{tikzcd}$$

Diagram 2.10

$$\begin{tikzcd} & {} & \\ {} \arrow[ur, "\sigma'\in\Sigma'"] & & {} \arrow[ul, "\sigma\in\Sigma"'] \\ & {} \arrow[ul, "\sigma_*"] \arrow[ur, "\sigma'_*"'] & \end{tikzcd}$$

Diagram 2.11

$$\begin{tikzcd} & u & \\ x \arrow[ur, "\sigma_1"] & & y \arrow[ul, "\sigma_2"'] \\ & d \arrow[ul, "\sigma_3"] \arrow[ur, "\sigma_4"'] & \end{tikzcd}$$

Diagram 2.12

From this last remark, we deduce a handy description of the notion of Σ-zigzag connectedness described in Definition 1.4.

2.12. LEMMA. *Let Σ be a* Yoneda-*system of a category $\mathcal{C}$. Given two objects x and y of $\mathcal{C}$, x and y are Σ-zigzag connected if and only if there are two objects u and d of $\mathcal{C}$ as well as four morphisms σ_1, σ_2, σ_3 and σ_4 respectively taken from $\Sigma[x,u]$, $\Sigma[y,u]$, $\Sigma[d,x]$ and $\Sigma[d,y]$ such that the diagram 2.12 commutes.*

PROOF. Easily follows from the previous facts. ■

3. Loop-free categories

3.1. DEFINITION. *A morphism γ of some category $\mathcal{C}$ is said to be* **without return** *when the homset $\mathcal{C}[t(\gamma), s(\gamma)]$ is empty. Otherwise, we say that γ* **admits a return** *or* **has a return**.

3.2. DEFINITION. *A category $\mathcal{C}$ is said to be* **loop-free** *when all its morphisms,* except *its identities, are* without *return.*

One can think of loop-free categories as a generalization of partially ordered sets since the lack of return can be reformulated in these terms : $\mathcal{C}[x,y] \neq \emptyset$ and $\mathcal{C}[y,x] \neq \emptyset$ implies $x = y$ and $\mathcal{C}[x,x] = \{\mathrm{id}_x\}$, which can be understood as a generalized antisymmetry.

The category of loop-free categories LfCat is an epireflective subcategory of the category of small categories whose left adjoint is given by $\mathcal{C}/_\Sigma$ (Proposition 1.8) where Σ is the collection of morphisms of $\mathcal{C}$ which have a return.

3.3. PROPOSITION. *Any sub-category of a loop-free category is loop-free. The isomorphisms and endomorphisms of a loop-free category are its identities. If the composite of two morphisms of a loop-free category in an isomorphism, then both of these morphisms are identities. In particular, any loop-free category is skeletal, in other words two given objects of a loop-free category are isomorphic if and only if they are equal.*

PROOF. Easily follows from Definition 3.2. ■

3.4. Remark. If $\mathcal{C}$ is a loop-free category and σ is a *Yoneda*-morphism of $\mathcal{C}$, then $\mathcal{C}[s(\sigma), t(\sigma)]$ is reduced to the singleton $\{\sigma\}$ because the map $\big(\mathbb{Y}_{\mathcal{C}}(y')\big)(\sigma)$ is a bijection from $\mathcal{C}[s(\sigma), s(\sigma)]$ onto $\mathcal{C}[s(\sigma), t(\sigma)]$ and, for the category $\mathcal{C}$ is loop-free, the homset $\mathcal{C}[s(\sigma), s(\sigma)]$ is a singleton. In regard of this remark, when x and y are two objects of $\mathcal{C}$ and the homset $\mathcal{C}[x, y]$ contains an element of a *Yoneda*-system Σ over $\mathcal{C}$, we will use the notation $x \xrightarrow{\Sigma} y$ to mean that the *single* element of $\mathcal{C}[x, y]$ belongs to Σ.

3.5. Definition. *A collection Σ of morphisms of a category $\mathcal{C}$ is said to be* **pure** *in $\mathcal{C}$ when for all morphisms γ and δ of $\mathcal{C}$ such that $s(\gamma) = t(\delta)$, if the morphism $\gamma \circ \delta$ belongs to Σ, then γ and δ also belong to Σ.*

3.6. Lemma. *Any* Yoneda-*system Σ of a* loop-free *category $\mathcal{C}$ is pure in $\mathcal{C}$.*

Proof. Consider an element σ of Σ and two morphisms δ and γ of $\mathcal{C}$ such that $\sigma = \gamma \circ \delta$.

From the point $(2b)$ of Definition 2.5, we have an element σ' of Σ and a morphism δ' of $\mathcal{C}$ which form a push-out square of $\mathcal{C}$, we also have a unique morphism ξ of $\mathcal{C}$ making the diagram 3.1 commute. Since $\mathcal{C}$ is a loop-free category and both morphisms δ' and ξ admit a return, both are identities. In particular, we have $\gamma = \sigma'$ therefore γ is in Σ. We prove the same way, using the point $(2a)$ instead of $(2b)$, that δ is in Σ.

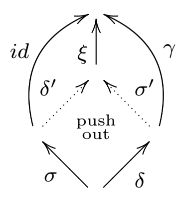

Diagram 3.1

■

We have the material to enhance Theorem 2.6, to this aim, we recall that a **locale**[8] [5, 26, 30, 32] is a complete lattice $(\mathcal{L}, \sqsubseteq)$ in which the generalized distributivity holds. Formally, it means that we have

$$y \wedge \Big(\bigvee_{i \in I} x_i\Big) \;=\; \bigvee_{i \in I} (y \wedge x_i)$$

for all families $(x_i)_{i \in I}$ of elements of $\mathcal{L}$ indexed by a set I and for all elements y of $\mathcal{L}$. In particular, the collection of open subsets of a topological space is a locale.

3.7. Proposition. *Given a* loop-free *category $\mathcal{C}$, the collection $\mathcal{L}_{\mathcal{C}}$ of all the* Yoneda-*systems of $\mathcal{C}$, ordered by inclusion, is a locale.*

Proof. We already know that $(\mathcal{L}_{\mathcal{C}}, \subseteq)$ is a complete lattice (Theorem 2.6) so we just have to prove the generalized distributivity: let Σ' be a *Yoneda*-system of $\mathcal{C}$ and $\big(\Sigma_i\big)_{i \in I}$ be a family of *Yoneda*-systems of $\mathcal{C}$,

the inclusion $\biguplus_{i \in I} (\Sigma' \cap \Sigma_i) \subseteq \Sigma' \cap \left(\biguplus_{i \in I} \Sigma_i\right)$ comes immediately from the stability under composition of Σ'.

Conversely, an element of the right hand side member can be written as a composite $\gamma = \sigma_n \circ \cdots \circ \sigma_1$ where n is a non zero natural number and $\{i_1, \ldots, i_n\}$ is and a finite

[8] Sometimes called “pointless” topology or topology without points.

subset of I such that for all k in $\{1, \dots, n\}$, the morphism σ_k is in Σ_{i_k}. By pureness of Σ' (Lemma 3.6), all the morphisms $\sigma_1, \dots, \sigma_n$ belong to Σ' and hence so does their composite.

■

In the next result, we describe the structure of each Σ-component of a given loop-free category $\mathcal{C}$.

3.8. THEOREM. [Structure of a Σ-component] *Let $\mathcal{C}$, Σ, K and $\mathcal{K}$ be respectively a loop-free category, a* Yoneda-*system of $\mathcal{C}$, a Σ-component of $\mathcal{C}$ and the* full *sub-category of $\mathcal{C}$ whose objects are the elements of K. The following properties are satisfied:*

1. *the category $\mathcal{K}$ is isomorphic to the poset $(K, \sqsubseteq)$, seen as a small category, where for all elements x and y of K, $x \sqsubseteq y$ means that the homset $\mathcal{C}[x, y]$ is not empty. In particular any diagram in $\mathcal{K}$ is* commutative.

2. *The poset $(K, \sqsubseteq)$ is a lattice: in other words, any pair $\{x, y\}$ of elements of K has a greatest lower bound and a least upper bound in $(K, \sqsubseteq)$ respectively denoted by $x \wedge y$ and $x \vee y$.*

3. *If two objects x and y of $\mathcal{C}$ are in the same Σ-component, then diagram 3.2 is both the* push-out *square and the* pull-back *square (in $\mathcal{C}$) of the diagrams 3.3 and 3.4, besides, all the arrows appearing in these three diagrams belong to Σ.*

In particular, any morphism of $\mathcal{C}$ whose source and target belong to the same Σ-component is a morphism of Σ.

$$\begin{tikzcd} & x \vee y & \\ x \arrow[ur] & & y \arrow[ul] \\ & x \wedge y \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Diagram 3.2

$$\begin{tikzcd} x & & y \\ & x \wedge y \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Diagram 3.3

$$\begin{tikzcd} & x \vee y & \\ x \arrow[ur] & & y \arrow[ul] \end{tikzcd}$$

Diagram 3.4

PROOF. The relation $\sqsubseteq$ is obviously reflexive and the transitive, it is also antisymmetric because $\mathcal{C}$ is loop-free. Let α be an element of $\mathcal{K}[x, y]$, for x and y are taken from the same Σ-component, there are four morphisms σ_1, σ_2, σ_3 and σ_4 in Σ which form, together with α, the commutative diagram 3.5 (Lemma 2.12 and remark 3.4). As a consequence, α is an element of Σ (Lemma 3.6) and therefore the homset $\mathcal{K}[x, y]$ is a singleton (remark 3.4). By the way, we have proved that any morphism of $\mathcal{C}$ between two objects of the same Σ-component is in Σ. Let x and y be two elements K. The diagram 3.6 (given by Lemma 2.12) admits a pull-back in $\mathcal{C}$ as shown by diagram 3.7 with σ_1' and σ_2' taken from Σ (remark 2.11). The object d clearly belongs to K. If d' is a lower bound of $\{x, y\}$ in $(K, \sqsubseteq)$, then $\mathcal{C}[d', x]$ and $\mathcal{C}[d', y]$ are two singletons whose respective elements γ and δ belong to Σ; so diagram 3.8 commutes (remark 3.4). The universal property of pull-back squares implies that $\mathcal{K}[d', d]$ is not empty, in other words $d' \sqsubseteq d$. We prove analogously the existence of the least upper bound of $\{x, y\}$. The third assertion immediately follows. ■

$$\begin{tikzcd} & {} & \\ x \arrow[ur, "\sigma_1"] \arrow[rr, "\alpha"] & & y \arrow[ul, "\sigma_2"'] \\ & {} \arrow[ul, "\sigma_3"] \arrow[ur, "\sigma_4"'] & \end{tikzcd}$$

Diagram 3.5

$$\begin{tikzcd} & u & \\ x \arrow[ur, "\sigma_1"] & & y \arrow[ul, "\sigma_2"'] \end{tikzcd}$$

Diagram 3.6

$$\begin{tikzcd} & u & \\ x \arrow[ur, "\sigma_1"] \arrow[rr, phantom, "\text{pull-back}"] & & y \arrow[ul, "\sigma_2"'] \\ & d \arrow[ul, "\sigma_2'"] \arrow[ur, "\sigma_1'"'] & \end{tikzcd}$$

Diagram 3.7

$$\begin{tikzcd} & u & \\ x \arrow[ur, "\sigma_1"] & & y \arrow[ul, "\sigma_2"'] \\ & d' \arrow[ul, "\gamma"] \arrow[ur, "\delta"'] & \end{tikzcd}$$

Diagram 3.8

From now on, $\mathcal{C}$ is a (small) loop-free category and Σ is a *Yoneda*-system of $\mathcal{C}$; besides, the ordered pair $(\sim_o, \sim_m)$ is the generalized congruence described in Proposition 1.8. In this framework, Σ admits a right calculus of fractions (Proposition 2.10) so we have a handy description of the category of fractions $\mathcal{C}[\Sigma^{-1}]$ (Proposition 1.3). Finally, the pureness of Σ provides a convenient characterization of identities of $\mathcal{C}/_\Sigma$.

3.9. Remark. Given a Σ-composable sequence $\overrightarrow{\gamma}:=(\gamma_n, \ldots, \gamma_0)$, an induction proves that if all the entries of $\overrightarrow{\gamma}$ are in Σ and $\overrightarrow{\gamma} \sim_m \overrightarrow{\delta}$, where $\overrightarrow{\delta}$ is another Σ-composable sequence, then all the entries of $\overrightarrow{\delta}$ are also in Σ. Indeed, for Σ is pure and stable under composition, the entries of the sequence $(\ldots, \gamma_{k+1} \circ \gamma_k, \ldots)$ are in Σ if and only if so are the ones of $(\ldots, \gamma_{k+1}, \gamma_k, \ldots)$.

Remark 3.9 can be rephrased saying that if $\overrightarrow{\gamma}$ and $\overrightarrow{\delta}$ are two $\sim_m$-equivalent Σ-composable sequences one of which possesses an element that is not in Σ, then so does the other.

3.10. Remark. Provided that $n \geq 1$, one can always, up to several $\sim^1_m$-transformations, remove from a Σ-composable sequence $\overrightarrow{\gamma} := (\gamma_n, \ldots, \gamma_0)$ an entry which is in Σ. More precisely, suppose that γ_k belongs to Σ, as suggested by diagram 4.2 where the inner commutative square comes from Theorem 3.8, we already have:

$$(\gamma_k) \sim^1_m (\mathrm{id}_y \circ \gamma_k) \sim^1_m (\mathrm{id}_y, \gamma_k) \sim^1_m \big((y \overset{\Sigma}{\to} y \vee x'), \gamma_k\big) \sim^1_m \big((y \overset{\Sigma}{\to} y \vee x') \circ \gamma_k\big) \ ,$$

moreover, since γ_k belongs to Σ so does $\big((y \overset{\Sigma}{\to} y \vee x') \circ \gamma_k\big)$ and it comes $\big((y \overset{\Sigma}{\to} y \vee x') \circ \gamma_k\big) \sim^1_m (\mathrm{id}_{y \vee x'})$, then $(\mathrm{id}_{y\vee x'}) \sim^1_m (x' \overset{\Sigma}{\to} y \vee x') \sim^1_m (\mathrm{id}_{x'})$ and finally $(\gamma_k) \sim_m (\mathrm{id}_{s(\gamma_{k+1})})$. It follows, since $(\sim_o, \sim_m)$ is a generalized congruence, that $(\ldots, \gamma_{k+1}, \gamma_k, \ldots) \sim_m$ $(\ldots, \gamma_{k+1}, \mathrm{id}_{s(\gamma_{k+1})}, \ldots) \sim_m (\ldots, \gamma_{k+1} \circ \mathrm{id}_{s(\gamma_{k+1})}, \gamma_{k-1}, \ldots)$.
In particular, the sequence $(\gamma_n, \ldots, \gamma_{k+1}, \gamma_{k-1}, \ldots, \gamma_0)$ is not empty, Σ-composable and we have $\overrightarrow{\gamma} \sim_m (\gamma_n, \ldots, \gamma_{k+1}, \gamma_{k-1}, \ldots, \gamma_0)$.

Note that remark 3.10 is still valid in the case where Σ has the right (respectively the left) extension property.

3.11. Lemma. *A Σ-composable sequence $\overrightarrow{\gamma}$ is $\sim_m$-equivalent to a sequence having for single element an identity if and only if all its elements are in Σ. In other words, the sequence $\overrightarrow{\gamma}$ represents an identity of $\mathcal{C}/_\Sigma$ if and only if all its elements are in Σ.*

Proof. If $(\sigma_n, \ldots, \sigma_0)$ is a Σ-composable sequence whose elements are in Σ, then we have $(\sigma_n, \ldots, \sigma_0) \sim_m (\sigma_0) \sim_m (\mathrm{id}_{s(\sigma_0)})$ by remark 3.10, the converse straightforwardly comes from remark 3.9. ∎

3.12. LEMMA. *Let γ be a morphism in $\mathcal{C}[x,y]$ and z be an object of $\mathcal{C}$ such that $x \sim_o z$ and $\mathcal{C}[z,y]$ is not empty, there exists a unique morphism γ' in $\mathcal{C}[x \vee z, y]$ making the diagram 3.10 commute.*

PROOF. Applying Theorem 3.8 and the fact that any element of Σ is a *Yoneda*-morphism, we construct the diagram 3.9 which provides the expected morphism γ'. ∎

$$\begin{tikzcd}
& x \arrow[dr, "\Sigma"] \arrow[drr, bend left, "\gamma"] & & \\
x \wedge z \arrow[ur, "\Sigma"] \arrow[dr, "\Sigma"'] & \text{push out} & x \vee z \arrow[r, "\gamma'"] & y \\
& z \arrow[ur, "\Sigma"'] \arrow[urr, bend right, "\gamma''"'] & &
\end{tikzcd}$$

Diagram 3.9

$$\begin{tikzcd}
x \arrow[d, "\Sigma"'] \arrow[dr, "\gamma"] & \\
x \vee z \arrow[r, "\gamma'"'] & y
\end{tikzcd}$$

Diagram 3.10

$$\begin{tikzcd}
\mathcal{C}[s(\sigma), y] \arrow[r] & \mathcal{C}[\Sigma^{-1}][x,y] \\
\gamma \arrow[r, mapsto] & I_\Sigma(\gamma)
\end{tikzcd}$$

Map 3.1

From now on and for the rest of the article, we denote the equivalence relation over $\mathcal{C}[x,y]$ defined in Proposition 1.3 by $\sim_{x,y}$.

3.13. LEMMA. *Let x and y be two objects of $\mathcal{C}[\Sigma^{-1}]$ such that $\mathcal{C}[\Sigma^{-1}][x,y]$ is not empty, then:*

1. *there exists a morphism σ of Σ such that $\mathcal{C}[s(\sigma), y]$ is not empty and $t(\sigma) = x$,*
2. *for any morphism σ in Σ such that $\mathcal{C}[s(\sigma), y]$ is not empty and $t(\sigma) = x$, the map 3.1 is a* bijection. *In particular, the functor I_Σ is* faithful.

PROOF. The first point is obvious from Propositions 1.3 and 2.10. We set $u := s(\sigma)$ and choose two morphisms γ and δ of $\mathcal{C}[u,y]$ and two morphisms τ_1 and τ_2 of $\Sigma[t,u]$ so that diagram 3.11 commutes (Proposition 1.3). Since $\tau_1 = \tau_2$ (Theorem 3.8) and τ_1 is a monomorphism (Proposition 2.4) we have $\gamma = \delta$. Let (δ, τ) represent an element of $\mathcal{C}[\Sigma^{-1}][x,y]$ and set $v := s(\tau)$, in particular we have $u \sim_o v$. By hypothesis the set $\mathcal{C}[u,y]$ is not empty, so there exists a unique morphism γ' in $\mathcal{C}[u \vee v, y]$ such that triangle $\boxed{1}$ of diagram 3.12 commutes (Lemma 3.12); triangles $\boxed{2}$ and $\boxed{3}$ also commute (Theorem 3.8) therefore diagram 3.12 commutes too. Now we set $\gamma := \gamma' \circ (u \xrightarrow{\Sigma} u \vee v)$ and thus obtain $(\delta, \tau) \sim_{x,y} (\gamma, \sigma)$. ∎

$$\begin{tikzcd}
& y & \\
u \arrow[ur, "\gamma"] \arrow[dr, "\sigma"'] & t \arrow[l, dotted, "\tau_1 \in \Sigma"'] \arrow[r, dotted, "\tau_2 \in \Sigma"] & u \arrow[ul, "\delta"'] \arrow[dl, "\sigma"] \\
& x &
\end{tikzcd}$$

Diagram 3.11

$$\begin{tikzcd}
& y \arrow[dr, phantom, "\boxed{1}"] & & u \vee v \arrow[ll, "\gamma'"'] \arrow[dr, phantom, "\boxed{2}"] & \\
v \arrow[ur, "\delta"] \arrow[urrr, "\Sigma"'] \arrow[drr, "\tau"'] & & u \wedge v \arrow[ll, dotted, "\Sigma"] \arrow[rr, dotted, "\Sigma"'] \arrow[d, phantom, "\boxed{3}"] & & u \arrow[ul, "\Sigma"'] \arrow[dll, "\sigma"] \\
& & x & &
\end{tikzcd}$$

Diagram 3.12

3.14. PROPOSITION. *Let $\mathcal{C}$ be a loop-free category and Σ be a* Yoneda-*system of $\mathcal{C}$. The isomorphisms of $\mathcal{C}[\Sigma^{-1}]$ are the morphisms of $\mathcal{C}[\Sigma^{-1}]$ which have a* return, *in addition, each of them can be written as $I_\Sigma(\sigma_2) \circ \big(I_\Sigma(\sigma_1)\big)^{-1}$ for some elements σ_1 and σ_2 of Σ. Furthermore, the collection of isomorphisms of $\mathcal{C}[\Sigma^{-1}]$ is pure in $\mathcal{C}[\Sigma^{-1}]$ and any homset of $\mathcal{C}[\Sigma^{-1}][x,y]$ which contains an isomorphism is a singleton.*

PROOF.
Let x and y be two objects of $\mathcal{C}[\Sigma^{-1}]$ and f be a morphism of $\mathcal{C}[\Sigma^{-1}][x,y]$ which has a return g. Let (γ,σ) and (γ',σ') respectively represent the morphisms f and g, the diagram 3.13 commutes and the morphisms σ, σ' and σ'' belong to Σ (Proposition 1.3). Actually γ, γ' and γ'' also belong to Σ (Theorem 3.8 and Lemma 3.6) therefore (γ,σ) represents the isomorphism $I_\Sigma(\gamma) \circ \big(I_\Sigma(\sigma)\big)^{-1}$. The collection of isomorphisms of $\mathcal{C}[\Sigma^{-1}]$ is pure for so is the collection of morphisms of $\mathcal{C}[\Sigma^{-1}]$ that have a return. Whatever γ is, the diagram 3.14 commutes provided the morphisms σ_1, σ_2 and σ_3 belong to Σ (Theorem 3.8), therefore $\mathcal{C}[\Sigma^{-1}][x,y]$ is a singleton.

$$\begin{tikzcd}
x & & \\
z_1 \arrow[u, "\sigma"] \arrow[r, "\gamma"] & y & \\
z_3 \arrow[u, dotted, "\sigma''"] \arrow[r, dotted, "\gamma''"'] & z_2 \arrow[u, "\sigma'"'] \arrow[r, "\gamma'"'] & x
\end{tikzcd}$$

Diagram 3.13

$$\begin{tikzcd}
x & & z_2 \arrow[ll, "\sigma_2"'] \arrow[dd, "\sigma_3"] \\
& z_1 \wedge z_2 \arrow[ur, "\Sigma"] \arrow[dl, "\Sigma"] & \\
z_1 \arrow[uu, "\sigma_1"] \arrow[rr, "\gamma"'] & & y
\end{tikzcd}$$

Diagram 3.14

■

4. The fibered equivalence from $\mathcal{C}[\Sigma^{-1}]$ to $\mathcal{C}/_\Sigma$

In this section, we prove the existence of a fibered equivalence from $\mathcal{C}[\Sigma^{-1}]$ to $\mathcal{C}/_\Sigma$, this fibered equivalence is indeed given by P_Σ, which is the unique functor from $\mathcal{C}[\Sigma^{-1}]$ to $\mathcal{C}/_\Sigma$ such that $Q_\Sigma = P_\Sigma \circ I_\Sigma$ (Proposition 1.8).

4.1. THEOREM. *The functor P_Σ is a* fibered equivalence*.*

4.2. COROLLARY. *Given a loop-free category $\mathcal{C}$, a* Yoneda-*system Σ over $\mathcal{C}$ and two objects x and y of $\mathcal{C}$ such that $\mathcal{C}[x,y]$ is not empty, the following map is a bijection. In particular, the functor Q_Σ is* faithful.

$$\begin{array}{ccc} \mathcal{C}[x,y] & \longrightarrow & \mathcal{C}/_\Sigma[Q_\Sigma(x), Q_\Sigma(y)] \\ \gamma & \longmapsto & Q_\Sigma(\gamma) \end{array}$$

PROOF. Follows from $Q_\Sigma = P_\Sigma \circ I_\Sigma$, Lemma 3.13 and Theorem 4.1. ■

4.3. COROLLARY. [Quotient of a loop-free category by a *Yoneda*-system] *If $\mathcal{C}$ is a loop-free category and Σ a* Yoneda-*system of $\mathcal{C}$, then the quotient category $\mathcal{C}/_\Sigma$ is loop-free. Besides, if the category $\mathcal{C}$ is a poset, that is to say $\mathcal{C}$ is loop-free and for all objects x and y of $\mathcal{C}$, the set $\mathcal{C}[x,y]$ contains at most one element, then $\mathcal{C}/_\Sigma$ is also a poset. In particular, the category of components of a loop-free category (respectively a poset) is also loop-free (respectively a poset).*

PROOF. Given two objects a and b of $\mathcal{C}/_{\Sigma}$ such that neither $\mathcal{C}/_{\Sigma}[a,b]$ nor $\mathcal{C}/_{\Sigma}[b,a]$ are empty, we have two objects x and y of $\mathcal{C}$ whose image by P_{Σ} are respectively a and b and neither $\mathcal{C}[\Sigma^{-1}][x,y]$ nor $\mathcal{C}[\Sigma^{-1}][y,x]$ are empty (Theorem 4.1). It follows from Proposition 3.14 that $\mathcal{C}[\Sigma^{-1}][x,y]$ has a single element which can be written as $I_{\Sigma}(\sigma_2)\circ\big(I_{\Sigma}(\sigma_1)\big)^{-1}$ for some elements σ_1 and σ_2 of Σ and therefore $\mathcal{C}/_{\Sigma}[a,b]$ has a single element which is

$$P_{\Sigma}\Big(I_{\Sigma}(\sigma_2)\circ\big(I_{\Sigma}(\sigma_1)\big)^{-1}\Big) = Q_{\Sigma}(\sigma_2)\circ\big(Q_{\Sigma}(\sigma_1)\big)^{-1} = \mathrm{id}_a = \mathrm{id}_b\ .$$

Furthermore, if we suppose that all the homsets of $\mathcal{C}$ contain at most one element, then so are the homsets of $\mathcal{C}[\Sigma^{-1}]$ (Lemma 3.13) and hence, by Theorem 4.1, so are the homsets of $\mathcal{C}/_{\Sigma}$. ■

4.4. COROLLARY. [Quotients and Skeleta] *Given a* Yoneda-*system of a loop-free category $\mathcal{C}$, the quotient category $\mathcal{C}/_{\Sigma}$ is isomorphic to the skeleton of $\mathcal{C}[\Sigma^{-1}]$.*

PROOF. The category $\mathcal{C}/_{\Sigma}$ is skeletal by Corollary 4.3 and equivalent to $\mathcal{C}[\Sigma^{-1}]$ by Theorem 4.1. ■

4.5. COROLLARY. *Given a* Yoneda-*system Σ over a loop-free category $\mathcal{C}$, the group* $\mathsf{Aut}(\mathcal{C}/_{\Sigma})$ freely *and* transitively *acts (on the left) on the set of equivalences from $\mathcal{C}[\Sigma^{-1}]$ to $\mathcal{C}/_{\Sigma}$ and all the equivalence from $\mathcal{C}[\Sigma^{-1}]$ to $\mathcal{C}/_{\Sigma}$ are* fibered.

PROOF. Given a skeletal category $\mathcal{S}$, the category of functors from any category $\mathcal{C}$ to $\mathcal{S}$ is still skeletal, the action is thus free and transitive since the category $\mathcal{C}/_{\Sigma}$ is skeletal (Corollary 4.4). Moreover, any equivalence whose codomain is skeletal is obviously fibered. ■

Given a category $\mathcal{C}$, we put $x \sqsubseteq y$ for $\mathcal{C}[x,y] \neq \emptyset$; in the case where $\mathcal{C}$ is loop-free, $\sqsubseteq$ is an order relation over the collection $|\mathcal{C}|$ of objects of $\mathcal{C}$. The next corollary describes the image of the morphism of groups given by Proposition 2.8.

4.6. COROLLARY. *Given a loop-free category $\mathcal{C}$ and an autofunctor Ψ of $\mathcal{C}/_{\overline{\Sigma}}$, there is an autofunctor Φ of $\mathcal{C}$ such that $Q_{\overline{\Sigma}}\circ\Phi = \Psi\circ Q_{\overline{\Sigma}}$ if and only if there is an automorphism ϕ of $(|\mathcal{C}|,\sqsubseteq)$ such that for all objects x and y of $\mathcal{C}$ and for all $\overline{\Sigma}$-component K, if x and y belong to K then $\phi(x)$ and $\phi(y)$ belong to $\Psi(K)$.*

PROOF. Suppose we have ϕ as in the statement of Corollary 4.6, for any object x of $\mathcal{C}$, we put $\Phi(x) := \phi(x)$. Furthermore, if $\mathcal{C}[x,y]$ is not empty, then the map $\xi_{x,y}$ from $\mathcal{C}[x,y]$ to $\big(\mathcal{C}/_{\overline{\Sigma}}\big)[Q_{\overline{\Sigma}}(x),Q_{\overline{\Sigma}}(y)]$ given by Corollary 4.2 is a bijection.

So, given a morphism γ in $\mathcal{C}[x,y]$, we set $\Phi(\gamma) := \xi^{-1}_{\phi(x),\phi(y)}\Big(\Psi\big(\xi_{x,y}(\gamma)\big)\Big)$ thus defining the expected automorphism of $\mathcal{C}$. The converse is obvious. ■

The rest of the section is devoted to intermediate results leading to the proof of Theorem 4.1.

4.7. Definition. *Given a non empty sequence* $(\gamma_n, \dots, \gamma_0)$ *of morphisms of* $\mathcal{C}$*, we write:* • $(\gamma_n, \dots, \gamma_{k+1}, \gamma_k, \gamma_{k-1}, \dots \gamma_0) {\sim'}^1_m (\gamma_n, \dots, \gamma_{k+1}, \gamma'_k, \gamma_{k-1}, \dots \gamma_0)$ *when there exists some* k *in* $\{0, \dots, n\}$ *and some morphism* γ'_k *in* $\mathcal{C}$ *such that the diagram 4.1 commutes (*$x \wedge x'$ *and* $y \vee y'$ *refer to Theorem 3.8), and* • $(\gamma_n, \dots, \gamma_{k+2}, \gamma_{k+1}, \gamma_k, \gamma_{k-1}, \dots, \gamma_0) {\sim'}^1_m (\gamma_n, \dots, \gamma_{k+2}, \gamma_{k+1} \circ \gamma_k, \gamma_{k-1}, \dots, \gamma_0)$ *when there exists some* k *in* $\{0, \dots, n-1\}$ *such that* $t(\gamma_k) = s(\gamma_{k+1})$*.*

$$\begin{tikzcd}
x \arrow[r, "\gamma_k"] & y \arrow[r, "\Sigma"] & y \vee y' \\
x \wedge x' \arrow[u, "\Sigma"] \arrow[r, "\Sigma"'] & x' \arrow[r, "\gamma'_k"'] & y' \arrow[u, "\Sigma"']
\end{tikzcd}$$

Diagram 4.1

$$\begin{tikzcd}
x \arrow[r, "\gamma_k"] & y \arrow[r, "\Sigma"] & y \vee x' & \\
& y \wedge x' \arrow[u, "\Sigma"] \arrow[r, "\Sigma"'] & x' \arrow[u, "\Sigma"'] \arrow[r, "\gamma_{k+1}"'] & y'
\end{tikzcd}$$

Diagram 4.2

Since Σ is pure in $\mathcal{C}$ (Lemma 3.6) and stable under composition, in either case, if one of the sequences on either side of ${\sim'}^1_m$ is Σ-composable (respectively Σ-composable *normalized*), then so is the other. Given two sequences of morphisms of $\mathcal{C}$, $\overrightarrow{\gamma}$ and $\overrightarrow{\delta}$ for instance, we write $\overrightarrow{\gamma} {\sim'}_m \overrightarrow{\delta}$ when there exists a sequence $(\overrightarrow{\gamma_0}, \dots, \overrightarrow{\gamma_K}, \dots, \overrightarrow{\gamma_N})$ of sequences of morphisms of $\mathcal{C}$ such that $\overrightarrow{\gamma_0} = \overrightarrow{\gamma}$, $\overrightarrow{\gamma_N} = \overrightarrow{\delta}$ and for all K in $\{0, \dots, N-1\}$ we have $\overrightarrow{\gamma_K} \, {\sim'}^1_m \, \overrightarrow{\gamma_{K+1}}$. Such a sequence $(\overrightarrow{\gamma_0}, \dots, \overrightarrow{\gamma_K}, \dots, \overrightarrow{\gamma_N})$ is called a sequence of ${\sim'}^1_m$-transformations. Given a Σ-composable sequence $\overrightarrow{\gamma}$ an element of which is not in Σ, we denote by $\overrightarrow{\gamma}^\times$ the subsequence of $\overrightarrow{\gamma}$ obtained by removing all the entries of $\overrightarrow{\gamma}$ that belong to Σ.

4.8. Lemma. *Let* $\overrightarrow{\gamma}$ *and* $\overrightarrow{\delta}$ *be two* Σ*-composable sequences having* at least *one element which is not in* Σ*, then the following statements are satisfied:*

1. *the sequence* $\overrightarrow{\gamma}^\times$ *is* Σ*-composable and normalized,*
2. $\overrightarrow{\gamma} \sim_m \overrightarrow{\gamma}^\times$*,*
3. *if* $\overrightarrow{\gamma} \sim^1_m \overrightarrow{\delta}$*, then* $\overrightarrow{\gamma}^\times {\sim'}^1_m \overrightarrow{\delta}^\times$ *or* $\overrightarrow{\gamma}^\times = \overrightarrow{\delta}^\times$ *and*
4. *if* $\overrightarrow{\gamma}^\times {\sim'}^1_m \overrightarrow{\delta}^\times$*, then* $\overrightarrow{\gamma} \sim_m \overrightarrow{\delta}$*.*

In particular we have $\overrightarrow{\gamma} \sim_m \overrightarrow{\delta}$ *if and only if* $\overrightarrow{\gamma}^\times {\sim'}_m \overrightarrow{\delta}^\times$*.*

Proof. Since one of the elements of $\overrightarrow{\gamma}$ is not in Σ, the sequence $\overrightarrow{\gamma}^\times$ is not empty, moreover the sequence $\overrightarrow{\gamma}^\times$ is Σ-composable and satisfies $\overrightarrow{\gamma} \sim_m \overrightarrow{\gamma}^\times$ (remark 3.10). For Σ is pure in $\mathcal{C}$ (Lemma 3.6) and stable under composition (Definition 2.5), we have the third point. If diagram 4.1 commutes, then $(\gamma_k) \sim_m (\gamma'_k)$, the fourth point follows because $\sim$ is a generalized congruence. ∎

4.9. Lemma. *Let* γ*,* γ'*,* δ *and* δ' *be four morphisms respectively taken from* $\mathcal{C}[y,z]$*,* $\mathcal{C}[y',z']$*,* $\mathcal{C}[x,y]$ *and* $\mathcal{C}[x',y']$*. If* $x \sim_o x'$*,* $y \sim_o y'$*,* $z \sim_o z'$ *and the diagrams 4.3 and 4.4 commute,* *then so does the diagram 4.5.*

$$\begin{tikzcd} x \arrow[r, "\delta"] & y \arrow[r, "\Sigma"] & y \vee y' \\ x \wedge x' \arrow[u, "\Sigma"] \arrow[r, "\Sigma"'] & x' \arrow[r, "\delta'"'] & y' \arrow[u, "\Sigma"'] \end{tikzcd}$$

Diagram 4.3

$$\begin{tikzcd} y \arrow[r, "\gamma"] & z \arrow[r, "\Sigma"] & z \vee z' \\ y \wedge y' \arrow[u, "\Sigma"] \arrow[r, "\Sigma"'] & y' \arrow[r, "\gamma'"'] & z' \arrow[u, "\Sigma"'] \end{tikzcd}$$

Diagram 4.4

$$\begin{tikzcd} x \arrow[r, "\gamma\circ\delta"] & z \arrow[r, "\Sigma"] & z \vee z' \\ x \wedge x' \arrow[u, "\Sigma"] \arrow[r, "\Sigma"'] & x' \arrow[r, "\gamma'\circ\delta'"'] & z' \arrow[u, "\Sigma"'] \end{tikzcd}$$

Diagram 4.5

PROOF. Since diagram 4.3 commutes and diagram 4.7 is a pull-back square (Theorem 3.8), there exists a unique morphism δ'' in $\mathcal{C}[x \wedge x', y \wedge y']$ making the diagram 4.6 commute. The same way, since diagram 4.7 is a push-out square (Theorem 3.8), there exists a unique morphism γ'' in $\mathcal{C}[y \vee y', z \vee z']$ making the diagram 4.8 commute and the expected result follows. ∎

$$\begin{tikzcd} & x \arrow[rr, "\delta"] & & y \arrow[dr, "\Sigma"] & \\ x \wedge x' \arrow[ur, "\Sigma"] \arrow[rr, "\delta''"] \arrow[dr, "\Sigma"'] & & y \wedge y' \arrow[ur, "\Sigma"] \arrow[dr, "\Sigma"'] & & y \vee y' \\ & x' \arrow[rr, "\delta'"'] & & y' \arrow[ur, "\Sigma"'] & \end{tikzcd}$$

Diagram 4.6

$$\begin{tikzcd} & y \arrow[dr, "\Sigma"] & \\ y \wedge y' \arrow[ur, "\Sigma"] \arrow[dr, "\Sigma"'] & & y \vee y' \\ & y' \arrow[ur, "\Sigma"'] & \end{tikzcd}$$

Diagram 4.7

$$\begin{tikzcd} & y \arrow[rr, "\gamma"] \arrow[dr, "\Sigma"] & & z \arrow[dr, "\Sigma"] & \\ y \wedge y' \arrow[ur, "\Sigma"] \arrow[dr, "\Sigma"'] & & y \vee y' \arrow[rr, "\gamma''"] & & z \vee z' \\ & y' \arrow[rr, "\gamma'"'] \arrow[ur, "\Sigma"'] & & z' \arrow[ur, "\Sigma"'] & \end{tikzcd}$$

Diagram 4.8

4.10. LEMMA. *Let two morphisms γ and γ' respectively taken from $\mathcal{C}[x, y]$ and in $\mathcal{C}[x', y']$ where x, x', y and y' are objects of $\mathcal{C}$ such that $x \sim_o x'$ and $y \sim_o y'$, then the diagram 4.9 is commutative if and only if there exists two objects x'' and y'' of $\mathcal{C}$ such that the sets $\Sigma[x'', x]$, $\Sigma[x'', x']$, $\Sigma[y, y'']$ and $\Sigma[y', y'']$ are not empty and the diagram 4.10 commutes.*

$$\begin{tikzcd} x \arrow[r, "\gamma"] & y \arrow[r, dashed, "\Sigma"] & y \vee y' \\ x \wedge x' \arrow[u, dashed, "\Sigma"] \arrow[r, dashed, "\Sigma"'] & x' \arrow[r, "\gamma'"'] & y' \arrow[u, dashed, "\Sigma"'] \end{tikzcd}$$

Diagram 4.9

$$\begin{tikzcd} x \arrow[r, "\gamma"] & y \arrow[r, dashed, "\Sigma"] & y'' \\ x'' \arrow[u, dashed, "\Sigma"] \arrow[r, dashed, "\Sigma"'] & x' \arrow[r, "\gamma'"'] & y' \arrow[u, dashed, "\Sigma"'] \end{tikzcd}$$

Diagram 4.10

PROOF. Suppose there exist x'' and y'' such that diagram 4.10 commutes, by Theorem 3.8 we have diagram 4.11 in which the dotted triangles as well as, by hypothesis, the outer shape, commute. Moreover, the morphisms $x'' \overset{\Sigma}{\dashrightarrow} x \wedge x'$ and $y \vee y' \overset{\Sigma}{\dashrightarrow} y''$ are bimorphisms (Proposition 2.4) so the inner rectangle of diagram 4.11 also commutes. ∎

$$\begin{tikzcd}
 & & & & y'' \\
 & x \arrow[r, "\gamma"] & y \arrow[r, dotted, "\Sigma"'] \arrow[urr, dotted, "\Sigma"] & y \vee y' \arrow[ur, dotted, "\Sigma"] & \\
 & x \wedge x' \arrow[u, dotted, "\Sigma"'] \arrow[r, dotted, "\Sigma"] & x' \arrow[r, "\gamma'"'] & y' \arrow[u, dotted, "\Sigma"] \arrow[uur, dotted, "\Sigma"'] & \\
x'' \arrow[ur, dotted, "\Sigma"] \arrow[urr, dotted, "\Sigma"'] \arrow[uur, dotted, "\Sigma"] & & & &
\end{tikzcd}$$

Diagram 4.11

$$\begin{tikzcd}
x \arrow[rr, "\gamma"] & & y \arrow[r, dotted, "\Sigma"] & y \vee y' \arrow[r, dotted, "\Sigma"] & y \vee y' \vee y'' \\
x \wedge x' \arrow[u, dotted, "\Sigma"] \arrow[r, dotted, "\Sigma"] \arrow[urrr, phantom, "\boxed{1}"] & x' \arrow[rr, "\gamma'"] & & y' \arrow[u, dotted, "\Sigma"] \arrow[r, dotted, "\Sigma"'] \arrow[ur, phantom, "\boxed{4}"] & y' \vee y'' \arrow[u, dotted, "\Sigma"'] \\
x \wedge x' \wedge x'' \arrow[u, dotted, "\Sigma"] \arrow[r, dotted, "\Sigma"'] \arrow[ur, phantom, "\boxed{3}"] & x' \wedge x'' \arrow[u, dotted, "\Sigma"'] \arrow[r, dotted, "\Sigma"'] \arrow[urrr, phantom, "\boxed{2}"] & x'' \arrow[rr, "\gamma''"'] & & y'' \arrow[u, dotted, "\Sigma"']
\end{tikzcd}$$

Diagram 4.12

4.11. Lemma. *Given three morphisms γ, γ' and γ'' respectively taken from $\mathcal{C}[x,y]$, $\mathcal{C}[x',y']$ and $\mathcal{C}[x'',y'']$ where x, x', x'', y, y' and y'' are objects of $\mathcal{C}$ such that $x \sim_o x'$, $x' \sim_o x''$, $y \sim_o y'$ and $y' \sim_o y''$; if the diagrams 4.13 and 4.14 commute, then so does the diagram 4.15.*

$$\begin{tikzcd}
x \arrow[r, "\gamma"] & y \arrow[r, "\Sigma"] & y \vee y' \\
x \wedge x' \arrow[u, "\Sigma"] \arrow[r, "\Sigma"'] \arrow[urr, phantom, "\boxed{1}"] & x' \arrow[r, "\gamma'"'] & y' \arrow[u, "\Sigma"']
\end{tikzcd}
\qquad
\begin{tikzcd}
x' \arrow[r, "\gamma'"] & y' \arrow[r, "\Sigma"] & y' \vee y'' \\
x' \wedge x'' \arrow[u, "\Sigma"] \arrow[r, "\Sigma"'] \arrow[urr, phantom, "\boxed{2}"] & x'' \arrow[r, "\gamma''"'] & y'' \arrow[u, "\Sigma"']
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, "\gamma"] & y \arrow[r, "\Sigma"] & y \vee y'' \\
x \wedge x'' \arrow[u, "\Sigma"] \arrow[r, "\Sigma"'] & x'' \arrow[r, "\gamma''"'] & y'' \arrow[u, "\Sigma"']
\end{tikzcd}$$

Diagram 4.13 *Diagram 4.14* *Diagram 4.15*

Proof. The rectangles $\boxed{1}$ and $\boxed{2}$ (on diagram 4.12) commute by hypothesis and the commutative rectangles $\boxed{3}$ and $\boxed{4}$ (on the same diagram) are given by thorem 3.8, we conclude by applying Lemma 4.10 to the outer shape of diagram 4.12. ∎

Now we come to the most technical fact of the paper though the underlying ideas remain simple and rather easy to grasp in any "concrete" example.

4.12. Lemma. [Interpolation] *Let $x, y, x_0, \ldots, x_n, y_0, \ldots, y_n$ be objects of $\mathcal{C}$. If the statements* **a**, **b** *and* **c** *are satisfied, then there exists a sequence of objects $z_1, \ldots, z_n$ of $\mathcal{C}$ satisfying the statements* **d** *and* **e**.

a $x \sqsubseteq y$, $x \sim_o x_0$, $y_n \sim_o y$,

b $\forall k \in \{1, \ldots, n\},\ y_{k-1} \sim_o x_k$,

c $\forall k \in \{0, \ldots, n\},\ x_k \sqsubseteq y_k$,

d $x \sqsubseteq z_1 \sqsubseteq \ldots \sqsubseteq z_n \sqsubseteq y$ *and*

e $\forall k \in \{1, \ldots, n\},\ z_k \sim_o x_k$

Diagramatically, if

$$\begin{tikzcd}
x_0 \arrow[r, "\sqsubseteq"] \arrow[d, squiggly, no head] & y_0 \arrow[dd, squiggly, no head] & x_2 \arrow[r, "\sqsubseteq"] \arrow[dd, squiggly, no head] & y_2 \arrow[dd, squiggly, no head] & x_4 \arrow[dd, squiggly, no head] & \cdots & x_{n-1} \arrow[r, "\sqsubseteq"] \arrow[dd, squiggly, no head] & y_{n-1} \arrow[dd, squiggly, no head] & \\
x \arrow[rrrrrrrr, "\sqsubseteq"] & & & & & & & & y \arrow[d, squiggly, no head] \\
 & x_1 \arrow[r, "\sqsubseteq"'] & y_1 & x_3 \arrow[r, "\sqsubseteq"'] & y_3 & \cdots & y_{n-2} & x_n \arrow[r, "\sqsubseteq"'] & y_n
\end{tikzcd}$$

then we have a sequence $z_1, \dots, z_n$ of objects of $\mathcal{C}$ such that

$$\begin{tikzcd}
x_0 \arrow[r, "\sqsubseteq"] \arrow[d, no head, squiggly] & y_0 \arrow[d, no head, squiggly] & x_2 \arrow[r, "\sqsubseteq"] \arrow[d, no head, squiggly] & y_2 \arrow[d, no head, squiggly] & x_4 \arrow[d, no head, squiggly] & \cdots & x_{n-1} \arrow[r, "\sqsubseteq"] \arrow[d, no head, squiggly] & y_{n-1} \arrow[d, no head, squiggly] & \\
x \arrow[r, "\sqsubseteq"] & z_1 \arrow[r, "\sqsubseteq"] \arrow[d, no head, squiggly] & z_2 \arrow[r, "\sqsubseteq"] \arrow[d, no head, squiggly] & z_3 \arrow[r, "\sqsubseteq"] \arrow[d, no head, squiggly] & z_4 \arrow[d, no head, squiggly] & & z_{n-1} \arrow[r, "\sqsubseteq"] \arrow[d, no head, squiggly] & z_n \arrow[r, "\sqsubseteq"] \arrow[d, no head, squiggly] & y \arrow[d, no head, squiggly] \\
 & x_1 \arrow[r, "\sqsubseteq"'] & y_1 & x_3 \arrow[r, "\sqsubseteq"'] & y_3 & \cdots & y_{n-2} & x_n \arrow[r, "\sqsubseteq"'] & y_n
\end{tikzcd}$$

where $\rightsquigarrow$ *represents* $\sim_o$ *and* $x \sqsubseteq y$ *means the homset* $\mathcal{C}[x,y]$ *is not empty.*

Proof. All along this proof, the symbols $\wedge$ and $\vee$ refer to Theorem 3.8. We choose a morphism γ_k in $\mathcal{C}[x_k, y_k]$ for each k in $\{0, \dots, n\}$ and set $a_0 := x \wedge x_0$ and $a_k := y_{k-1} \wedge x_k$ for each k in $\{1, \dots, n\}$.

For every k in $\{0, \dots, n\}$ we denote by σ_k the single element of $\Sigma[a_k, x_k]$. Finally we set $\delta_k := \gamma_k \circ \sigma_k$ for each k in $\{0, \dots, n\}$.

Then we recursively construct the push-out squares $\boxed{0}, \boxed{1}, \boxed{2}, \dots, \boxed{n}$ as indicated below.

$$\begin{tikzcd}
x \arrow[r, "\gamma'_0"] \arrow[dr, phantom, "\boxed{0}"] & x'_1 \arrow[r, "\gamma'_1"] \arrow[dr, phantom, "\boxed{1}"] & x'_2 \arrow[r, "\gamma'_2"] \arrow[ddr, phantom, "\boxed{2}"] & x'_3 \arrow[r, "\gamma'_3"] \arrow[ddddr, phantom, "\boxed{3}"] & x'_4 & \cdots & x'_{n-1} \arrow[r, "\gamma'_{n-1}"] \arrow[dddddr, phantom, "\boxed{\text{n-1}}"] & x'_n \arrow[r, "\gamma'_n"] \arrow[ddddddr, phantom, "\boxed{\text{n}}"] & x'_{n+1} \\
a_0 \arrow[u, "\Sigma"] \arrow[r, "\delta_0"'] & y_0 \arrow[u, "\Sigma"'] & & & & & & & \\
 & a_1 \arrow[u, "\Sigma"'] \arrow[r, "\delta_1"'] & y_1 \arrow[uu, "\Sigma"'] & & & & & & \\
 & & a_2 \arrow[u, "\Sigma"'] \arrow[r, "\delta_2"'] & y_2 \arrow[uuu, "\Sigma"'] & & & & & \\
 & & & a_3 \arrow[u, "\Sigma"'] \arrow[r, "\delta_3"'] & y_3 \arrow[uuuu, "\Sigma"'] & & & & \\
 & & & & & & a_{n-1} \arrow[uuuuu, "\Sigma"] \arrow[r, "\delta_{n-1}"'] & y_{n-1} \arrow[uuuuu, "\Sigma"'] & \\
 & & & & & & & a_n \arrow[u, "\Sigma"'] \arrow[r, "\delta_n"'] & y_n \arrow[uuuuuu, "\Sigma"']
\end{tikzcd}$$

Since $x'_{n+1} \sim_o y$, we can also recursively construct the pull-back squares $\boxed{0}'$, $\boxed{1}'$, $\boxed{2}'$, $\dots \boxed{\text{n-1}}'$, $\boxed{\text{n}}'$ as indicated below.

$$\begin{tikzcd}
z_0 \arrow[r, "\gamma''_0"] \arrow[d, "\Sigma"'] \arrow[dr, phantom, "\boxed{\text{n}}'"] & z_1 \arrow[r, "\gamma''_1"] \arrow[d, "\Sigma"'] \arrow[dr, phantom, "\boxed{\text{n-1}}'"] & z_2 \arrow[r, "\gamma''_2"] \arrow[d, "\Sigma"'] \arrow[dr, phantom, "\boxed{\text{n-2}}'"] & z_3 \arrow[d, "\Sigma"] & \cdots & z_{n-1} \arrow[r, "\gamma''_{n-1}"] \arrow[d, "\Sigma"'] \arrow[dr, phantom, "\boxed{1}'"] & z_n \arrow[r, "\gamma''_n"] \arrow[d, "\Sigma"'] \arrow[dr, phantom, "\boxed{0}'"] & x'_{n+1} \wedge y \arrow[d, "\Sigma"] \\
x \arrow[r, "\gamma'_0"'] & x'_1 \arrow[r, "\gamma'_1"'] & x'_2 \arrow[r, "\gamma'_2"'] & x'_3 & \cdots & x'_{n-1} \arrow[r, "\gamma'_{n-1}"'] & x'_n \arrow[r, "\gamma'_n"'] & x'_{n+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
x'_{n+1} \wedge y \arrow[r, "\Sigma"] \arrow[d, "\Sigma"'] & y \arrow[d, "\Sigma"] \\
x'_{n+1} \arrow[r, "\Sigma"'] & x'_{n+1} \vee y
\end{tikzcd}$$

We paste these squares to obtain the pull-back depicted on diagram 4.16. Since the arrow $y \xrightarrow{\Sigma} x'_{n+1} \vee y$ is a *Yoneda*-morphism and $\mathcal{C}[x,y]$ is not empty, there exists a unique morphism δ in $\mathcal{C}[x,y]$ such that diagram 4.17 commutes.

$$\begin{tikzcd}
z_0 \arrow[r, "\gamma''_n\circ\cdots\circ\gamma''_0"] \arrow[d, "\Sigma"'] & x'_{n+1}\wedge y \arrow[r, "\Sigma"] & y \arrow[d, "\Sigma"] \\
x \arrow[r, "\gamma'_n\circ\cdots\circ\gamma'_0"'] & x'_{n+1} \arrow[r, "\Sigma"'] & x'_{n+1}\vee y
\end{tikzcd}$$

Diagram 4.16

$$\begin{tikzcd}
 & & y \arrow[d, "\Sigma"] \\
x \arrow[urr, "\delta"] \arrow[r, "\gamma'_n\circ\cdots\circ\gamma'_0"'] & x'_{n+1} \arrow[r, "\Sigma"'] & x'_{n+1}\vee y
\end{tikzcd}$$

Diagram 4.17

Then we apply the universal property of pull-back squares to obtain the unique morphism ξ of $\mathcal{C}[x,z_0]$ that makes the diagram below commute.

$$\begin{tikzcd}
x \arrow[dr, "\xi"] \arrow[drrr, bend left, "\delta"] \arrow[ddr, bend right, "\mathrm{id}_x"'] & & & \\
 & z_0 \arrow[r, "\gamma''_n\circ\cdots\circ\gamma''_0"] \arrow[d, "\Sigma"'] & x'_{n+1}\wedge y \arrow[r, "\Sigma"] & y \arrow[d, "\Sigma"] \\
 & x \arrow[r, "\gamma'_n\circ\cdots\circ\gamma'_0"'] & x'_{n+1} \arrow[r, "\Sigma"'] & x'_{n+1}\vee y
\end{tikzcd}$$

In particular we have $x \sqsubseteq z_0 \sqsubseteq x$ which implies, since $\mathcal{C}$ is loop-free, that $\xi = \mathrm{id}_x$. Finally, it comes

$$\begin{tikzcd}
x \arrow[r, "\gamma''_0"] & z_1 \arrow[r, "\gamma''_1"] \arrow[d, "\Sigma"'] & z_2 \arrow[r, "\gamma''_2"] \arrow[d, "\Sigma"'] & z_3 \arrow[d, "\Sigma"'] & \cdots & z_{n-1} \arrow[r, "\gamma''_{n-1}"] \arrow[d, "\Sigma"] & z_n \arrow[r, "\gamma''_n"] \arrow[d, "\Sigma"] & x'_{n+1}\wedge y \arrow[r, "\Sigma"] \arrow[d, "\Sigma"] & y \\
 & x'_1 & x'_2 & x'_3 & \cdots & x'_{n-1} & x'_n & x'_{n+1} & \\
 & y_0 \arrow[u, "\Sigma"] & y_1 \arrow[u, "\Sigma"] & y_2 \arrow[u, "\Sigma"] & \cdots & y_{n-2} \arrow[u, "\Sigma"'] & y_{n-1} \arrow[u, "\Sigma"'] & y_n \arrow[u, "\Sigma"'] & \\
 & a_1 \arrow[u, "\Sigma"] & a_2 \arrow[u, "\Sigma"] & a_3 \arrow[u, "\Sigma"] & \cdots & a_{n-1} \arrow[u, "\Sigma"'] & a_n \arrow[u, "\Sigma"'] & &
\end{tikzcd}$$

and $z_1,\dots,z_n$ is the expected sequence of objects. ∎

4.13. Lemma. [Translation] *Let γ be a morphism in $\mathcal{C}[x,y]$ and two objects x' and y' of $\mathcal{C}$ such that $x' \sim_o x$, $y' \sim_o y$ and the homset $\mathcal{C}[x',y']$ is not empty. There exists a* unique *morphism γ' in $\mathcal{C}[x',y']$ such that the diagram 4.18 commutes.*

$$\begin{tikzcd}
x \arrow[r, "\gamma"] & y \arrow[r, "\Sigma"] & y\vee y' \\
x\wedge x' \arrow[u, "\Sigma"] \arrow[r, "\Sigma"'] & x' \arrow[r, "\gamma'"'] & y' \arrow[u, "\Sigma"']
\end{tikzcd}$$

Diagram 4.18

$$\begin{tikzcd}
 & y\vee y' \\
x\wedge x' \arrow[ur, "\gamma_0"] \arrow[r, "\gamma_1"'] & y' \arrow[u, "\Sigma"']
\end{tikzcd}$$

Diagram 4.19

$$\begin{tikzcd}
x\wedge x' \arrow[r, "\Sigma"'] \arrow[rr, bend left, "\gamma_1"] & x' \arrow[r, "\gamma'"'] & y'
\end{tikzcd}$$

Diagram 4.20

Proof. Denote by γ_0 the following composite of morphisms $x \wedge x' \xrightarrow{\Sigma} x \xrightarrow{\gamma} y \xrightarrow{\Sigma} y \vee y'$. Since the arrows $y \xrightarrow{\Sigma} y \vee y'$ and $x \wedge x' \xrightarrow{\Sigma} x'$ are *Yoneda*-morphisms, there exist a *unique* morphism γ_1 in $\mathcal{C}[x \wedge x', y']$ that makes the diagram 4.19 commute and a *unique* morphism γ' in $\mathcal{C}[x', y']$ that makes the diagram 4.20 commute. ∎

As suggested by the terminology of linear algebra and the dotted segments on diagram 4.18, we say that we **translate** γ (to γ') along x, x' and y, y'. We also say the translation of γ along x, x' and y, y' is γ'.

4.14. Lemma. [Assembly] *If $(\gamma_n, \ldots, \gamma_0)$ is a Σ-composable sequence whose elements are respectively taken from $\mathcal{C}[x_k, y_k]$ (for $0 \leq k \leq n$) and x, y are two objects of $\mathcal{C}$ such that, $x \sim_o x_0$, $y \sim_o y_n$ and the homset $\mathcal{C}[x, y]$ is not empty, then we have a* composable *sequence $(\zeta_n, \ldots, \zeta_0)$ such that $s(\zeta_0) = x$, $t(\zeta_n) = y$ and for each k in $\{0, \ldots, n\}$, the morphism ζ_k belongs to $\mathcal{C}[z_k, z_{k+1}]$ where z_k and z_{k+1} are objects of $\mathcal{C}$ such that $x_k \sim_o z_k$, $y_k \sim_o z_{k+1}$ and the diagram 4.21 commutes.*

$$\begin{tikzcd}
x_k \arrow[r, "\gamma_k"] & y_k \arrow[r, "\Sigma"] & y_k \vee z_{k+1} \\
x_k \wedge z_k \arrow[u, "\Sigma"] \arrow[r, "\Sigma"'] & z_k \arrow[r, "\zeta_k"'] & z_{k+1} \arrow[u, "\Sigma"']
\end{tikzcd}
\qquad
\begin{tikzcd}
x_0 \arrow[r, "\gamma_0"] & y_0 \arrow[r, "\Sigma"] & y_0 \vee y \\
x_0 \wedge x \arrow[u, "\Sigma"] \arrow[r, "\Sigma"'] & x \arrow[r, "\zeta_0"'] & y \arrow[u, "\Sigma"']
\end{tikzcd}
\qquad
\begin{tikzcd}
x_k \arrow[r, "\gamma_k"] & y_k \arrow[r, "\Sigma"] & y_k \vee y'_k \\
x_k \wedge x'_k \arrow[u, "\Sigma"] \arrow[r, "\Sigma"'] & x'_k \arrow[r, "\gamma'_k"'] & y'_k \arrow[u, "\Sigma"']
\end{tikzcd}$$

Diagram 4.21 *Diagram 4.22* *Diagram 4.23*

Moreover, if $(\zeta'_n, \ldots, \zeta'_0)$ is another such composable sequence, then $\zeta'_n \circ \cdots \circ \zeta'_0 = \zeta_n \circ \cdots \circ \zeta_0$.

Proof. There is a sequence $z_1, \ldots, z_n$ of objects of $\mathcal{C}$ such that for each k in $\{1, \ldots, n\}$, $z_k \sim_o x_k$ and $x \sqsubseteq z_1 \sqsubseteq \ldots \sqsubseteq z_n \sqsubseteq y$ (interpolation Lemma 4.12). We extend this sequence setting $z_0 := x$ and $z_{n+1} := y$. For each k in $\{0, \ldots, n\}$ we translate γ_k to ζ_k along x_k, z_k and y_k, z_{k+1} (diagram 4.21 and translation Lemma 4.13). The composable sequence $(\zeta_n, \ldots, \zeta_0)$ has the expected properties. Besides, if $(\zeta'_n, \ldots, \zeta'_0)$ is another such sequence, we prove that $\zeta'_n \circ \cdots \circ \zeta'_0 = \zeta_n \circ \cdots \circ \zeta_0$ by recursively applying Lemma 4.9. ∎

Given a Σ-composable sequence $\overrightarrow{\gamma}$, any sequence $\overrightarrow{\zeta}$ satisfying the conclusions of Lemma 4.14 is called an **assembly** of $\overrightarrow{\gamma}$ from x to y. Thus, Lemma 4.14 allows us to define the **value** of $\overrightarrow{\gamma}$ from x to y as $\mathrm{Val}_{x,y}(\overrightarrow{\gamma}) := \zeta_n \circ \cdots \circ \zeta_0$ where $\overrightarrow{\zeta}$ is any assembly of $\overrightarrow{\gamma}$.

4.15. Corollary. *Given a Σ-composable sequence $\overrightarrow{\gamma} := (\gamma_n, \ldots, \gamma_0)$ whose elements γ_k respectively belong to $\mathcal{C}[x_k, y_k]$ and two objects x and y of $\mathcal{C}$ such that $x \sim_o x_0$, $y \sim_o y_n$ and $\mathcal{C}[x, y]$ is not empty, we have the following statements:*

1. *if $x = s(\gamma_0)$, $y = t(\gamma_n)$ and $(\gamma_n, \ldots, \gamma_0)$ is composable,*
   then $\mathrm{Val}_{x,y}(\overrightarrow{\gamma}) := \gamma_n \circ \cdots \circ \gamma_0$,

2. *if $n = 0$, which means that the sequence $(\gamma_n, \ldots, \gamma_0)$ is reduced to (γ_0),*
   then $\mathrm{Val}_{x,y}(\overrightarrow{\gamma})$ *is given by the translation of γ_0 along x_0, x and y_0, y (diagram 4.22),*

3. *if $x \sim_o y$, then* $\mathrm{Val}_{x,y}(\overrightarrow{\gamma})$ *belongs to* $\Sigma[x, y]$ *and finally*

4. *if the sequence* $\overrightarrow{\gamma}$ *is normalized and* $\overrightarrow{\delta}$ *is a sequence of morphisms of* $\mathcal{C}$ *such that* $\overrightarrow{\gamma} \sim'_m \overrightarrow{\delta}$, *then* $\mathrm{Val}_{x,y}(\overrightarrow{\gamma}) = \mathrm{Val}_{x,y}(\overrightarrow{\delta})$.

Proof. The first point is obvious since we can take $\zeta_n := \gamma_n, \dots, \zeta_0 := \gamma_0$ as an assembly of $(\gamma_n, \dots, \gamma_0)$. The second point follows from translation Lemma (4.13) and the third one from Theorem 3.8.

By definition of $\sim'_m$, we can suppose that $\overrightarrow{\gamma} \sim'^1_m \overrightarrow{\delta}$ and by definition of $\sim'^1_m$ (4.7), we have either:

1. $\overrightarrow{\delta} = (\gamma_n, \dots, \gamma_{k+1} \circ \gamma_k, \dots, \gamma_0)$ for some $k \in \{0, \dots, n-1\}$ or
2. $\overrightarrow{\delta} = (\gamma_n, \dots, \gamma_{k+1}, \xi_1, \xi_0, \gamma_{k-1}, \dots, \gamma_0)$ for some $k \in \{0, \dots, n\}$ where $\xi_1 \circ \xi_0 = \gamma_k$ or
3. $\overrightarrow{\delta} = (\gamma_n, \dots, \gamma_{k+1}, \gamma'_k, \gamma_{k-1}, \dots, \gamma_0)$ for some $k \in \{0, \dots, n\}$ where γ'_k has been obtained by translating γ_k along x_k, x'_k and y_k, y'_k (diagram 4.23).

In the first case, the sequence $(\zeta_n, \dots, \zeta_{k+1} \circ \zeta_k, \cdots, \zeta_0)$ is an assembly of the sequence $(\gamma_n, \dots, \gamma_{k+1} \circ \gamma_k, \dots, \gamma_0)$ (Lemma 4.9) so $\mathrm{Val}_{x,y}(\gamma_n, \dots, \gamma_0) = \zeta_n \circ \cdots \circ (\zeta_{k+1} \circ \zeta_k) \circ \cdots \circ \zeta_0 = \mathrm{Val}_{x,y}(\overrightarrow{\delta})$.

In the second case, one just has to exchange the roles of $\overrightarrow{\delta}$ and $(\gamma_n, \dots, \gamma_0)$.

In the third one, the diagram 4.24 commutes because $(\zeta_n, \dots, \zeta_0)$ is an assembly of $(\gamma_n, \dots, \gamma_0)$, then by Lemma 4.11, the diagram 4.25 also commutes. Thus the sequence $(\zeta_n, \dots, \zeta_0)$ is also an assembly of the sequence $\overrightarrow{\delta}$ and then $\mathrm{Val}_{x,y}(\overrightarrow{\delta}) = \zeta_n \circ \cdots \circ \zeta_0 = \mathrm{Val}_{x,y}(\gamma_n, \dots, \gamma_0)$. ■

$$\begin{tikzcd} x_k \arrow[r, "\gamma_k"] & y_k \arrow[r, "\Sigma"] & y_k \vee z_{k+1} \\ x_k \wedge z_k \arrow[u, "\Sigma"] \arrow[r, "\Sigma"'] & z_k \arrow[r, "\zeta_k"'] & z_{k+1} \arrow[u, "\Sigma"'] \end{tikzcd}$$

Diagram 4.24

$$\begin{tikzcd} x'_k \arrow[r, "\gamma'_k"] & y'_k \arrow[r, "\Sigma"] & y'_k \vee z_{k+1} \\ x'_k \wedge z_k \arrow[u, "\Sigma"] \arrow[r, "\Sigma"'] & z_k \arrow[r, "\zeta_k"'] & z_{k+1} \arrow[u, "\Sigma"'] \end{tikzcd}$$

Diagram 4.25

$$\begin{tikzcd} x \arrow[r, "\gamma"] & y \arrow[r, "\Sigma"] & y \vee y' \\ x \wedge x' \arrow[u, "\Sigma"] \arrow[r, "\Sigma"'] & x' \arrow[r, "\gamma'"'] & y' \arrow[u, "\Sigma"'] \end{tikzcd}$$

Diagram 4.26

We can finally gives an "intuitive" and "handy" description of the relation $\sim_m$.

4.16. Proposition. *Given a* Yoneda-*system* Σ *on a loop-free category* $\mathcal{C}$ *and two morphisms* γ *and* γ' *respectively taken from* $\mathcal{C}[x, y]$ *and* $\mathcal{C}'[x', y']$, *we have* $(\gamma) \sim_m (\gamma')$ *if and only if the translation of* γ *along* x, x' *and* y, y' *is* γ', *that is to say* $x \sim_o x'$, $y \sim_o y'$ *and the diagram 4.26 commutes.*

Proof. Suppose $(\gamma) \sim_m (\gamma')$. We first suppose that $x = x'$ and $y = y'$. Two situations may arise, in the first one, we have $\gamma \in \Sigma$ or $\gamma' \in \Sigma$ so $\gamma = \gamma'$ (remark 3.4). In the second one, none of these two morphisms belong to Σ, therefore (γ) and (γ') are two normalized Σ-composable sequences such that $(\gamma) \sim_m (\gamma')$, hence we have $(\gamma) \sim'_m (\gamma')$ (Lemma 4.8) and then $\gamma = \mathrm{Val}_{x,y}\big((\gamma)\big) = \mathrm{Val}_{x,y}\big((\gamma')\big) = \gamma'$ (Corollary 4.15).

In the general case, as $(\gamma) \sim_m (\gamma')$ we have $x \sim_o x'$ and $y \sim_o y'$ (because $(\sim_o, \sim_m)$ is a generalized congruence) and the morphism γ' belongs to $\mathcal{C}[x', y']$. Since $\mathcal{C}[x', y']$ is not empty, we translate γ to δ along x, x' and y, y' (Lemma 4.13) that is to say diagram 4.27 commutes, therefore $(\gamma) \sim_m (\delta)$ (Lemma 4.8) and finally $(\gamma') \sim_m (\delta)$ because $\sim_m$ is transitive. Then we have $\gamma' = \delta$ by applying the first case.

The converse immediately follows from Proposition 1.8. ∎

$$\begin{tikzcd} x \arrow[r, "\gamma"] & y \arrow[r, "\Sigma"] & y \vee y' \\ x \wedge x' \arrow[u, "\Sigma"] \arrow[r, "\Sigma"'] & x' \arrow[r, "\delta"'] & y' \arrow[u, "\Sigma"'] \end{tikzcd}$$

Diagram 4.27

$$\begin{tikzcd} u \arrow[r, "\gamma"] & y \arrow[r, "\mathrm{id}_y"] & y \vee y = y \\ u = u \wedge u \arrow[u, "\mathrm{id}_u"] \arrow[r, "\mathrm{id}_u"'] & u \arrow[r, "\gamma'"'] & y \arrow[u, "\mathrm{id}_y"'] \end{tikzcd}$$

Diagram 4.28

Recall that with the notations of Propositions 1.1 and 1.8, P_Σ is the unique functor from $\mathcal{C}[\Sigma^{-1}]$ to $\mathcal{C}/_\Sigma$ such that $Q_\Sigma = P_\Sigma \circ I_\Sigma$.

4.17. LEMMA. *Given a* Yoneda-*system* Σ *on a loop-free category* $\mathcal{C}$*, the functor* P_Σ *is* faithful.

PROOF. Let x and y be two objects of $\mathcal{C}[\Sigma^{-1}]$ such that $\mathcal{C}[\Sigma^{-1}][x, y]$ is not empty, by Lemma 3.13, there exists an object u of $\mathcal{C}$ and a morphism σ in $\Sigma[u, x]$ such that any element of $\mathcal{C}[\Sigma^{-1}][x, y]$ (which is a $\sim_{x,y}$-equivalence class by Proposition 1.3) has a unique representative written as (γ, σ) where γ belongs to $\mathcal{C}[u, y]$. Then let γ and γ' be two morphisms of $\mathcal{C}[u, y]$ such that $P_\Sigma(\gamma, \sigma) = P_\Sigma(\gamma', \sigma)$, which means that $(\gamma) \sim_m (\gamma')$. By Proposition 4.16, the diagram 4.28 commutes, in other words $\gamma = \gamma'$. ∎

Now we can complete the proof of the main result of the article:

PROOF OF THEOREM 4.1. Let a and b be two objects of $\mathcal{C}[\Sigma^{-1}]$, let f be a morphism of $\mathcal{C}/_\Sigma[P_\Sigma(a), P_\Sigma(b)]$ (if it exists) and the Σ-composable sequence $(\gamma_n, \ldots, \gamma_0)$ be a $\sim_m$-representative of f. We translate γ_n to γ'_n in such way that $t(\gamma'_n) = b$, then γ_{n-1} to γ'_{n-1} in such way that $t(\gamma'_{n-1}) = s(\gamma'_n)$ and so on till we have the composable sequence $(\gamma'_n, \ldots, \gamma'_0)$. Necessarily, $s(\gamma'_0) \sim_o a$ because $(\gamma_n, \ldots, \gamma_0) \sim_m (\gamma'_n, \ldots, \gamma'_0)$ (Proposition 4.16). Up to the replacement of $s(\gamma'_0)$ by $s(\gamma'_0) \wedge a$, we can suppose that $\mathcal{C}[s(\gamma'_0), a] \neq \emptyset$ (Theorem 3.8).

Therefore we have a unique morphism σ from $s(\gamma'_0)$ to a and it belongs to Σ (Theorem 3.8), besides, the composite $\gamma := \gamma'_n \circ \cdots \circ \gamma'_0$ is the value of $(\gamma_n, \ldots, \gamma_0)$ from $s(\gamma'_0)$ to b. Thus we have $P_\Sigma\Big(I_\Sigma(\gamma) \circ \big(I_\Sigma(\sigma)\big)^{-1}\Big) = Q_\Sigma(\gamma) = f$ and $I_\Sigma(\gamma) \circ \big(I_\Sigma(\sigma)\big)^{-1}$ belongs to $\mathcal{C}[\Sigma^{-1}][a, b]$.

Let y be an object of $\mathcal{C}/_\Sigma$, by Proposition 1.6, there exists an object x of $\mathcal{C}$ such that $Q_\Sigma(x) = y$, furthermore, by Proposition 1.1, we have $I_\Sigma(x) = x$. Thus, x is an object of $\mathcal{C}[\Sigma^{-1}]$ such that $P_\Sigma(x) = y$. Finally, P_Σ is faithful (Lemma 4.17), therefore it is a fibered equivalence (Definition 1.12). ∎

5. Examples

We focus on some examples of categories of components to show that the methods we have developed actually provide the expected results.

5.1. If $\mathbb{P}$ is a partition of a set X and x, y are two points of X, we write $x \sim_{\mathbb{P}} y$ when x and y belong to the same element P of $\mathbb{P}$. A partition $\mathbb{P}'$ of the same set X is said to be **finer** than $\mathbb{P}$ when for each element P' of $\mathbb{P}'$ there exists an element P of $\mathbb{P}$ such that $P' \subseteq P$. Given a poset $(X, \sqsubseteq)$ seen as a category denoted by $\mathcal{X}$, the collection of *Yoneda*-systems of $\mathcal{X}$ is in one-to-one correspondence with the collection of partitions $\mathbb{P}$ of X such that for all points x and y of X,

1) if there exists z in X such that $z \sqsubseteq x$, $z \sqsubseteq y$ and $z \sim_{\mathbb{P}} x$, then the least upper bound $x \vee y$ of $\{x, y\}$ exists (in $(X, \sqsubseteq)$) and $y \sim_{\mathbb{P}} x \vee y$ and

2) if there exists z in X such that $x \sqsubseteq z$, $y \sqsubseteq z$ and $x \sim_{\mathbb{P}} z$, then the greatest lower bound $x \wedge y$ of $\{x, y\}$ exists (in $(X, \sqsubseteq)$) and $x \wedge y \sim_{\mathbb{P}} y$.

The preceding bijection easily comes from the facts that push-out and pull-back in $\mathcal{X}$ correspond to least upper bound and greatest lower bound in $(X, \sqsubseteq)$ and every morphism of $\mathcal{X}$ is a *Yoneda*-morphism of $\mathcal{X}$.

In particular, given two *Yoneda*-systems Σ and Σ' of $\mathcal{X}$ and their corresponding partitions $\mathbb{P}$ and $\mathbb{P}'$, one has $\Sigma \subseteq \Sigma'$ if and only if $\mathbb{P}$ is finer than $\mathbb{P}'$. Hence the locale of *Yoneda*-systems of $\mathcal{X}$ is isomorphic to the collection of partitions of X satisfying 1) and 2) ordered by the following relation.

$$\left\{ (\mathbb{P}, \mathbb{P}') \,\middle|\, \mathbb{P}, \mathbb{P}' \text{ partitions of } X \text{ satisfying 1) and 2) ; } \mathbb{P} \text{ is finer than } \mathbb{P}' \right\}$$

In particular, each element P of a partition $\mathbb{P}$ satisfying 1) and 2) is an **order convex sub-lattice** of $(X, \sqsubseteq)$. In other words, for all elements p_1 and p_2 of P, the set $\{x \in X \mid p_1 \sqsubseteq x \sqsubseteq p_2\}$ (which may be empty) is contained in P, the least upper and the greatest lower bound of the pair $\{p_1, p_2\}$ exist in $(X, \sqsubseteq)$ and both of them belong to P. This fact immediately follows from the pureness of the *Yoneda*-system corresponding to the partition $\mathbb{P}$ (Lemma 3.6) and Theorem 3.8. If $(X, \sqsubseteq)$ is a **chain** (i.e. for all x and y in X, one has $x \sqsubseteq y$ or $y \sqsubseteq x$), then any partition $\mathbb{P}$ of X whose elements are order convex sub-lattices of $(X, \sqsubseteq)$ satisfies 1) and 2).

5.2. We recall that a **lattice** is a non empty poset in which any pair of elements admits a least upper bound and a greatest lower bound. With the preceding example in mind, one easily verifies that the collection of *all* morphisms of a lattice is its greatest *Yoneda*-system; consequently, the category of components of any lattice is reduced to the terminal object of Cat. Conversely, let $\mathcal{C}$ be a loop-free category and $\overline{\Sigma}$ be the greatest *Yoneda*-system of $\mathcal{C}$, if the category of components of $\mathcal{C}$ is reduced to the terminal object of Cat, then $\mathcal{C}$ has a single $\overline{\Sigma}$-component which is a lattice (Theorem 3.8), in other words, the category $\mathcal{C}$ is a lattice.

5.3. Let us now consider a geometric example: the set of objects of the category $\mathcal{C}$ is $X{:=}[0,1]^2 \backslash]\frac{1}{3}, \frac{2}{3}[^2$ and the description of the morphisms of $\mathcal{C}$ is based on the partition

$X = A \cup B_1 \cup B_2 \cup C$ given below. Indeed, each morphism is given a "type" depending on which parts its extremities belong to.

$$A{:=}\left[0,\tfrac{1}{3}\right]^2,\ B_1{:=}\left(\left]\tfrac{1}{3},1\right]\times\left[0,\tfrac{2}{3}\right[\right)\backslash\left]\tfrac{1}{3},\tfrac{2}{3}\right[^2,\ B_2{:=}\left(\left[0,\tfrac{2}{3}\right[\times\left]\tfrac{1}{3},1\right]\right)\backslash\left]\tfrac{1}{3},\tfrac{2}{3}\right[^2 \text{ and } C{:=}\left[\tfrac{2}{3},1\right]^2.$$

For all objects $x := (x_1, x_2)$ and $y := (y_1, y_2)$ of $\mathcal{C}$, we write $x \sqsubseteq y$ when $x_1 \leq x_2$ and $y_1 \leq y_2$; if $x \not\sqsubseteq y$ then by definition the homset $\mathcal{C}[x, y]$ is empty. Moreover, for each object x of $\mathcal{C}$, the homset $\mathcal{C}[x, x]$ is the singleton $\{\mathrm{id}_x\}$. For the rest of the description of $\mathcal{C}$, we suppose that $x \sqsubset y$, in this case, the table 1 gives an extensive description of each homset $\mathcal{C}[x, y]$. Besides, the composition law of $\mathcal{C}$ is given by table 2: writing $\gamma \circ \delta$, the morphism δ is read vertically while γ is read horizontally.

$x \in$	$y \in$	$\mathcal{C}[x,y]$
A	A	$\{\sigma_{x,y}\}$
B_1	B_1	$\{\sigma_{x,y}\}$
B_2	B_2	$\{\sigma_{x,y}\}$
C	C	$\{\sigma_{x,y}\}$
A	B_1	$\{r_{x,y}\}$
A	B_2	$\{h_{x,y}\}$
B_1	C	$\{h'_{x,y}\}$
B_2	C	$\{r'_{x,y}\}$
B_1	B_2	$\emptyset$
B_2	B_1	$\emptyset$
A	C	$\{u_{x,y}\ ,\ d_{x,y}\}$

Table 5.1

In table 5.2, the indices indicate the source and the target. When composition makes sense the resulting morphism is given in the corresponding entry, a blank means that the component of y *does not meet* the one of w.

$\circ$	$\sigma_{w,z}$	$h_{w,z}$	$r_{w,z}$	$h'_{w,z}$	$r'_{w,z}$	$u_{w,z}$	$d_{w,z}$
$\sigma_{x,y}$	$\sigma_{x,z}$	$h_{x,z}$	$r_{x,z}$	$h'_{x,z}$	$r'_{x,z}$	$u_{x,z}$	$d_{x,z}$
$h_{x,y}$	$h_{x,z}$				$u_{x,z}$		
$r_{x,y}$	$r_{x,z}$			$d_{x,z}$			
$h'_{x,y}$	$h'_{x,z}$						
$r'_{x,y}$	$r'_{x,z}$						
$u_{x,y}$	$u_{x,z}$						
$d_{x,y}$	$d_{x,z}$						

Table 5.2

Tables 5.1 and 5.2 completely define the category $\mathcal{C}$. Yet, they suggest that in some sense, the morphisms $\sigma_{x,y}$, that is to say "those which do not cross any frontier", do not influence the "type" of the morphism they are composed with. Hence, the morphisms $\sigma_{x,y}$ behave as "neutral elements" with respect to the "type" of morphism. Then, the greatest *Yoneda*-system of $\mathcal{C}$, denoted by $\overline{\Sigma}$, is the family of morphisms $\sigma_{x,y}$ and the $\overline{\Sigma}$-components of $\mathcal{C}$ are A, B_1, B_2 and C. The category of components of $\mathcal{C}$ is *freely* generated by the graph on figure 5.1.

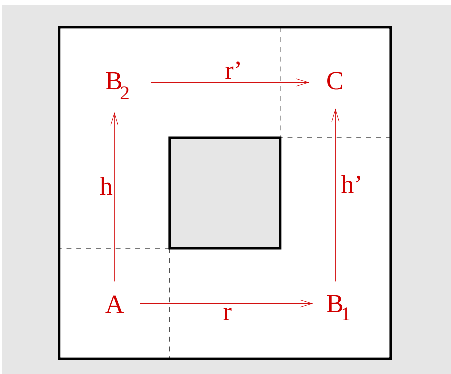

Figure 5.1

Besides, the group of autofunctors of the category of components of $\mathcal{C}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, hence there are exactly two equivalences from $\mathcal{C}[\overline{\Sigma}^{\,-1}]$ to $\mathcal{C}/_{\Sigma}$ (Corollary 4.5). Furthermore, the map from X to X sending each point (x, y) to (y, x) implies, together with Corollary 4.6, that the group morphism from $\mathsf{Aut}(\mathcal{C})$ to $\mathsf{Aut}(\mathcal{C}/_{\overline{\Sigma}})$ given by Proposition 2.8 is surjective.

5.4. We come back to the case where the set X is $[0,1]^2\backslash]\frac{1}{3},\frac{2}{3}[^2$ with the product order $\sqsubseteq$ and denote by $\mathcal{X}$ the induced category. The partition of X corresponding to the greatest *Yoneda*-system of $\mathcal{X}$ is

$$A_{n,p}{:=}\left[\tfrac{n}{3},\tfrac{n+1}{3}\right]\times\left[\tfrac{p}{3},\tfrac{p+1}{3}\right],\ B_n{:=}\left[\tfrac{n}{3},\tfrac{n+1}{3}\right]\times\left]\tfrac{1}{3},\tfrac{2}{3}\right[\text{ and } C_n{:=}\left]\tfrac{1}{3},\tfrac{2}{3}\right[\times\left[\tfrac{n}{3},\tfrac{n+1}{3}\right]$$

where n and p range over $\{0,2\}$. Hence the category of components of $\mathcal{X}$ is isomorphic to the poset $\{0<1<2\}\times\{0<1<2\}\backslash\{(1,1)\}$ which is *not* a lattice.

5.5. The geometric example 5.3 is an instance of a general situation we now describe, more details are available in [10, 11, 14, 23, 31, 33]. A **pospace**, denoted by $\overrightarrow{X}$, is a topological space X equipped with an order relation $\sqsubseteq$ whose graph is *closed* in $X\times X$. Pospaces and their morphisms, namely the continuous and increasing maps, form a complete and co-complete category[9] denoted by PoTop. The segment $[0,1]$ equiped with the natural order is an object of PoSpc denoted by $\overrightarrow{\mathbb{I}}$. Given two points a and b of $\overrightarrow{X}$, a **directed path** from a to b on $\overrightarrow{X}$ is an element γ of $\mathsf{PoTop}[\overrightarrow{\mathbb{I}},\overrightarrow{X}]$ such that $\gamma(0)=a$ and $\gamma(1)=b$. Given γ and δ, two directed paths on $\overrightarrow{X}$ such that $\gamma(0)=\delta(1)$, we defined the **concatenation** $\gamma\cdot\delta$ of δ followed by γ as in classical algebraic topology [20, 24]. Denoting by U the forgetful functor from PoTop to the category of *Hausdorff* spaces (the underlying topological space of any pospace is *Hausdorff* [33]), we say that an element h of $\mathsf{PoTop}[\overrightarrow{\mathbb{I}}\times\overrightarrow{\mathbb{I}},\overrightarrow{X}]$ is a **directed homotopy** from the directed path γ_1 to the directed path γ_2 when $U(h)$ is a usual homotopy [20, 24] from $U(\gamma_1)$ to $U(\gamma_2)$. Writing $\gamma_1\sqsubseteq_{dih}\gamma_2$ when there exists a directed homotopy from γ_1 to γ_2, we define an *order relation* over the set of directed paths on $\overrightarrow{X}$. Let $\mathcal{F}_{\overrightarrow{X}}$ be the free category spanned by the graph whose vertices and arrows are respectively the points of $\overrightarrow{X}$ and the dipaths on $\overrightarrow{X}$, the head and tail of an arrow γ being respectively $\gamma(0)$ and $\gamma(1)$. Then we define the congruence $\sim_{dih}$ on $\mathcal{F}_{\overrightarrow{X}}$ as the one induced by the relation $\alpha\sim^1_{dih}\gamma\circ\delta$ when $\alpha\sqsubseteq_{dih}\gamma\cdot\delta$[10] and the **fundamental category** of $\overrightarrow{X}$, denoted by $\overrightarrow{\pi_1}(\overrightarrow{X})$, as the quotient $\mathcal{F}_{\overrightarrow{X}}/{\sim_{dih}}$. The fundamental category of a pospace is loop-free, in regard of Theorem 4.1 this obvious fact is crucial because we define the **category of components** of a pospace as the category of components of its fundamental category. From the example where $X{:=}[0,1]^2\backslash]\frac{1}{3},\frac{2}{3}[^2$, we know that the category of components $\mathcal{C}$ of a *pospace* may differ from the category of components $\mathcal{P}$ of its underlying *poset*. In fact, it may happen that none of the categories $\mathcal{C}$ and $\mathcal{P}$ can be embedded in the other. Aside from the problem of components, the construction of the fundamental category we have given is a special instance of a general one [21] which also encompasses the fundamental groupoid construction.

5.6. Let us give an example in dimension 3: given a real number ε such that $0\leq\varepsilon<\frac{1}{2}$, let $\overrightarrow{X}_\varepsilon$ be the sub-pospace of $\overrightarrow{\mathbb{R}}^3$ whose underlying set is the unit cube $[0,1]^3$ from which we have removed the 3 following subsets $\mathbb{I}\times]\varepsilon,1\text{-}\varepsilon[\times]\varepsilon,1\text{-}\varepsilon[$, $]\varepsilon,1\text{-}\varepsilon[\times\mathbb{I}\times]\varepsilon,1\text{-}\varepsilon[$ and $]\varepsilon,1\text{-}\varepsilon[\times]\varepsilon,1\text{-}\varepsilon[\times\mathbb{I}$; see figure 5.2. The category of components of $\overrightarrow{X}_\varepsilon$ is freely generated

[9]A proof of the co-completeness of PoTop is given in [23].

[10]Note that here, $\circ$ and $\cdot$ are formally distinct since they are respectively put for the composition law of $\mathcal{F}_{\overrightarrow{X}}$ and the concatenation of dipaths: one of the purpose of $\sim_{dih}$ is precisely to identify them.

by a graph (pictured on figure 5.3) which has a planar representation given on figure 5.4. For any parameter ε taken from $]0, \frac{1}{2}[$, each $\overline{\Sigma}_\varepsilon$-component of $\overrightarrow{X}_\varepsilon$ is 3 dimensional in the sense that its interior, as a subset of $\mathbb{R}^3$, is not empty. On the other hand, in the case of $\overrightarrow{X}_0$, some components are reduced to a singleton (0-dimensional) while the others are segments (1-dimensional). The group of autofunctors of the category of components of $\overrightarrow{X}_\varepsilon$ is isomorphic to the group of permutations over a 3 elements set. Still, one can take $\frac{1}{2}$ for the parameter ε; in this case, $\overrightarrow{X}_{\frac{1}{2}}$ is just $\overrightarrow{\mathbb{I}}^3$ (product of 3 copies of $\overrightarrow{\mathbb{I}}$ in PoSpc) and, since the fundamental category of $\overrightarrow{X}_{\frac{1}{2}}$ is the lattice $[0,1]^3$ (together with the product order), its category of components is the terminal object of Cat.

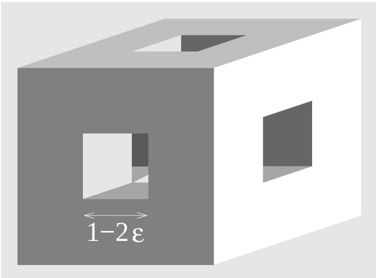

Figure 5.2

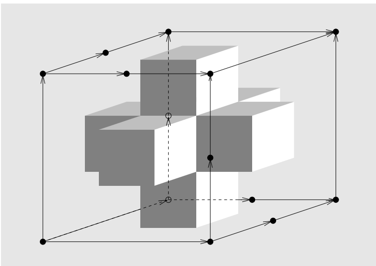

Figure 5.3

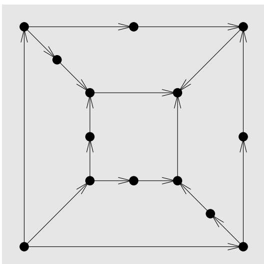

Figure 5.4

5.7. Consider the set T whose elements are points (x_1, x_2) of $[0,1]^2$ such that $x_2 \leq x_1$. The usual product order over $\mathbb{R}^2$ induces an order over T denoted by $\sqsubseteq$. The category of components of $(T, \sqsubseteq)$ is $(T, \sqsubseteq)$. Indeed, every element P of a partition of T satisfying properties 1) and 2) of example 5.1 is necessarily a singleton. By the way, the poset $(T, \sqsubseteq)$ is isomorphic to the set of *non empty* compact intervals of $[0,1]$; if we add the empty set, we obtain a lattice, thus making the category of components trivial.

5.8. For the last example, we consider the poset $(X, \sqsubseteq)$ where $X{:=}\big(\{0\}\times\mathbb{R}\big) \cup \big(\mathbb{R}\times\{0\}\big)$ and $\sqsubseteq$ is the order induced on X by the product order on $\mathbb{R}^2$. By the way, the set X has a pospace structure $\overrightarrow{X}$ inherited from the product pospace structure of $\mathbb{R}^2$ and its fundamental category is isomorphic to the poset $(X, \sqsubseteq)$. Moreover, the category of components of $\overrightarrow{X}$ and the one of its underlying poset $(X, \sqsubseteq)$ are isomorphic to the poset $\{(\text{-}1,0),(0,\text{-}1),(1,0),(0,1),(0,0)\}$ equipped with the order induced by the usual product order on $\mathbb{R}^2$. The five $\overline{\Sigma}$-components of $(X, \sqsubseteq)$ are respectively the open half-lines $\{0\}\times]0,+\infty[$, $\{0\}\times]\text{-}\infty,0[$, $]0,+\infty[\times\{0\}$, $]\text{-}\infty,0[\times\{0\}$ and the singleton $\{(0,0)\}$.

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