Introduction to Directed Algebraic Topology with a view towards modelling Concurrency I

Mathematical Structures of Computations - Lyon 2014

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Summary

Different kinds of parallelism

Virtual Machines
- Middle-End Representation
- Execution model

Concurrency
- Generalizing graphs
- Control flow precubical set
- The extended PV language

Distributed computation

- Variable amount of available resources
- Variable population of parallel processes
- e.g. SETI@home, Bitcoin, e-shopping
- Usual requirements: availability, coherence, fault tolerance

Fine grain parallelism

- Constant amount of available resources
- Constant population of parallel processes
- e.g. control-command, graphic rendering
- Usual requirements: deterministic output, nonblocking, as fast as possible

Expressions and values

$\mathcal{V}$: variables $\mathcal{E}$: expressions built on the following operators

v	content of $v \in \mathcal{V}$	$x \in \mathbb{R}$	constant
$\wedge$	minimum	$\vee$	maximum
$+$	addition	$-$	substraction
$*$	multiplication	$/$	division
$\leqslant$	less or equal	$\geqslant$	greater of equal
$<$	strictly less	$>$	strictly greater
$\neg$	complement	$=$	equal
$\perp$	bottom		

nullary	unary
$\perp$, $x \in \mathbb{R}$, $v \in \mathcal{V}$	$\neg$
binary	
$\wedge$, $\vee$, $+$, $-$, $*$, $/$, $<$, $>$, $\leqslant$, $\geqslant$, $=$	

Interpretation of expressions

$[\![\_]\!] : (\mathcal{V} \to \mathbb{R}_\perp) \to \mathcal{E} \to \mathbb{R}_\perp$

- distribution: $\delta : \mathcal{V} \to \mathbb{R}_\perp$

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- $[\![v]\!]_\delta = \delta(v)$

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- 0 stands for false any value in $\mathbb{R} \setminus \{0\}$ stands for true

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- $[\![\neg]\!] : \mathbb{R}_\perp \to \mathbb{R}_\perp$,
  $[\![\neg]\!](0) = 1$, and
  $[\![\neg]\!](x) = 0$ for all $x \in \mathbb{R} \setminus \{0\}$

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- $[\![e]\!] = \perp$ for all expression e in which $\perp$ occurs

Interpretation of actions

$[\![\_]\!] : (\mathcal{V} \to \mathbb{R}_\perp) \to \mathcal{V} \to \mathcal{E} \to (\mathcal{V} \to \mathbb{R}_\perp)$

- v: variable, e: expression, δ: distribution
- $v := e$ is called an action, $\mathcal{A}$ set of all the actions

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  $[\![v := e]\!]_\delta(v) = [\![e]\!]_\delta$

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  $[\![v := e]\!]_\delta(v) = [\![e]\!]_\delta$

  $[\![v := e]\!]_\delta(v') = \delta(v')$ for $v' \neq v$

Control Flow Graphs

A: arrows, V: control points, $\mathcal{A}$: actions

$$G : A \underset{\partial^+}{\overset{\partial^-}{\rightrightarrows}} V \quad \text{and} \quad \lambda : A \to \mathcal{A}$$

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- $\Phi : \mathcal{V} \to (\mathcal{E} \times A)^*$
  if $\Phi(v) = [(e_1, \alpha_1), \ldots, (e_k, \alpha_k)]$
  then $\partial^- \alpha_i = v$ for all $v \in \mathcal{V}$ and all $i \in \{1, \ldots, k\}$

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- $v_0 \in V$ the starting point
- (G, λ, Φ, v_0) is the middle-end representation

- δ_0 : initial state (with the starting point v_0)

Sequential

Virtual Machine

- δ_0 : initial state (with the starting point v_0)
- (v_n, δ_n): current state
  suppose $\Phi(v_n) = [(e_1, \alpha_1), \ldots, (e_k, \alpha_k)]$

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  define $i = \min\{j \in \{1, \ldots, k\} \mid [\![e_j]\!]_{\delta_n} \text{ is true}\}$

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  if i exists then $v_{n+1} = \partial^+\alpha_i$ and $\delta_{n+1} = [\![\lambda(\alpha_i)]\!]_{\delta_n}$

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  otherwise the induction stops

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  if i exists then $v_{n+1} = \partial^+\alpha_i$ and $\delta_{n+1} = [\![\lambda(\alpha_i)]\!]_{\delta_n}$

  otherwise the induction stops
- deterministic behavior and output

An example

The Hasse/Syracuse algorithm

•
input x;

while x≠1

do

if x mod 2 = 0

then x:=x/2

else x:=3∗x+1

done

An example

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input x;
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An example

The Hasse/Syracuse algorithm

```
input x;
while x≠1
    do
 if x mod 2 = 0
  then x:=x/2
 else x:=3*x+1
    done
```

$x = 1$

An example

The Hasse/Syracuse algorithm

input x;

while x≠1

$x \neq 1$

do

if x mod 2 = 0

then x:=x/2

else x:=3∗x+1

done

An example

The Hasse/Syracuse algorithm

```
input x;
while x≠1
    do
  if x mod 2 = 0
   then x:=x/2
  else x:=3*x+1
    done
```

x is even

An example

The Hasse/Syracuse algorithm

```
input x;
while x≠1
    do
 if x mod 2 = 0
  then x:=x/2
 else x:=3*x+1
    done
```

x is odd

An example

The Hasse/Syracuse algorithm

```
input x;
while x≠1
    do
  if x mod 2 = 0
  then x:=x/2
  else x:=3*x+1
    done
```

An example

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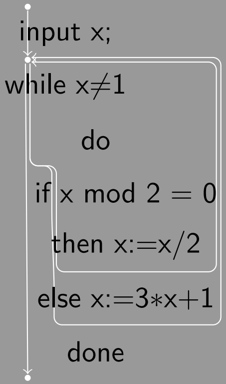
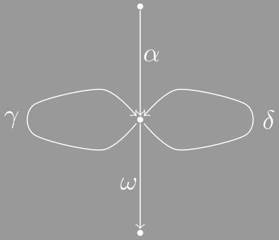
The Hasse/Syracuse algorithm





α stands for `input x` γ stands for `x:=x/2`

ω stands for "`exit`" δ stands for `x:=3*x+1`

An execution trace

Hasse/Syracuse algorithm

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input x;
while x≠1
  do
 if x mod 2 = 0
  then x:=x/2
 else x:=3*x+1
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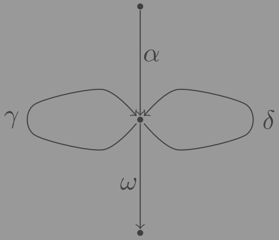
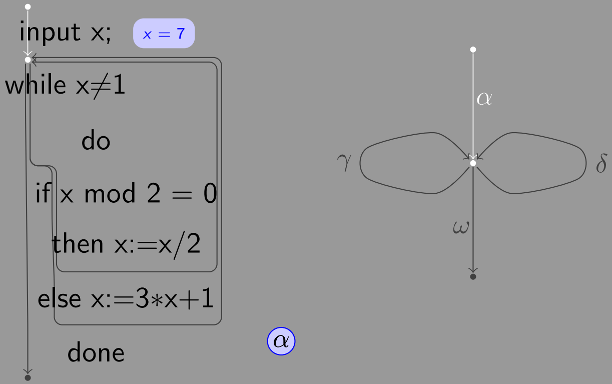
```



An execution trace

Hasse/Syracuse algorithm

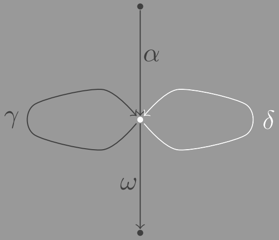


An execution trace

Hasse/Syracuse algorithm

```
input x;
while x≠1
   do
 if x mod 2 = 0
  then x:=x/2
 else x:=3*x+1
   done
```

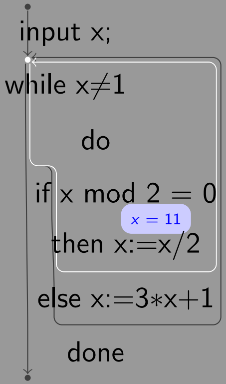
$x = 22$

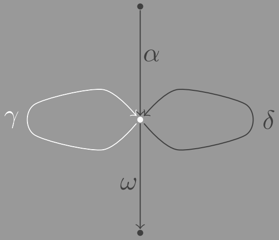


$\alpha\delta$

An execution trace

Hasse/Syracuse algorithm





$\alpha\ \delta\ \gamma$

An execution trace

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input x;
while x≠1
    do
  if x mod 2 = 0
    then x:=x/2
  else x:=3*x+1
    done
```
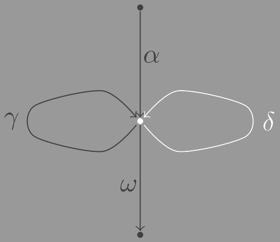

$x = 34$



$\alpha\ \delta\ \gamma\ \delta$

An execution trace

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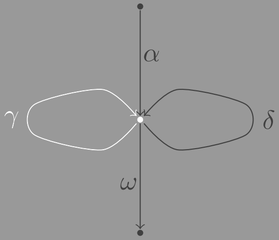
```
input x;
while x≠1
   do
 if x mod 2 = 0
  then x:=x/2
 else x:=3*x+1
   done
```

$x = 17$



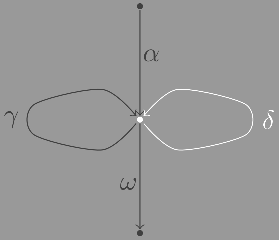
$\alpha\ \delta\ \gamma\ \delta\ \gamma$

An execution trace

Hasse/Syracuse algorithm

```
input x;
while x≠1
    do
  if x mod 2 = 0
   then x:=x/2
  else x:=3*x+1
    done
```

$x = 52$



$\alpha\ \delta\ \gamma\ \delta\ \gamma\ \delta$

An execution trace

Hasse/Syracuse algorithm

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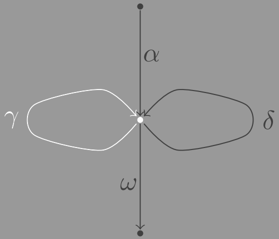
input x;
while x≠1
    do
  if x mod 2 = 0
    then x:=x/2
  else x:=3*x+1
    done
```

$x = 26$



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input x;
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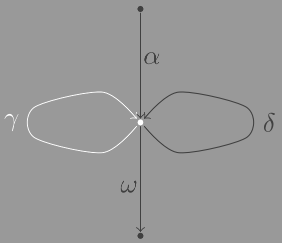
else x:=3∗x+1
done

$x = 13$



$\alpha\ \delta\ \gamma\ \delta\ \gamma\ \delta\ \gamma\ \gamma$

An execution trace

Hasse/Syracuse algorithm

input x;

while x≠1

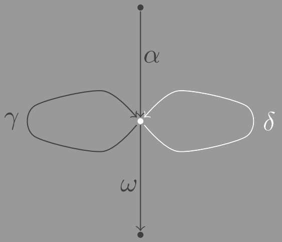
do

if x mod 2 = 0

then x:=x/2

else x:=3∗x+1

done $x = 40$



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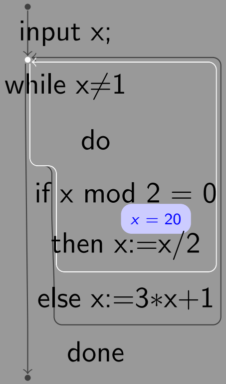

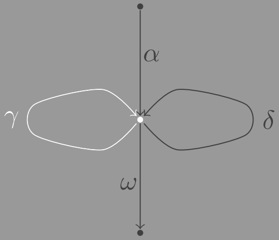

$\alpha\ \delta\ \gamma\ \delta\ \gamma\ \delta\ \gamma\ \gamma\ \delta\ \gamma$

An execution trace

Hasse/Syracuse algorithm

```
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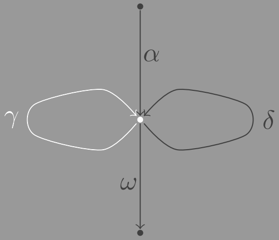
    then x:=x/2
  else x:=3*x+1
  done
```

$x = 10$



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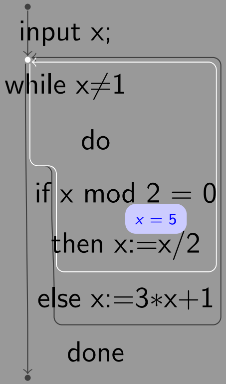

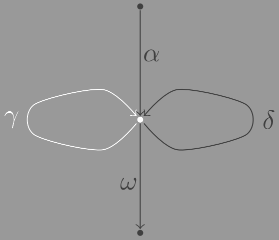

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    do
  if x mod 2 = 0
    then x:=x/2
  else x:=3*x+1
    done
```
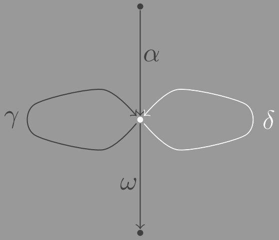

$x = 16$



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An execution trace

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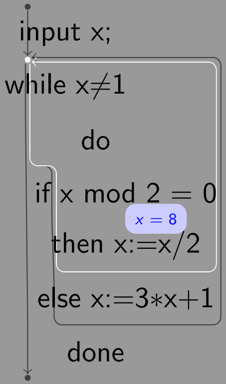

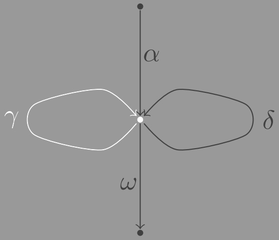

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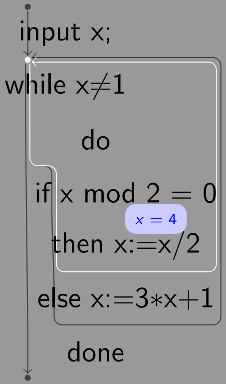

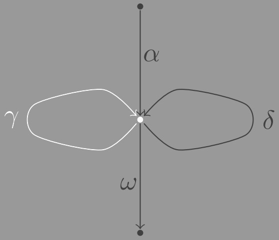

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An execution trace

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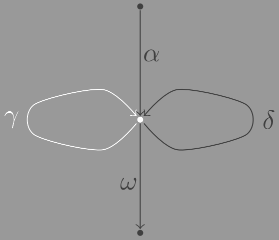
```
input x;
while x≠1
    do
  if x mod 2 = 0
   then x:=x/2
  else x:=3*x+1
    done
```

$x = 2$



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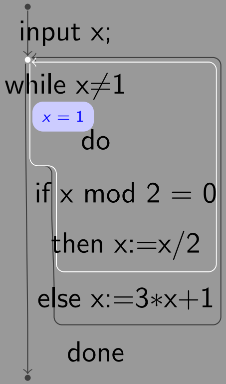

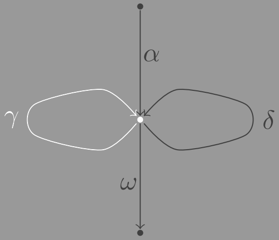

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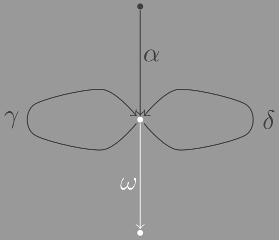

$\alpha\ \delta\ \gamma\ \delta\ \gamma\ \delta\ \gamma\ \gamma\ \delta\ \gamma\ \gamma\ \gamma\ \delta\ \gamma\ \gamma\ \gamma\ \gamma\ \omega$

Execution traces of a program

as paths over its control flow graph

- Any execution trace induces a path
- Some paths do not come from an execution trace

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Therefore the collection of all paths provides a (strict) overapproximation of the collection of execution traces

The (infinite) collection of paths is entirely determined by the (finite) control flow graph

The overall idea

of Static Analysis

The model of a program should be the finite representation of an overapproximation of the collection of all its execution traces.

The parallel composition operator

Enabling several actions to be performed at the same time

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Enabling several actions to be performed at the same time

- Middle-end: d-sequence of control flow graphs

The parallel composition operator

Enabling several actions to be performed at the same time

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- The virtual machine has to be adapted accordingly

Interleaving
Virtual Machine

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- neither behavior nor output is deterministic e.g.

```
x:=0 | x:=1
```

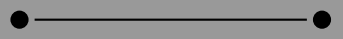
Precubical sets

higher dimensional graphs

dimension 0

Precubical sets

higher dimensional graphs



dimension 1

Precubical sets

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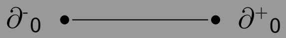

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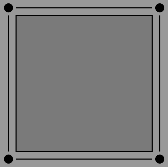

dimension 2

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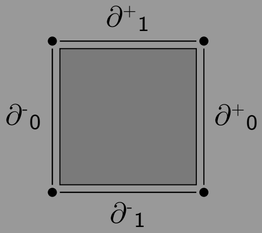

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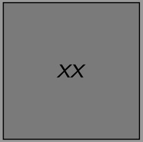

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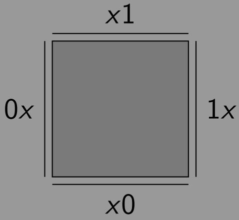

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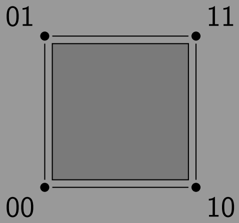

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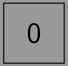
- if $w : a \to b$ and $w' : b \to c$ then $w'w$ is obtained
  replacing the i^{th} occurrence of x in w'
  by the i^{th} letter of w.

Precubical sets

The $\square^+$ category pictured



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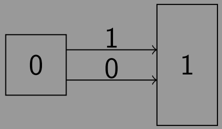

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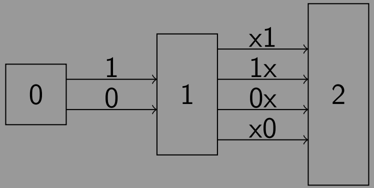

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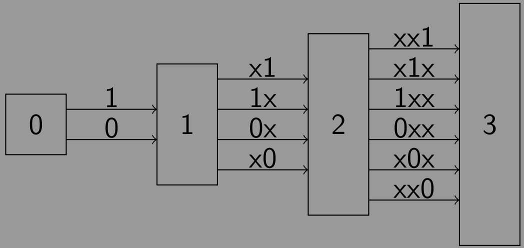

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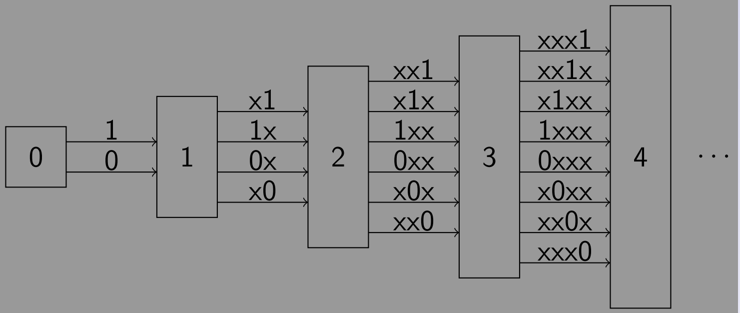
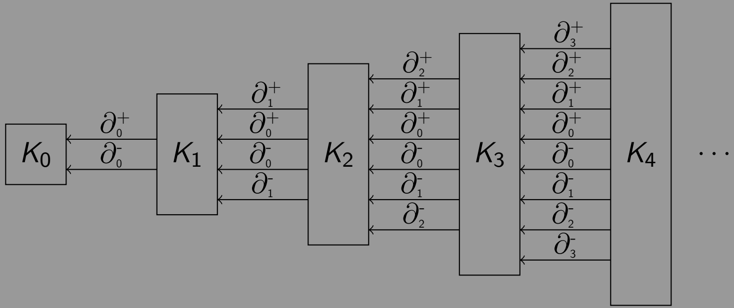


Precubical sets

as presheaves over $\square^+$



Tensor product

of precubical sets

Given precubical sets K and K' of dimension p and q, the set of n-cubes for $0 \leqslant n \leqslant p + q$

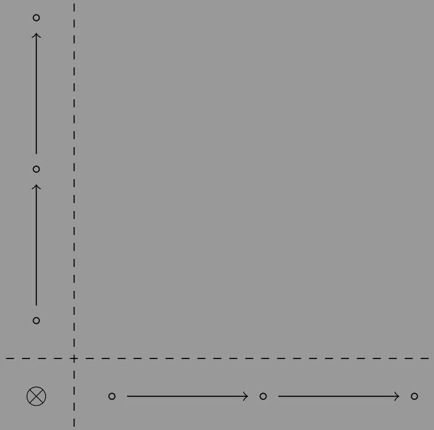
$$(K \otimes K')_n = \bigsqcup_{i+j=n} K_i \times K_j$$

For $x \otimes y \in K_i \times K'_j$ with $i + j = n$ the k^{th} face map, with $0 \leqslant k < n$, is given by

$$\partial^{\pm}_k(x \otimes y) = \begin{cases} \partial^{\pm}_k(x) \otimes y & \text{if } 0 \leqslant k < i \\ x \otimes \partial^{\pm}_{k-p} y) & \text{if } i \leqslant k < n \end{cases}$$

Example of tensor product

of precubical sets



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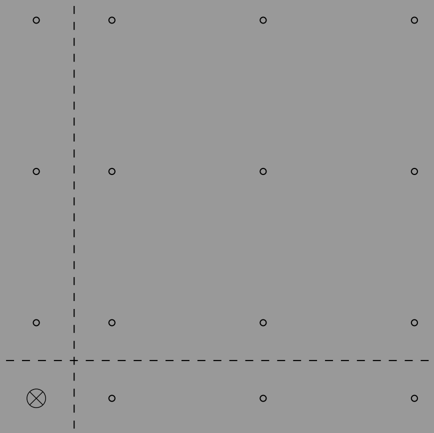

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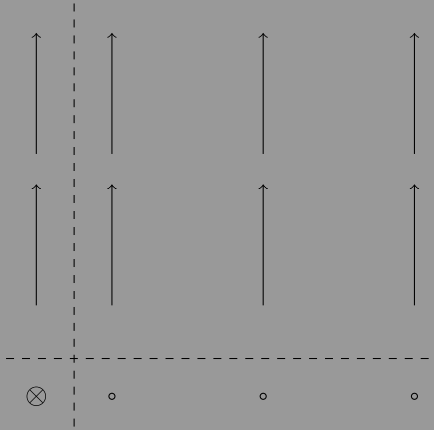

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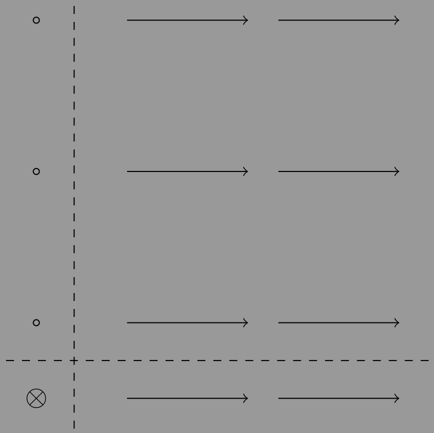

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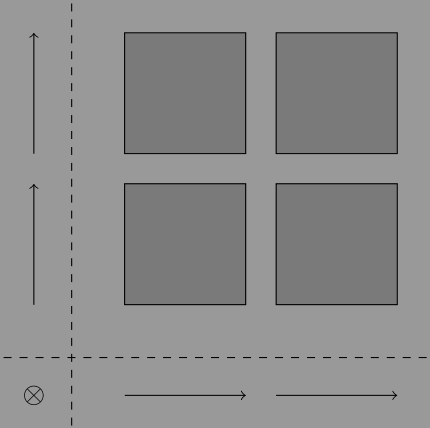

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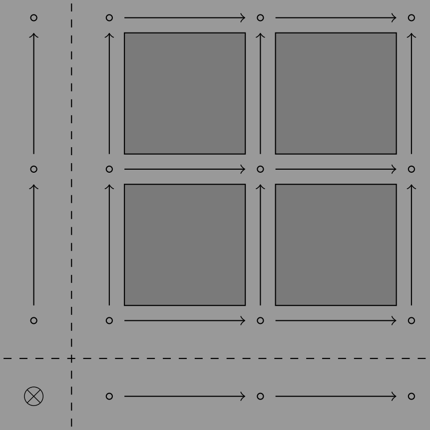

True concurrency - discrete version
Virtual Machine

- get rid of the global clock

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- interleaving model only allows M such that $|M| = 1$

True concurrency - discrete version

Virtual Machine

- performing M only makes sense under the sheaf condition:
  for all finite sequences s of length ℓ
  with elements in $\{1, \ldots, d\}$ and satisfying
  $\#\{i \mid s_i = k\} \leqslant M(k)$ for all $k \in \{1, \ldots, d\}$,
  the intermediate state of the interleaving execution at step
  ℓ from the inital state $(v_1, \ldots, v_d, \delta)$ and according to
  the global choice s, only depends on the multiset
  $k \mapsto \#\{i \mid s_i = k\}$.

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Control flow from tensor product of control flow graphs

$(\mathit{process}\ \ p)$ `x:=0 ; x:=2|`
$(\mathit{process}\ \ q)$ `x:=1 ; x:=2`

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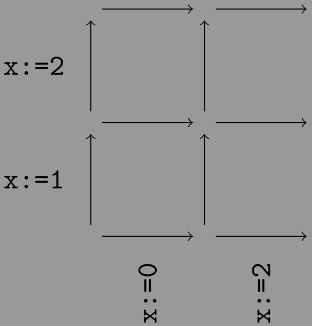

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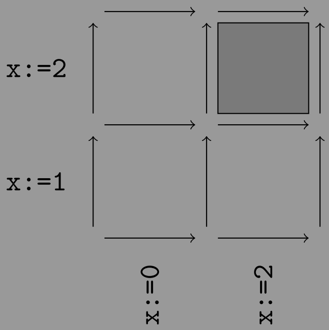

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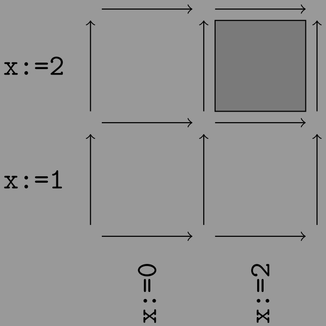

- $p + q \subseteq 2p + 2q$
- $2p + 2q$ is compatible yet $p + q$ is not

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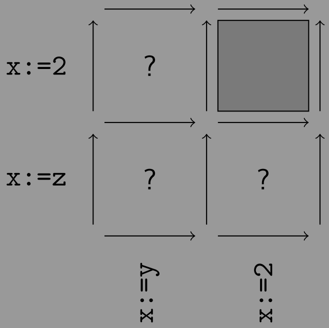

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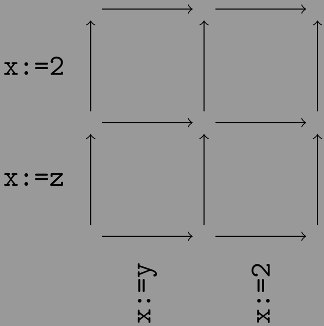

- filling square may depend on the current distribution
- solution: actions with disjoint sets of occuring variables

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Middle-end representation taking race conditions into account

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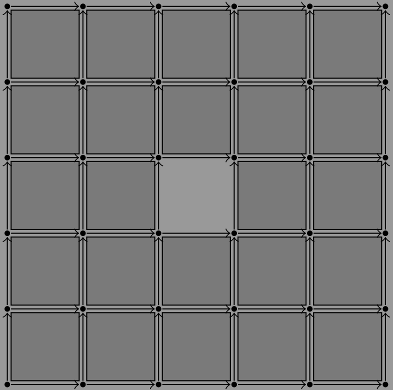
- remove all cubes $\alpha_1 \otimes \cdots \otimes \alpha_k$ s.t. there are $1 \leqslant i < j \leqslant k$ whose actions $\lambda_i(\alpha_i)$ and $\lambda_j(\alpha_j)$ share some variable

Dynamics



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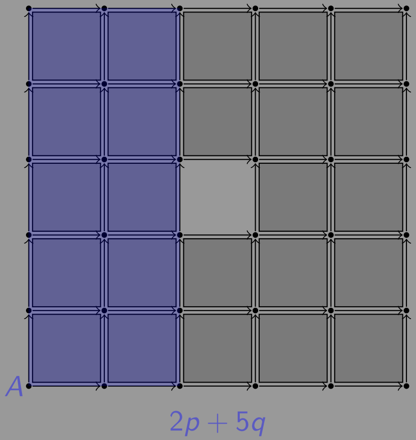

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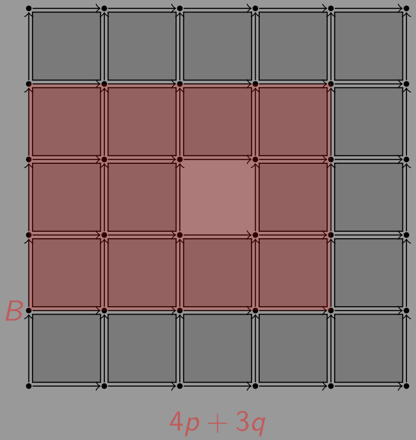

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- how do we deal with nondeterminacy?

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Dijkstra 68 - Input language for ALCOOL in an extended form

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- Instruction `W(x)` blocks the execution of the process until n (arity of `x`) processes are blocked by `x` then all the execution are resumed at the same time

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Potential function along a path

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$$[\![\mathtt{W(a)}]\!] = \text{ignored}$$

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Conservative process

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  is the action of the path γ on the distribution δ
- A process is conservative when for all paths γ, γ' on its cfg, all $x \in \mathcal{R}$ and all distributions δ

$$\partial^- \gamma = \partial^- \gamma' \text{ and } \partial^+ \gamma = \partial^+ \gamma' \;\Rightarrow\; [\![\gamma]\!] \cdot \delta(x) = [\![\gamma']\!] \cdot \delta(x)$$

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is decidable

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  Assuming s_n is built, s_{n+1} is defined for all $v \in V$ by

$$s_{n+1}(v) = s_n(v) \cup \bigcup_{f \in A;\ \partial^+ f = v;\ \lambda(f) \in \{\mathtt{P},\mathtt{V}\}} f \cdot s_n(\partial^- f)$$

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- in the first case we have the potential function
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  $F(v, x) = \delta(x)$ where $s_n(v) = \{\delta\}$
  note that if $s_n(v) = \emptyset$ then v is unreachable

The potential function

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- The potential function $F : K_0 \times \mathcal{R} \to \mathbb{N}$ is

$$F(v_1, \ldots, v_d, x) = \sum_{k=1}^{d} F_k(v_k, x)$$

The control flow precubical set

taking `P`, `V`, and `W` into account

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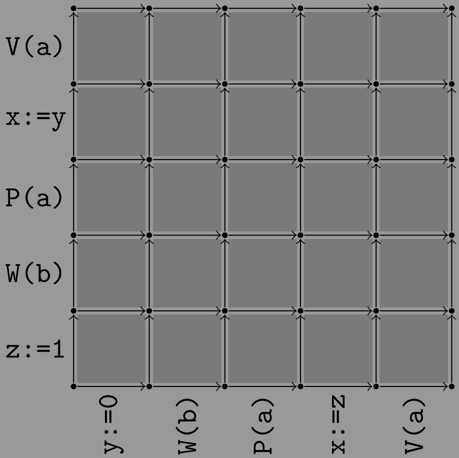
Control Flow Precubical Set: an example

y:=0.W(b).P(a).x:=z.V(a)|z:=0.W(b).P(a).x:=y.V(a)



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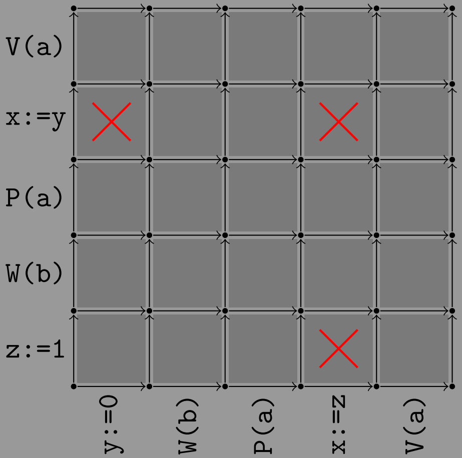

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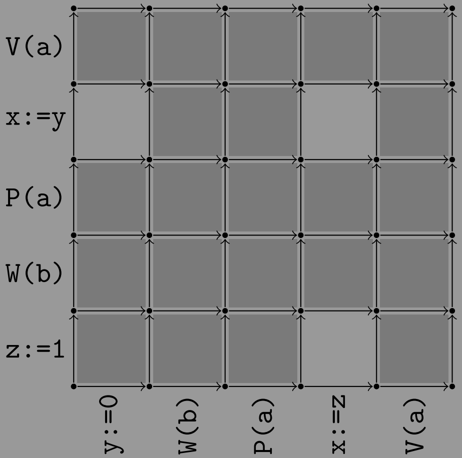

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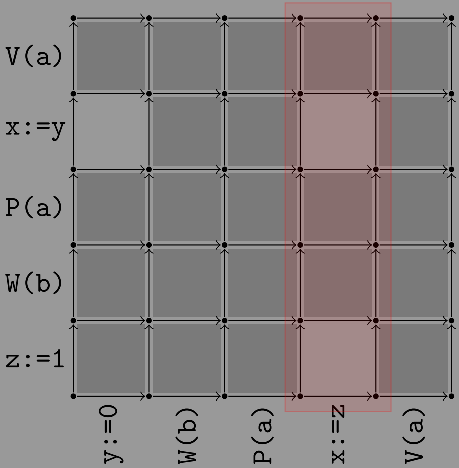

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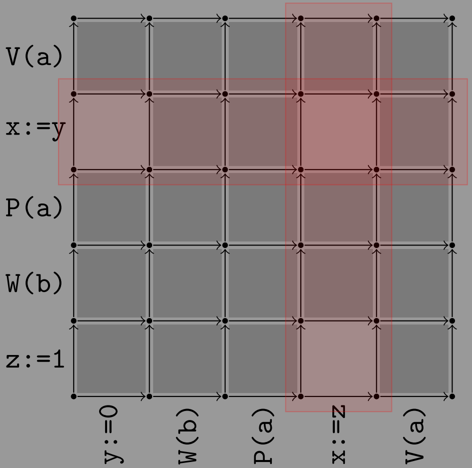

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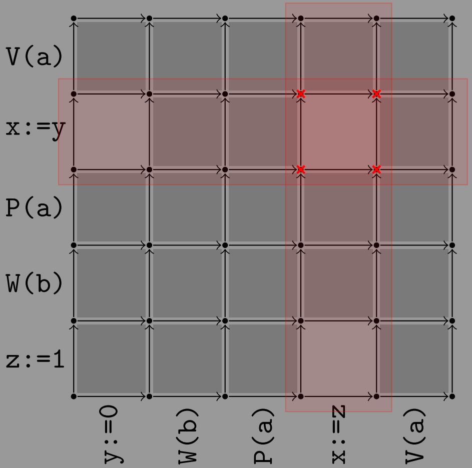

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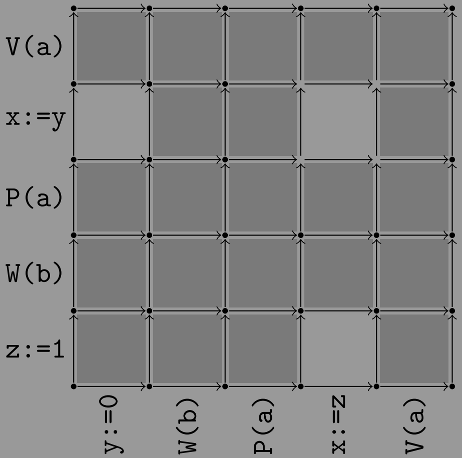

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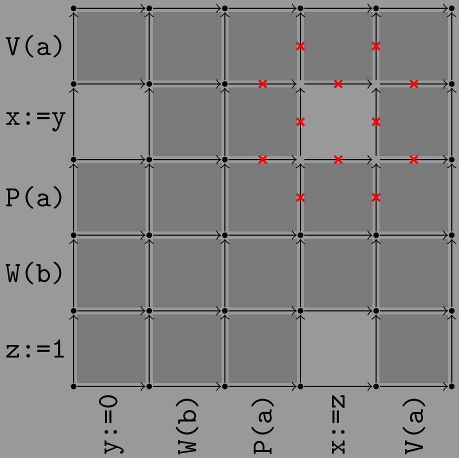

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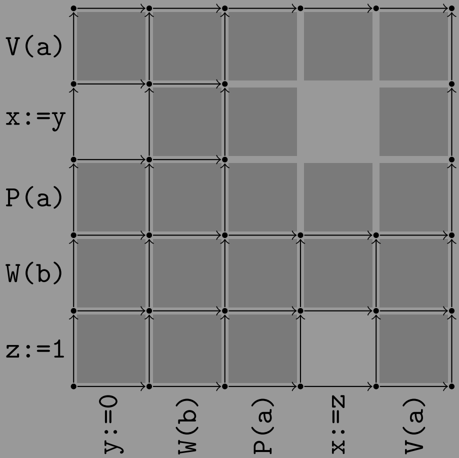

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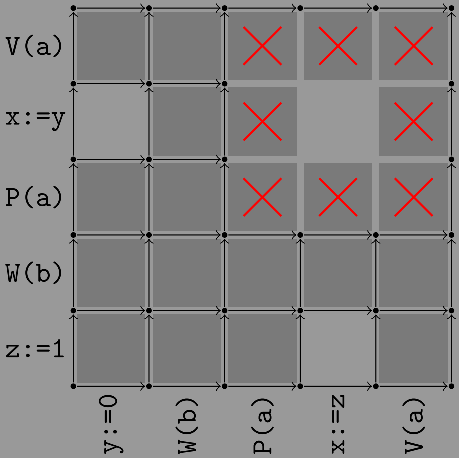

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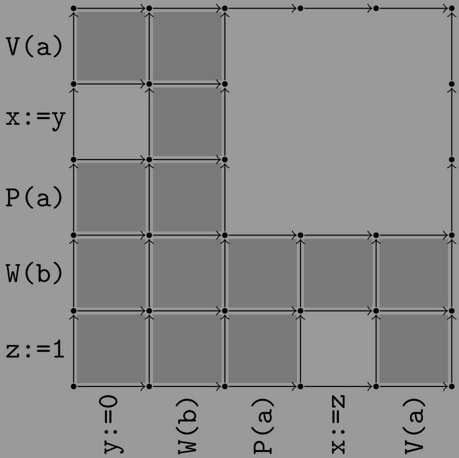

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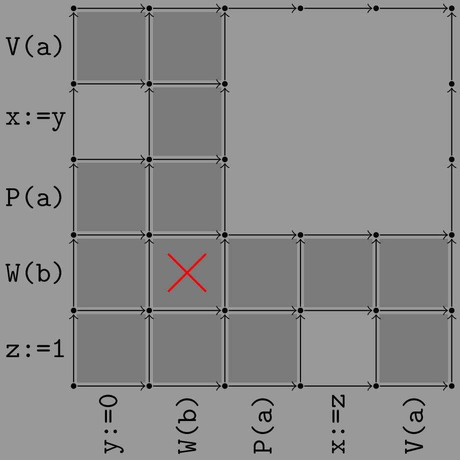

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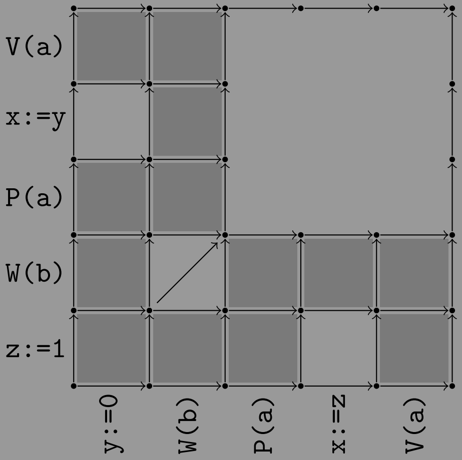

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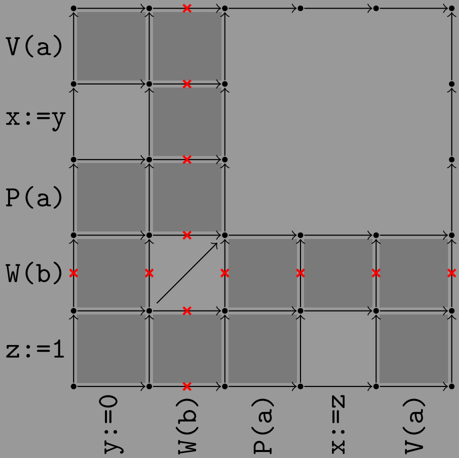

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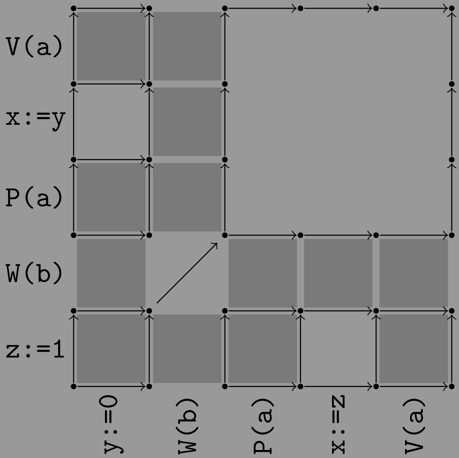

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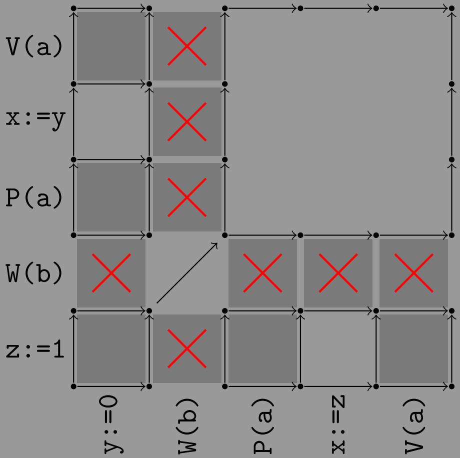

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