NON-HAUSDORFF SMOOTH MANIFOLDS

FOR

MODELLING CONCURRENT PROGRAMS

MOTIVATION

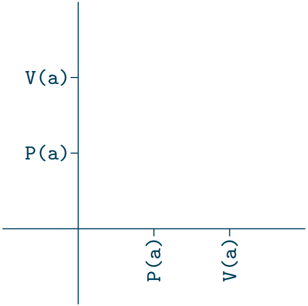
```
sem 1 a
proc:  p = P(a);V(a)
init:  2p
```

```
sem 1 a
proc:  p = P(a);V(a)
init:  2p
```

horizontal

vertical

```
sem 1 a
proc:  p = P(a);V(a)
init:  2p
```



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sem 1 a
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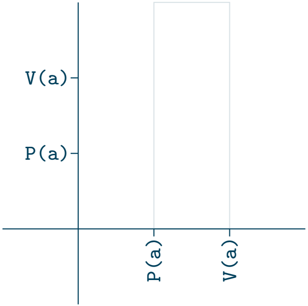

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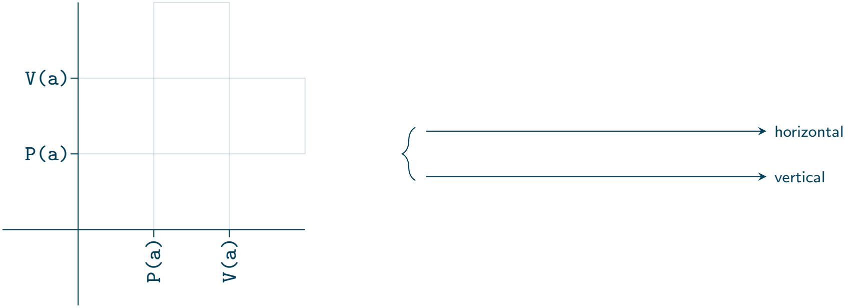

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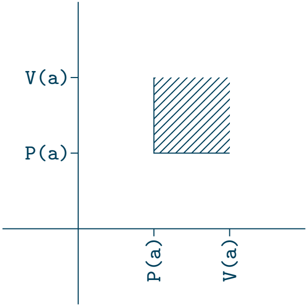

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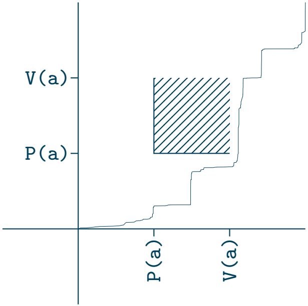

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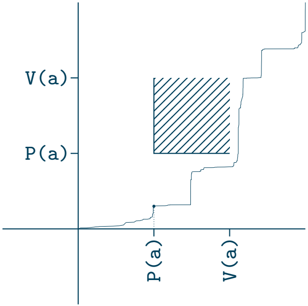

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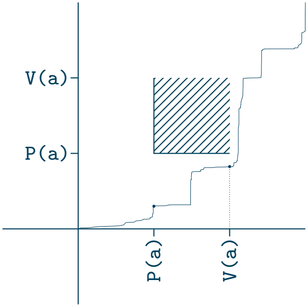

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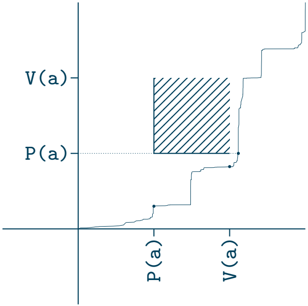

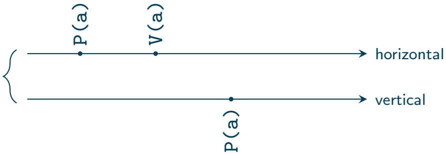

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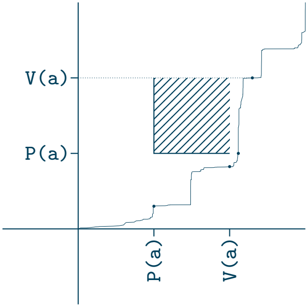

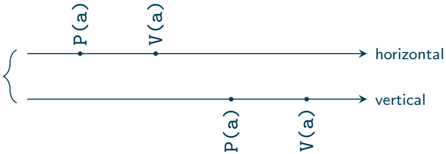

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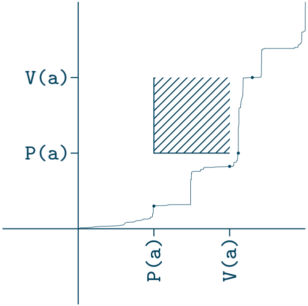

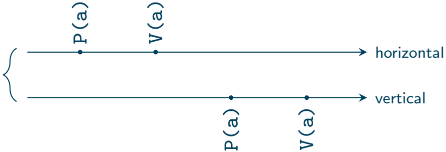

The geometry of semaphore programs. Carson, S. D., and Reynolds, P. F., ACM Trans. 1987.

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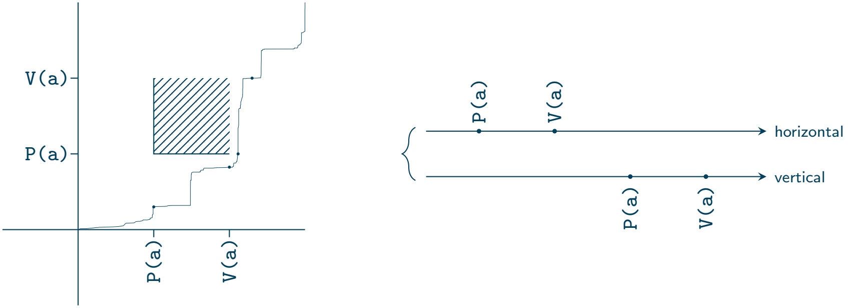

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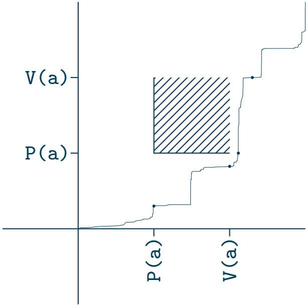

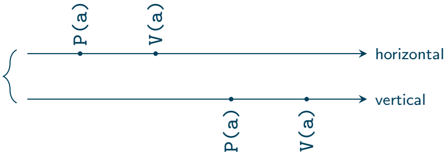

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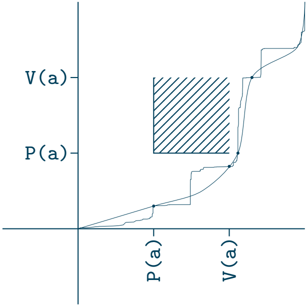

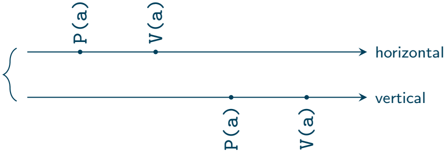

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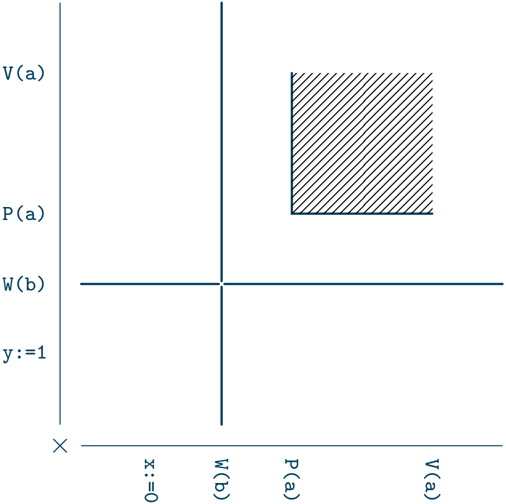
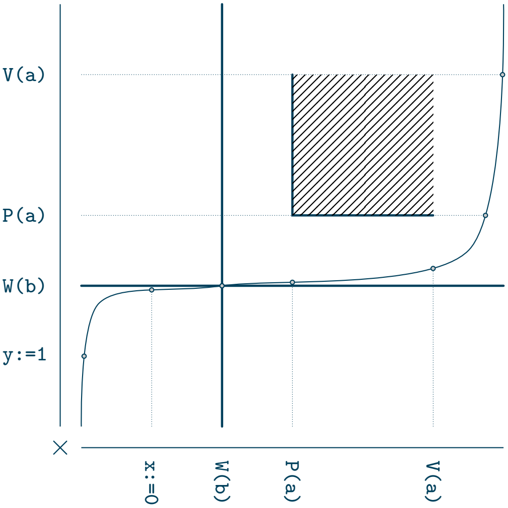
The geometry of conservative programs. Haucourt E., MSCS, 2018.

Non-Hausdorff parallelized manifolds over geometric models of conservative programs. Haucourt E., MSCS, 2025. (to appear)



V(a)
P(a)
W(b)
y:=1
x:=0
W(b)
P(a)
V(a)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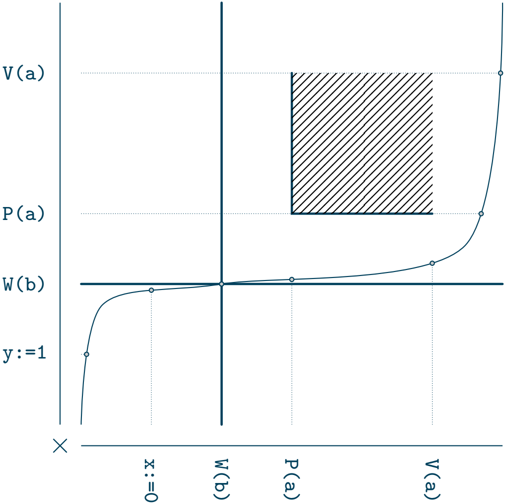
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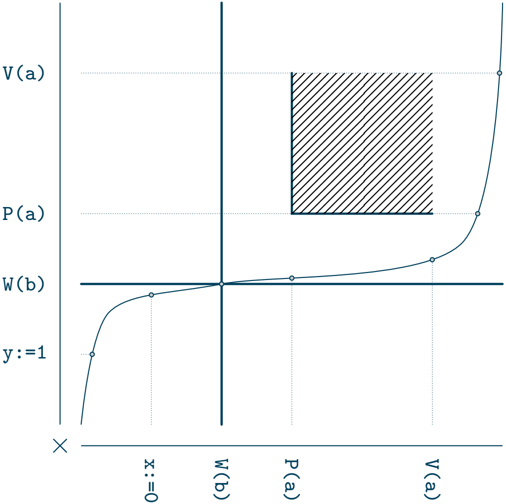
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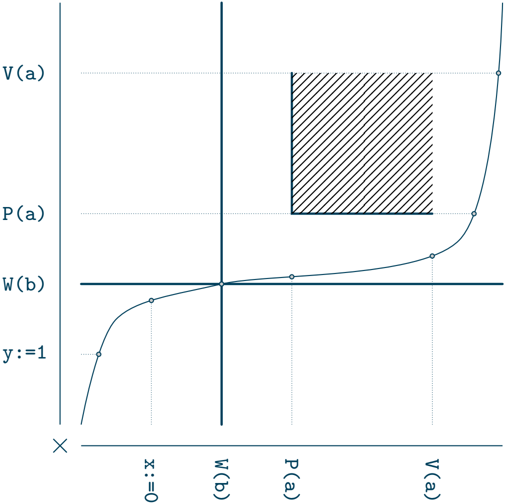

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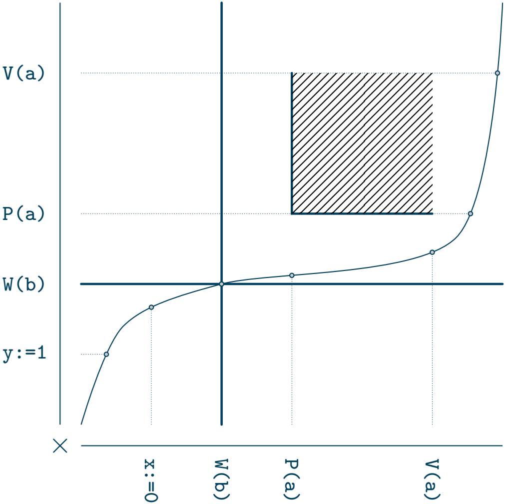
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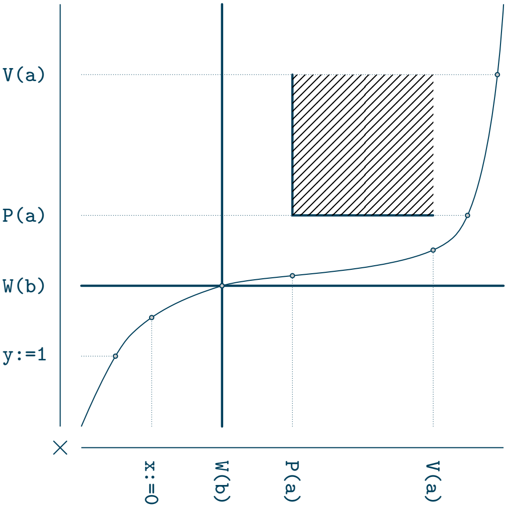

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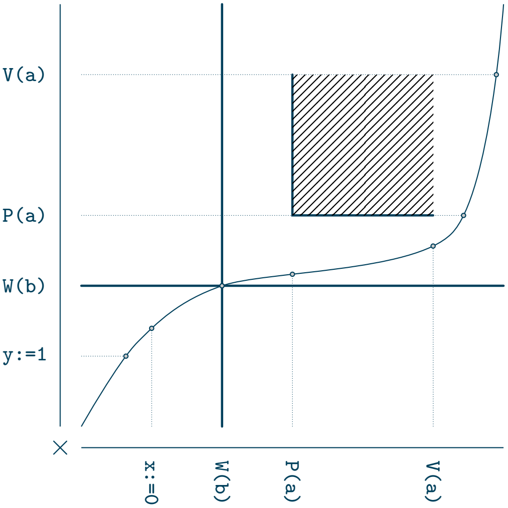

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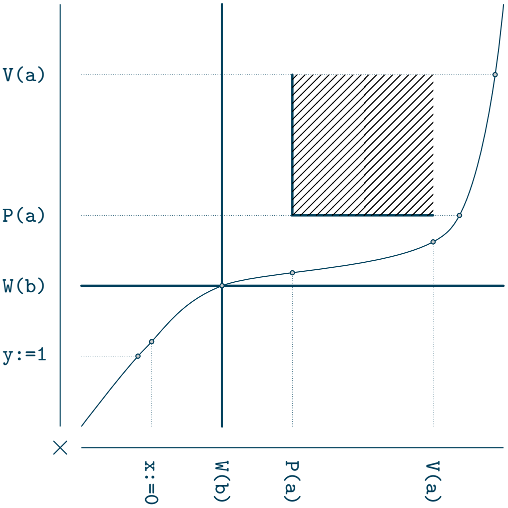

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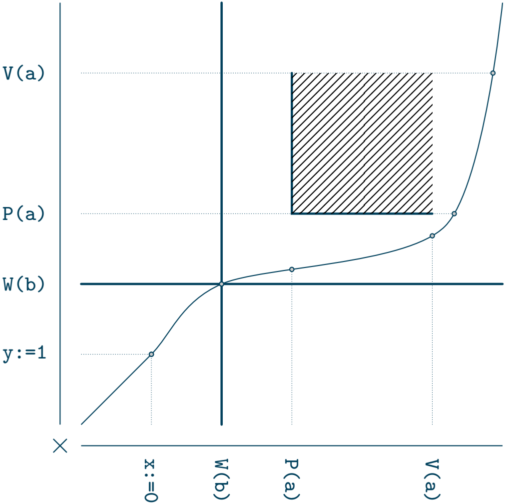

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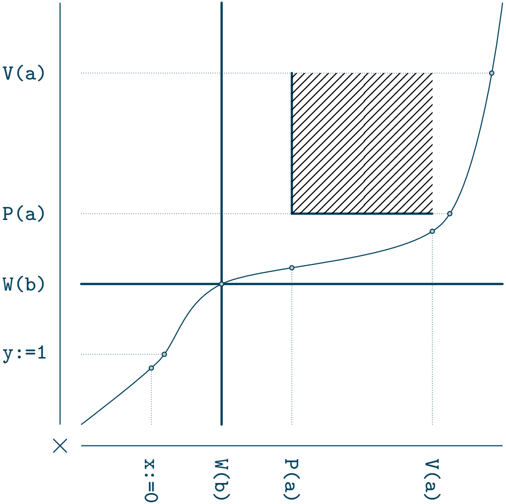

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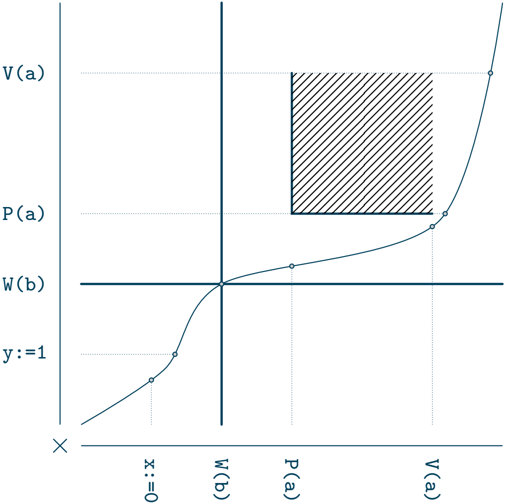

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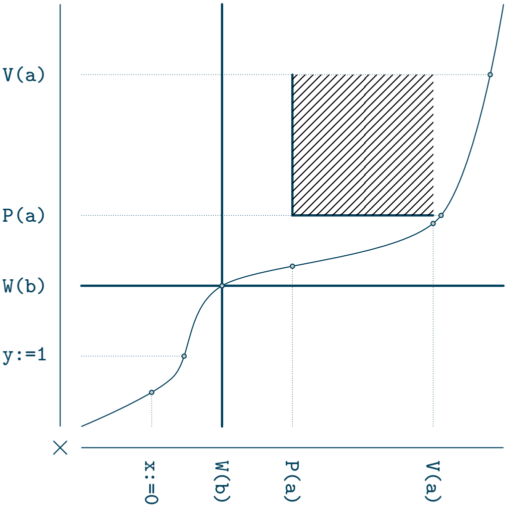

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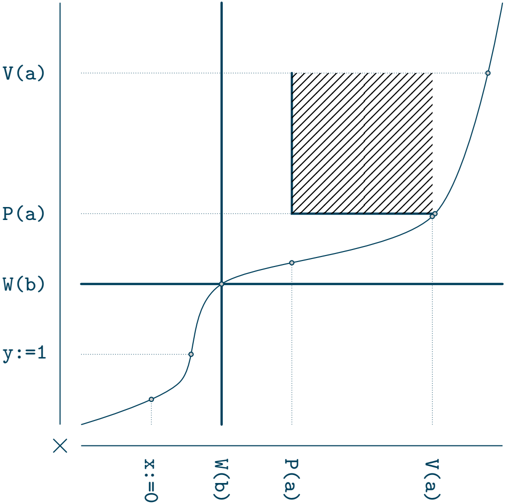
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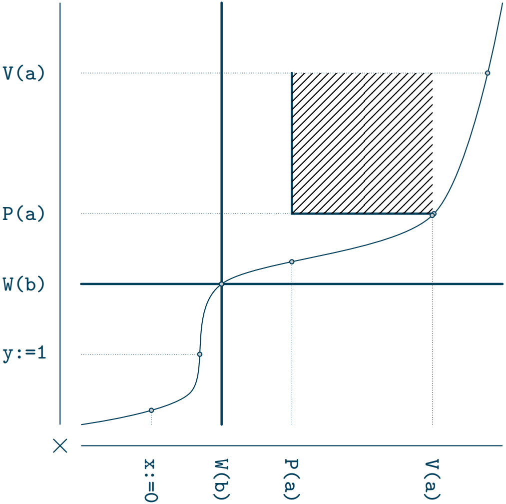
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THE BIG PICTURE

$$\underbrace{P_1 \quad | \quad \cdots \quad | \quad P_n}_{\text{program } P}$$

$$\underbrace{G_1 \quad , \quad \ldots \quad , \quad G_n}_{\text{graphs}}$$

$$\underbrace{P_1 \quad | \quad \cdots \quad | \quad P_n}_{\text{program } P}$$

$$\underbrace{|G_1| \quad \times \quad \cdots \quad \times \quad |G_n|}_{\text{sets}}$$

$$\underbrace{G_1 \quad , \quad \ldots \quad , \quad G_n}_{\text{graphs}}$$

$$\underbrace{P_1 \quad | \quad \cdots \quad | \quad P_n}_{\text{program } P}$$

$$|P| \subseteq \underbrace{|G_1| \times \cdots \times |G_n|}_{\text{sets}}$$

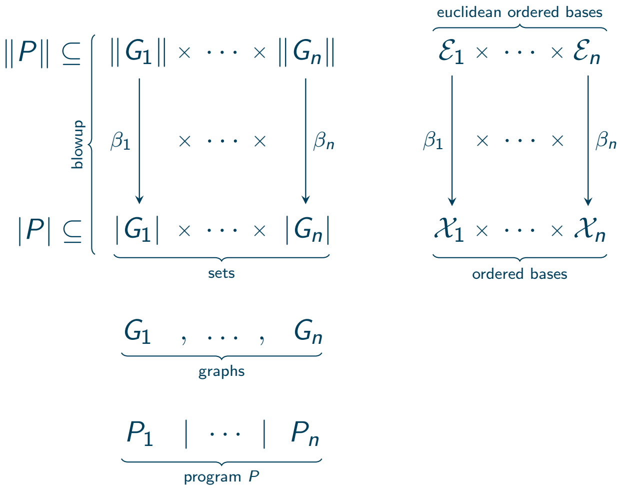
$$\underbrace{G_1 , \ldots , G_n}_{\text{graphs}}$$

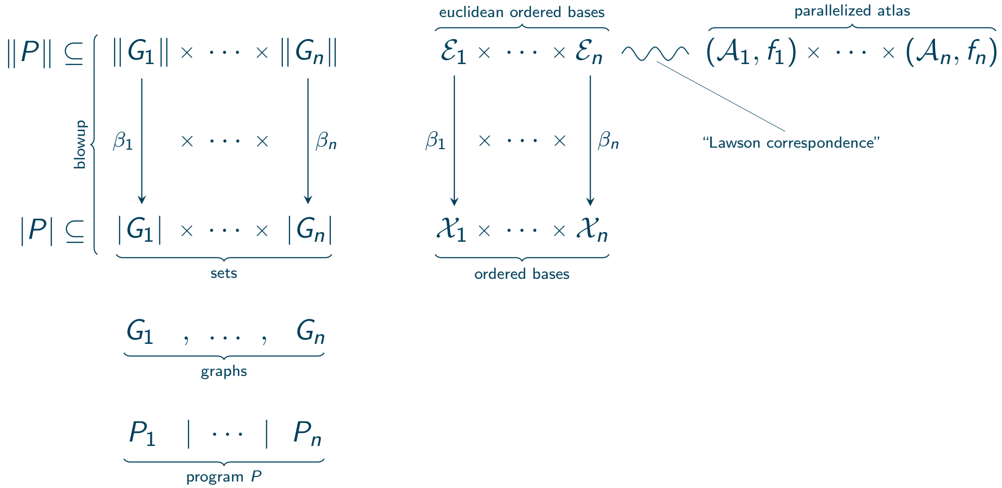
$$\underbrace{P_1 \mid \cdots \mid P_n}_{\text{program } P}$$

$$|P| \subseteq \underbrace{|G_1| \times \cdots \times |G_n|}_{\text{sets}} \qquad \underbrace{\mathcal{X}_1 \times \cdots \times \mathcal{X}_n}_{\text{ordered bases}}$$

$$\underbrace{G_1 \quad , \quad \ldots \quad , \quad G_n}_{\text{graphs}}$$

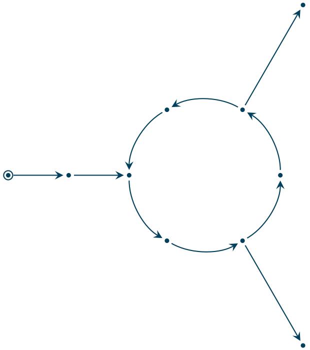
$$\underbrace{P_1 \quad | \quad \cdots \quad | \quad P_n}_{\text{program } P}$$

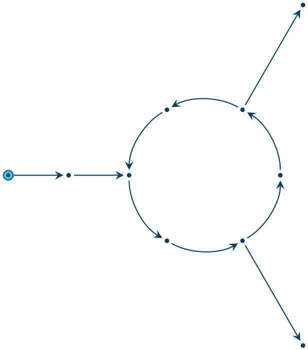

euclidean ordered bases
‖P‖ ⊆ ‖G1‖ × ⋯ × ‖Gn‖
E1 × ⋯ × En
blowup
β1 × ⋯ × βn
|P| ⊆ |G1| × ⋯ × |Gn|
X1 × ⋯ × Xn
sets
ordered bases
G1 , … , Gn
graphs
P1 | ⋯ | Pn
program P

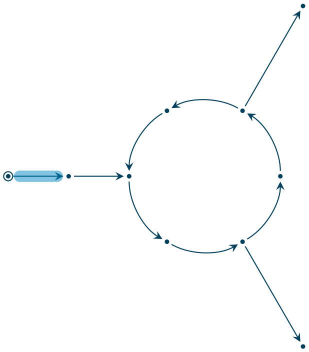


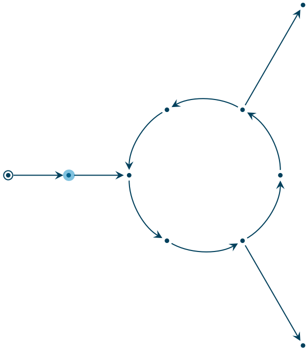
Ordered manifolds, invariant cone fields, and semigroups. Lawson, J. D., Forum Mathematicum, 1989.

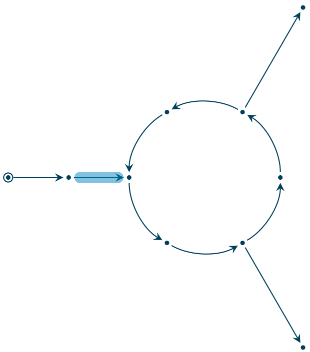
FROM DISCRETE TO CONTINUOUS

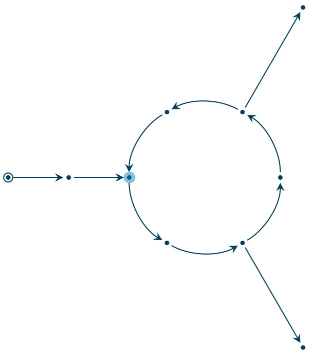


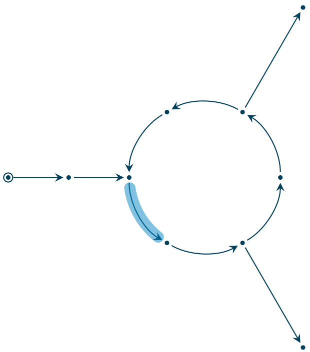


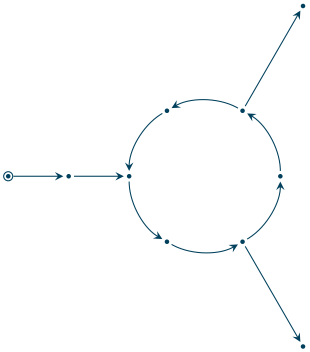


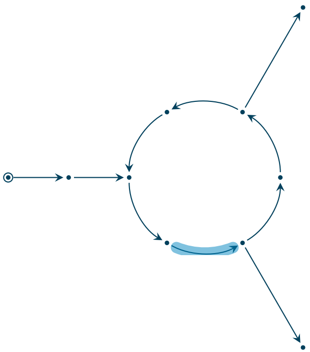


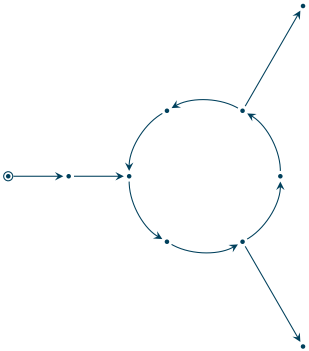


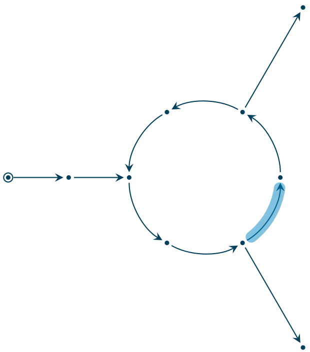


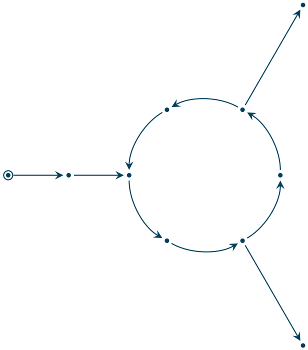


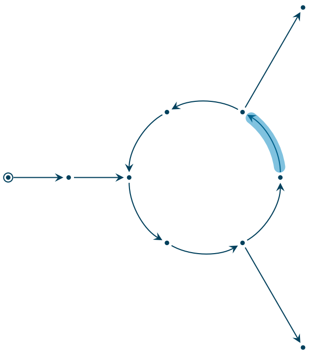


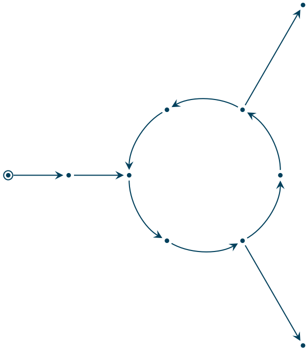


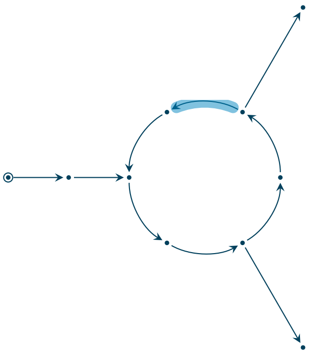


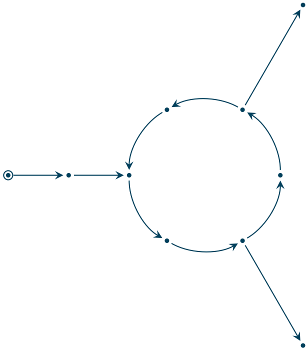


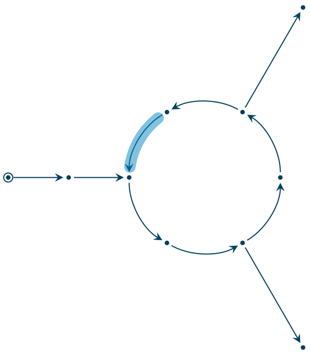


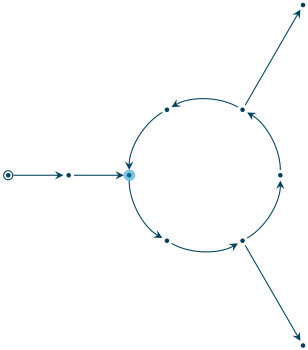


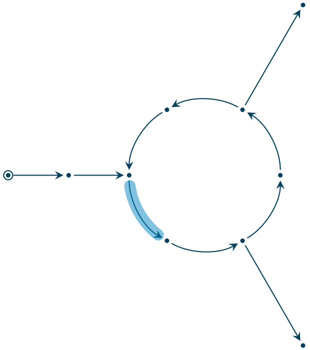


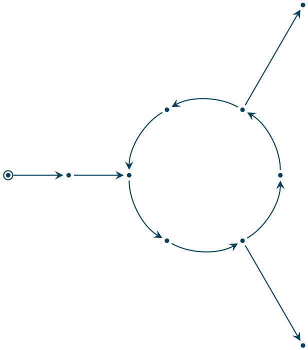


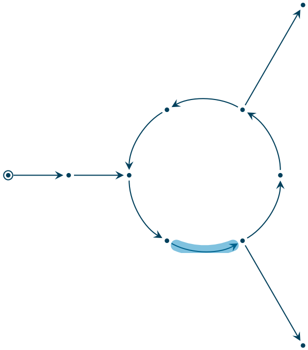


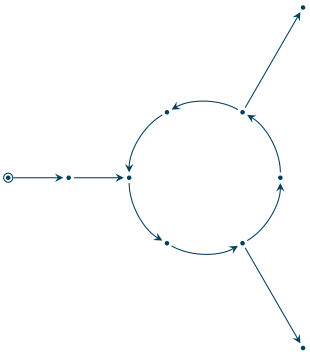


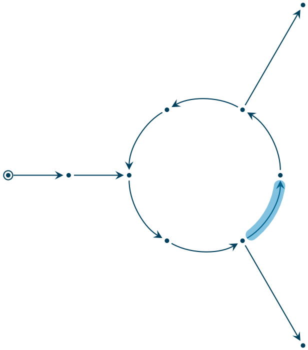


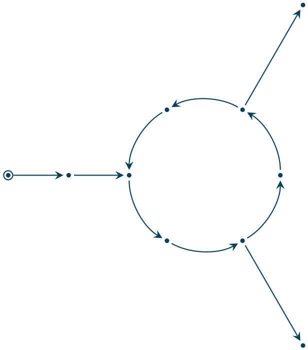


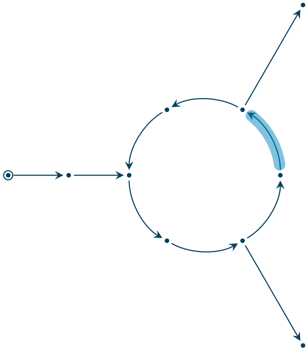


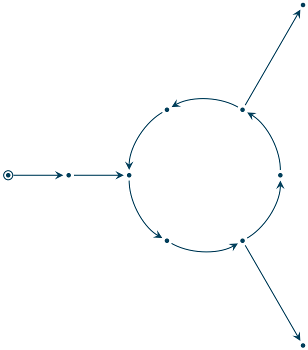


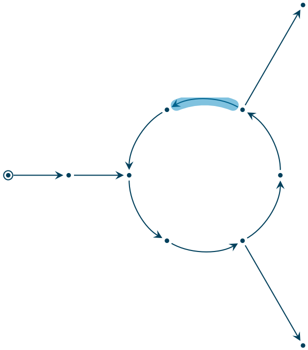


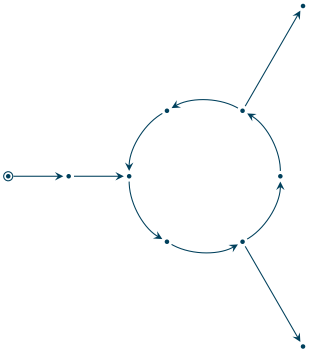


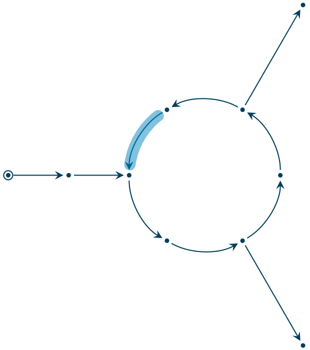


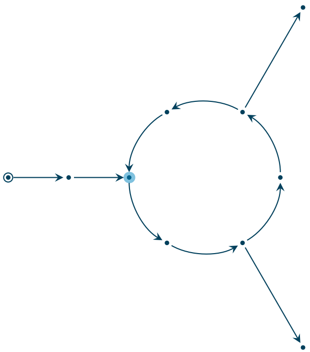


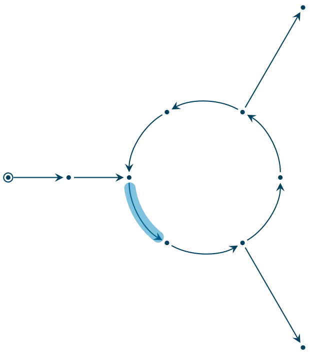


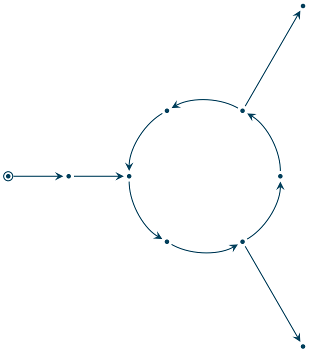


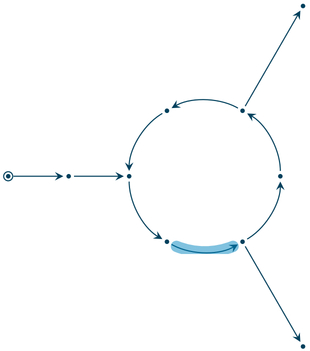


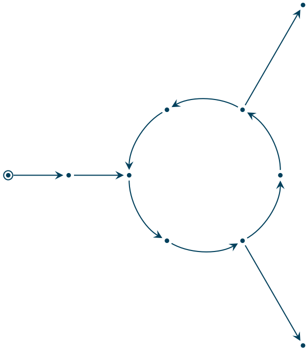


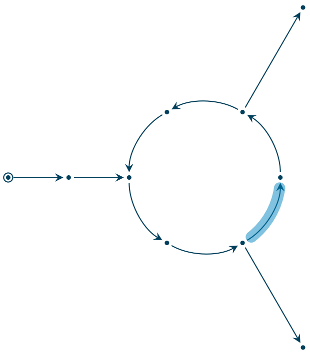


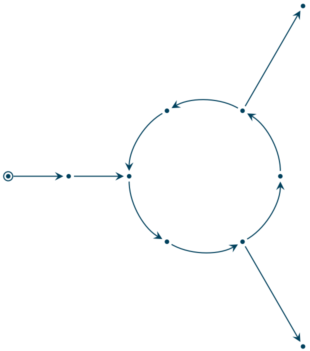


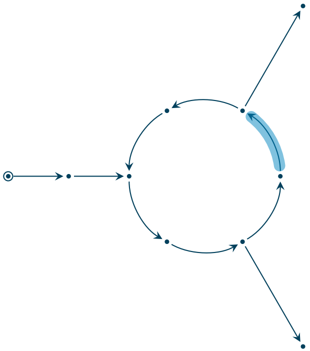


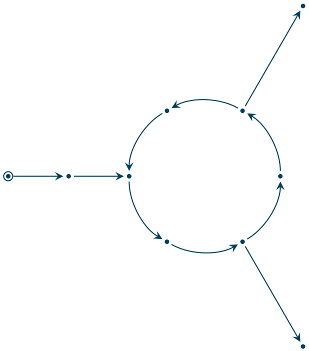


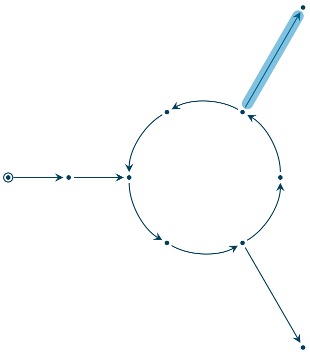


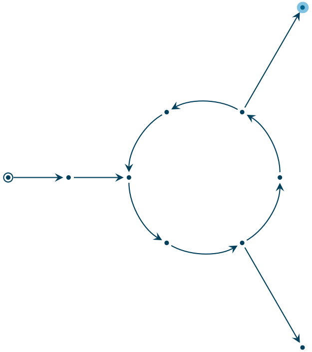


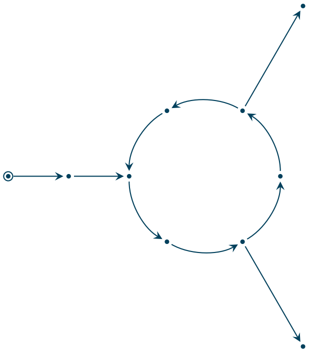


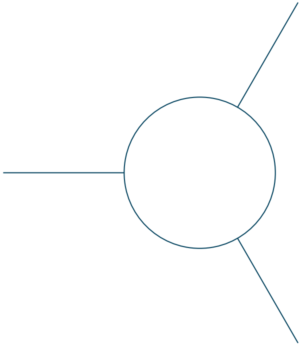












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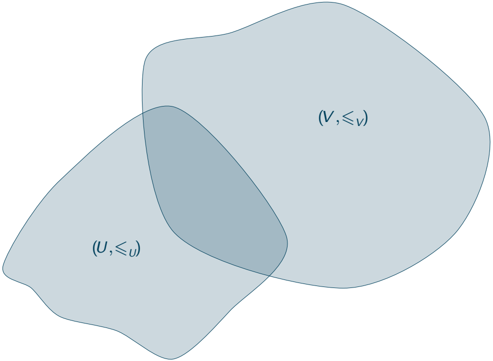

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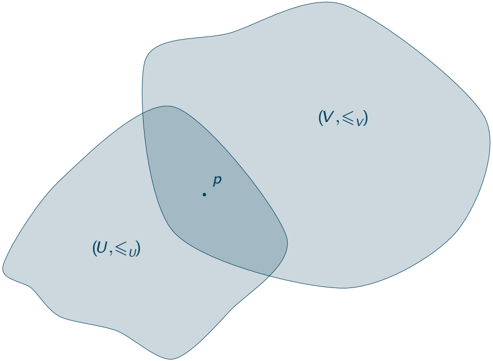

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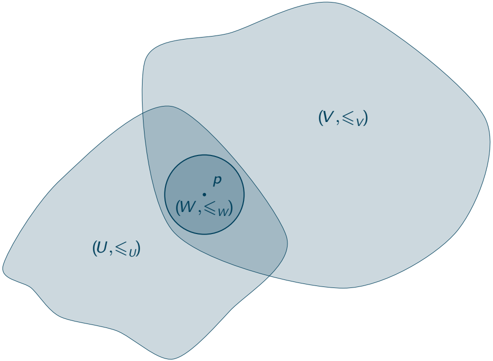

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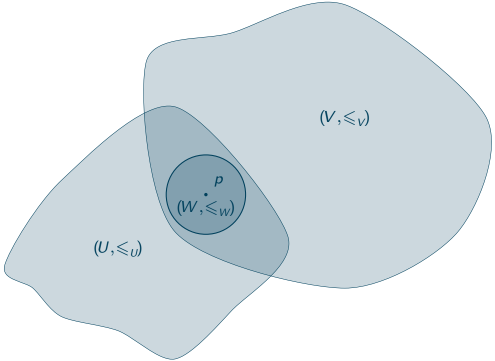

An ordered base $\mathcal{E}$ is said to be *euclidean* of dimension $n \in \mathbb{N}$ when every point p of $\mathcal{E}$ is contained in some $E \in \mathcal{E}$ with $E \cong \mathbb{R}^n$ (as ordered spaces).

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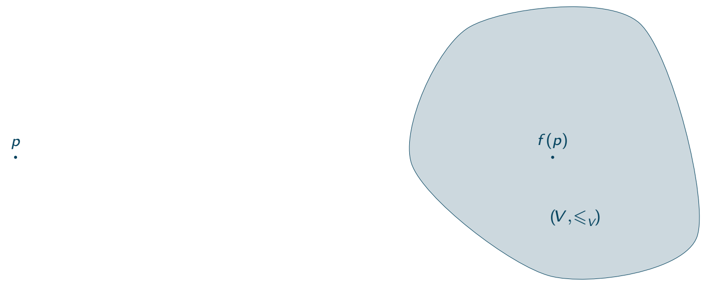

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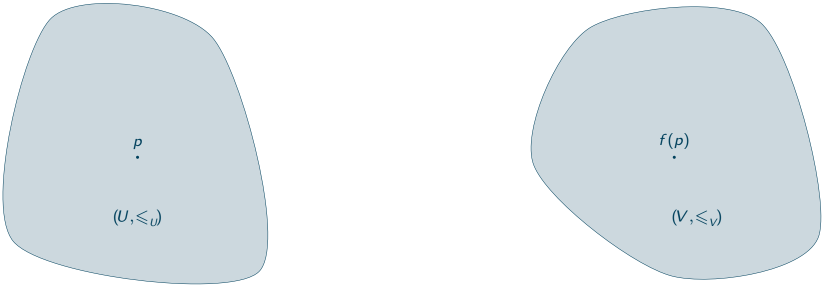

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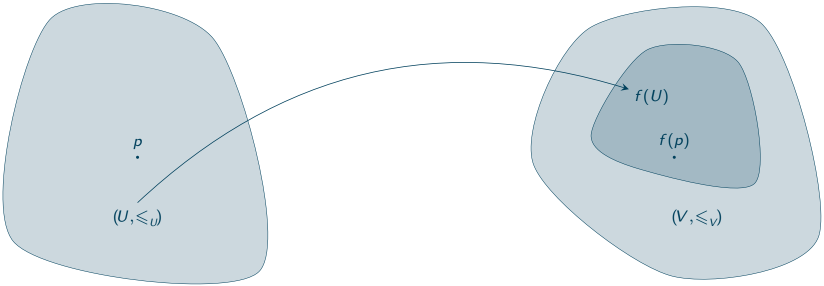

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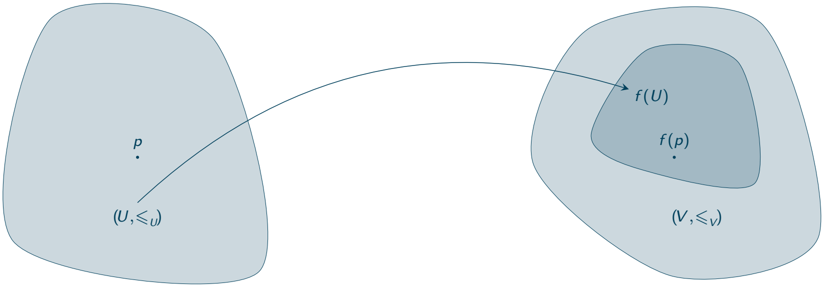

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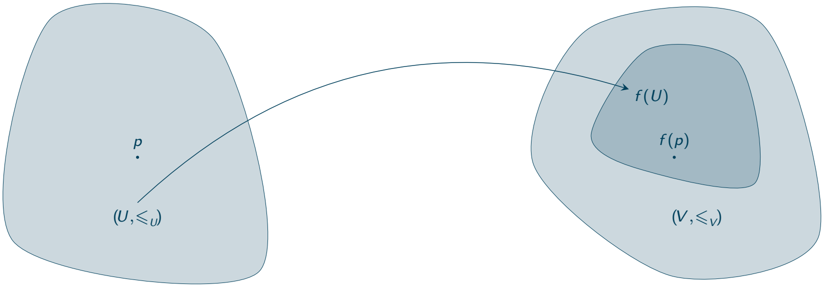

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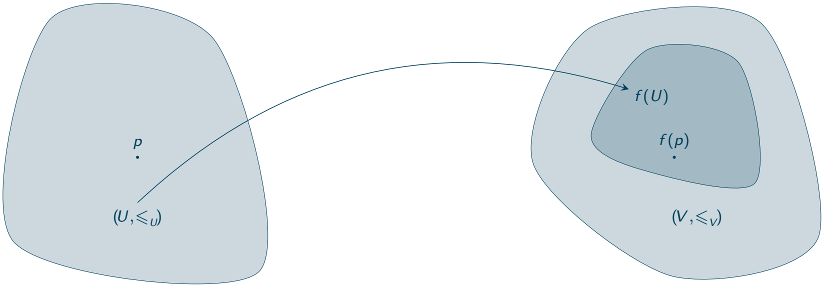

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If $\mathcal{U}$ is a directed compact interval, then f is said to be a *directed path* on $\mathcal{V}$.

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$$\begin{aligned} \pi_G \quad : \quad |G| &\rightarrow G \\ v &\mapsto v \\ (a,t) &\mapsto a \end{aligned}$$

Proposition

There exists a (unique) intrinsic metric d_G *on* $|G|$ *such that the open balls of radii* $\varepsilon > 0$ *about* (a, t) *and* v *are*
$\{a\} \times \,]t - \varepsilon, t + \varepsilon[$ *if* $\varepsilon \leqslant \min(t, 1 - t)$*, and*
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$d_G((a,t),(a,t')) = \min\{t'-t, 1-(t'-t)\}$	$(a,t') \sqsubseteq (a,t)$	if $t \in]0,\varepsilon[$ and $t' \in]1-\varepsilon, 1[$
$d_G((a,t),(b,t')) = d_G((a,t),v) + d_G(v,(b,t'))$		if $a \neq b$
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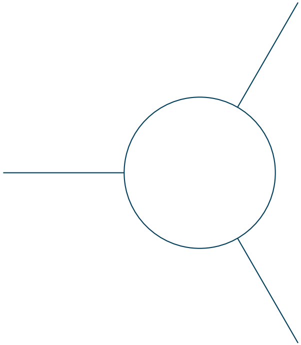
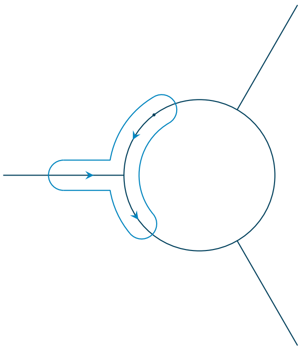
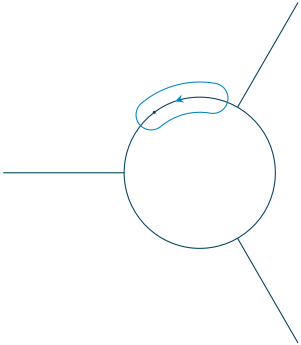
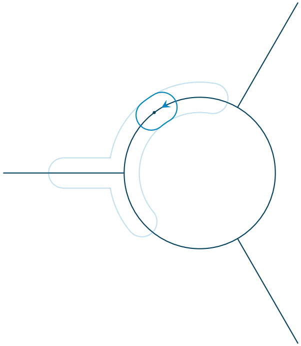
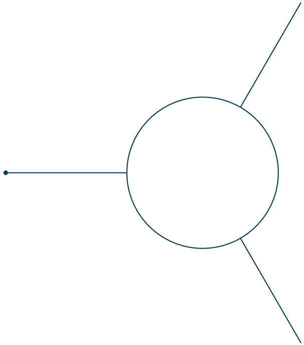
The *standard ordered base* of G is the collection of ordered open balls of radii $\varepsilon \leqslant \frac{1}{2}$ with their 'canonical' partial order.

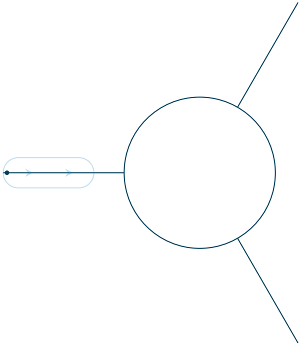


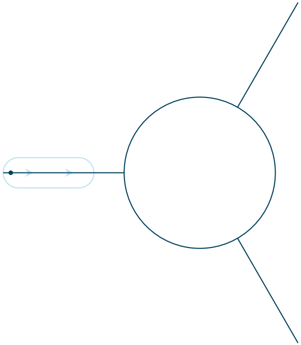


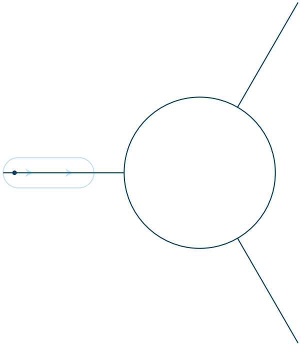


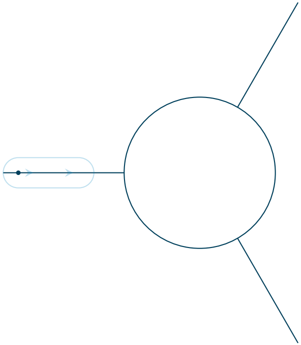


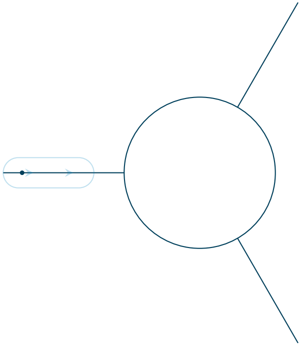


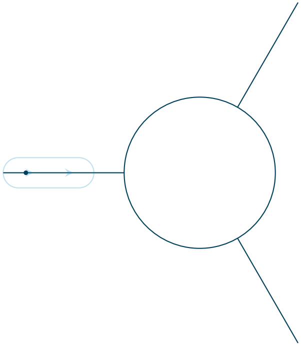


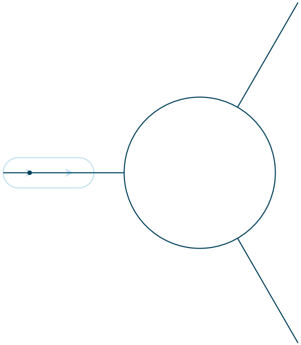


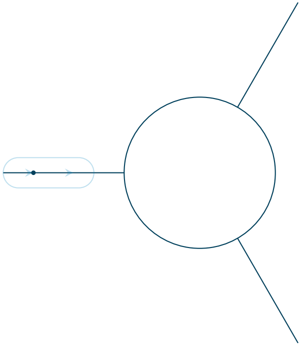


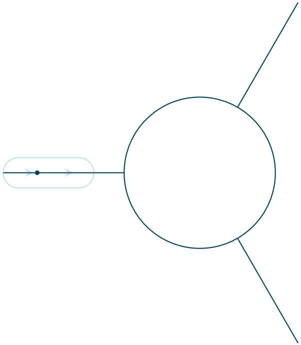


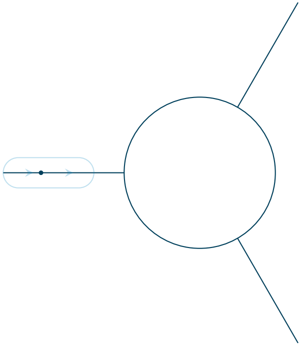


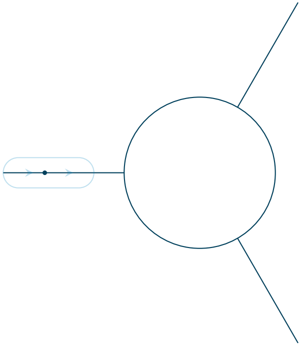


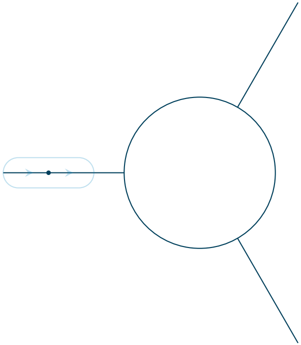


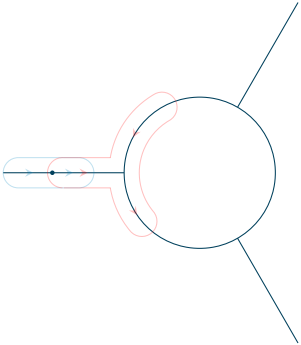


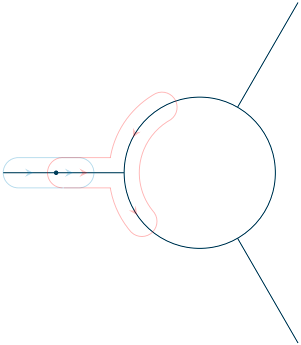


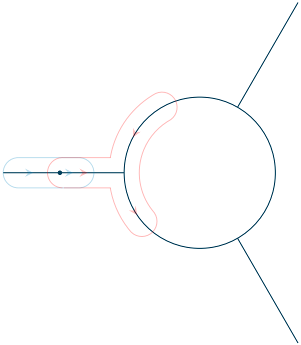


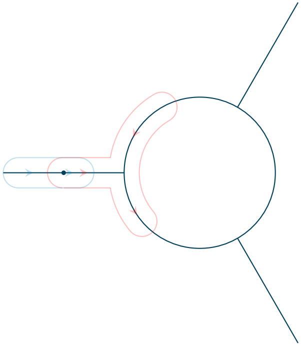


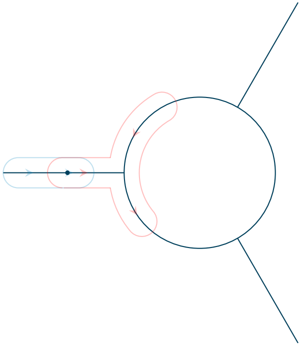


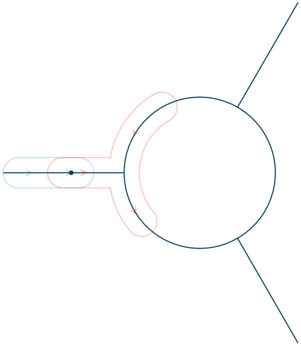


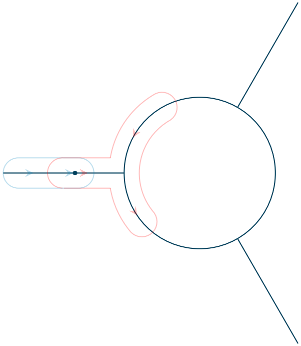


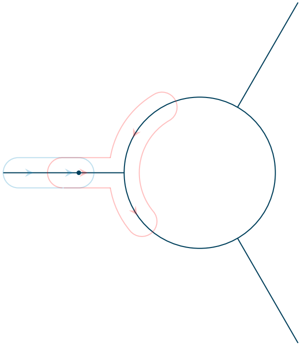


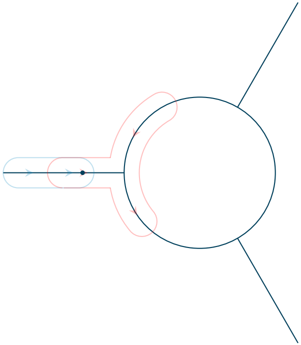


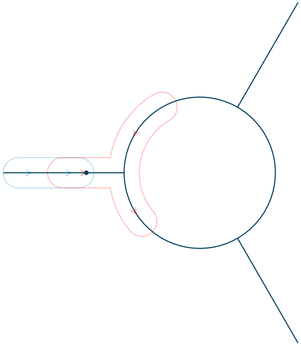


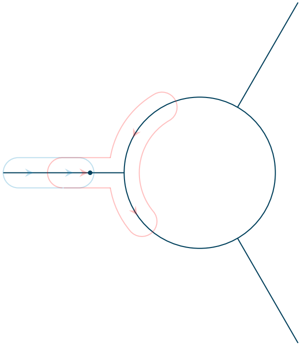


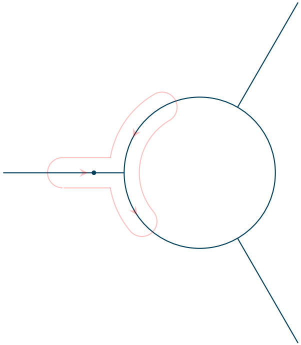


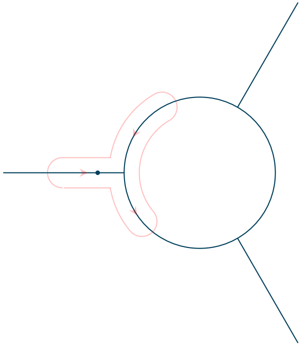


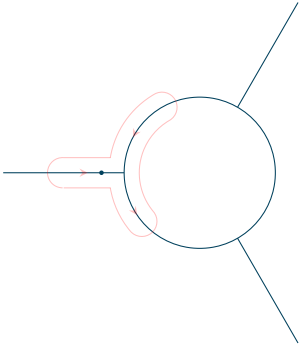


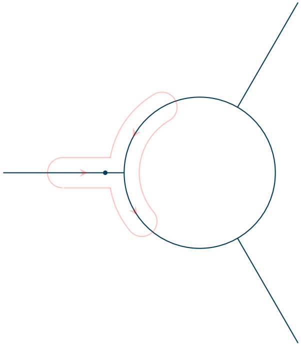


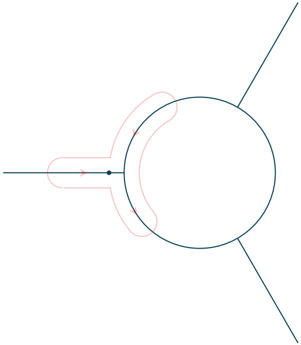


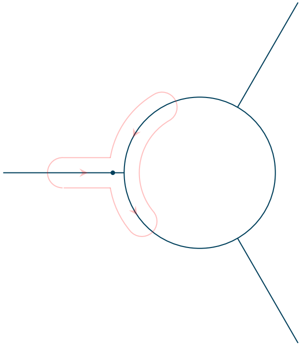


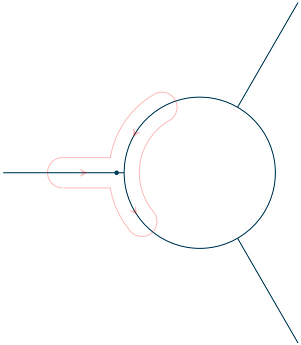


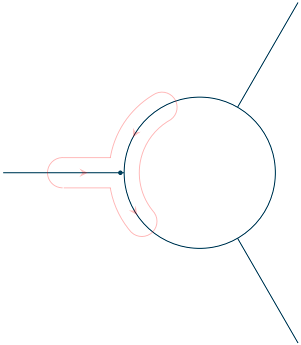


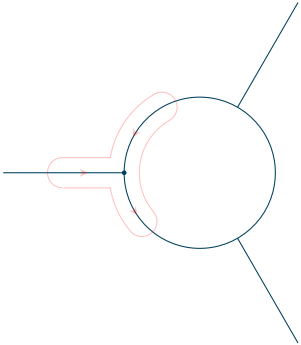


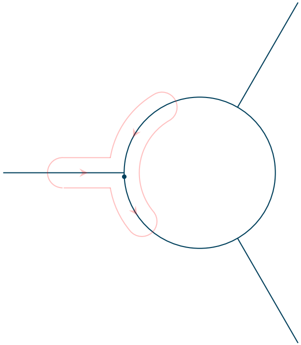


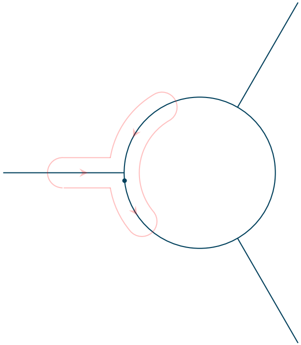


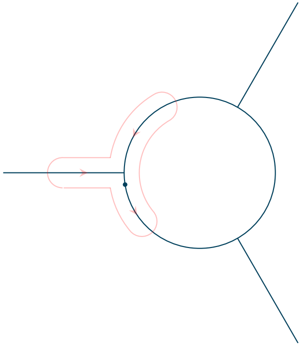


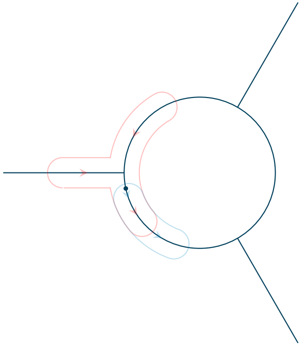


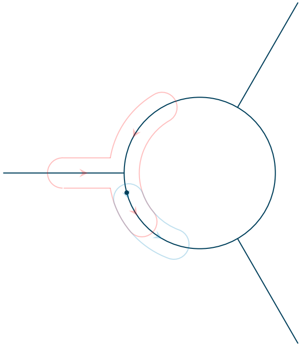


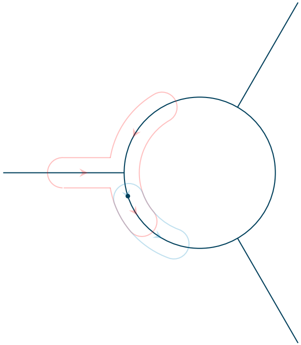


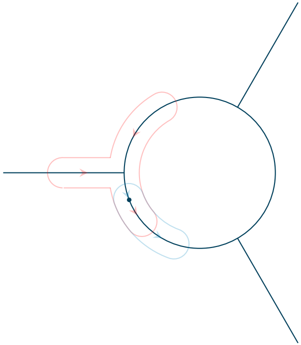


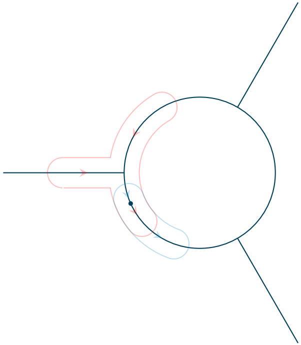


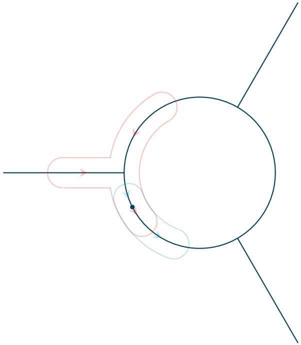


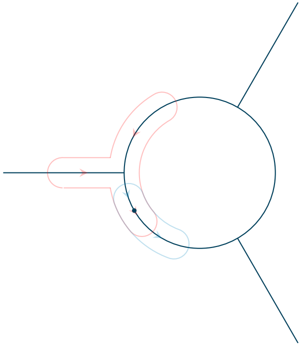


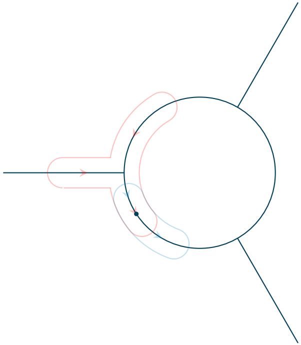


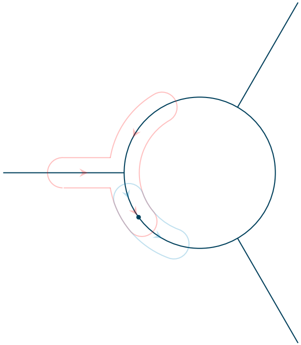


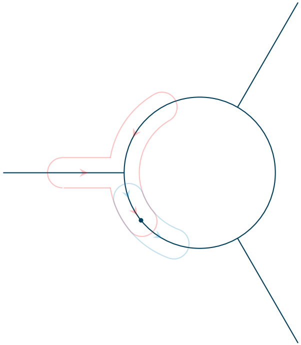


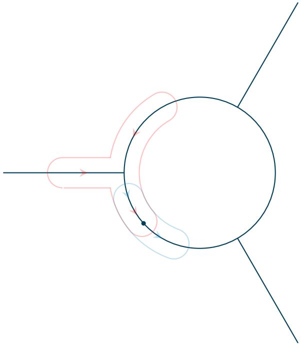


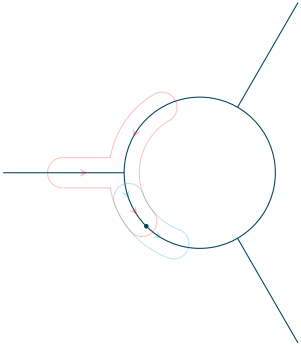


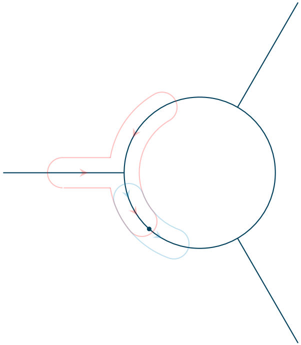


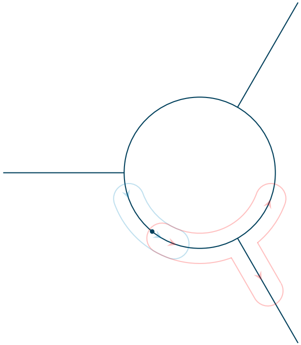


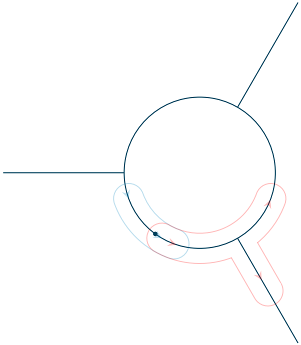


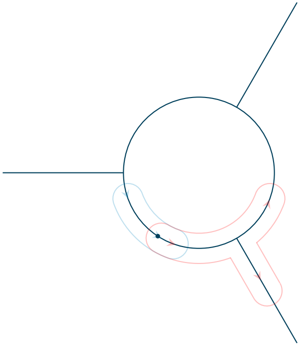


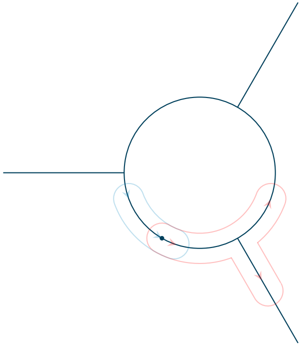


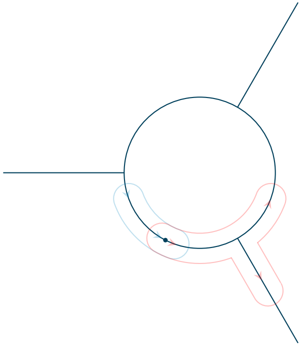


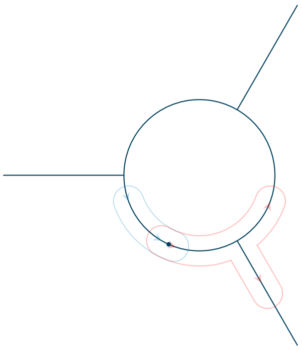


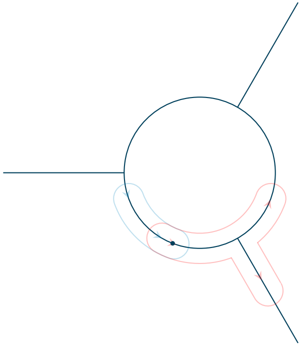


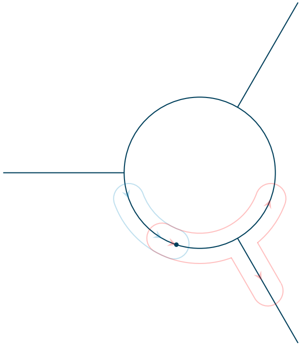


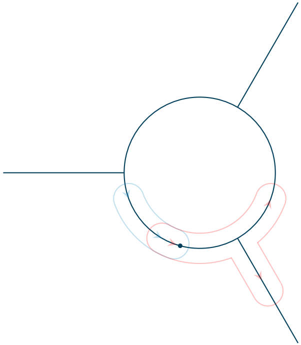


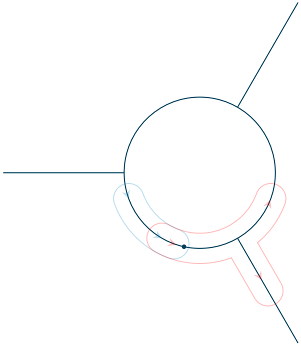


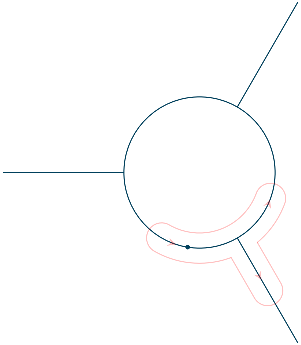


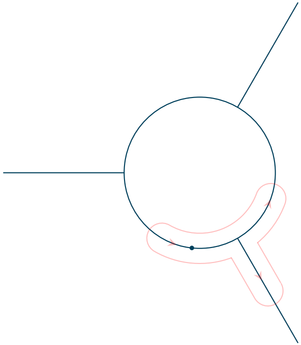


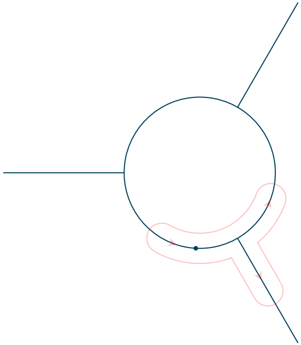


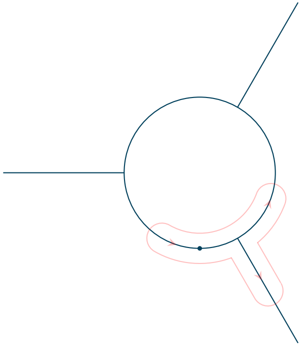


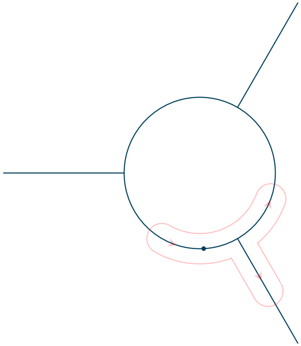


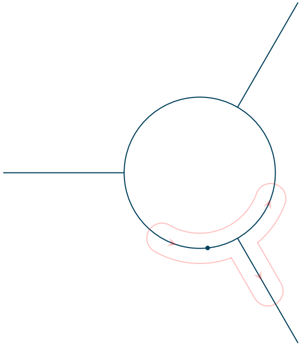


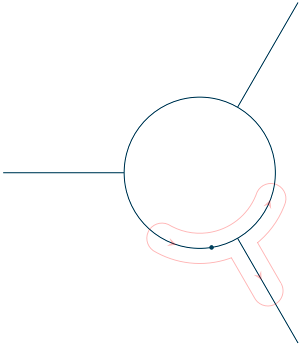


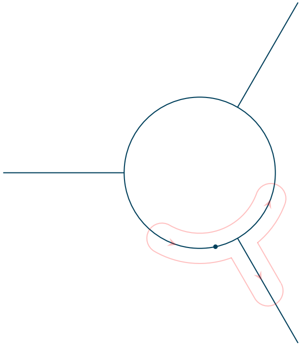


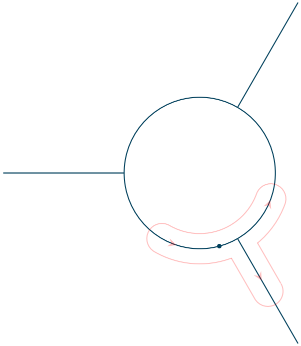


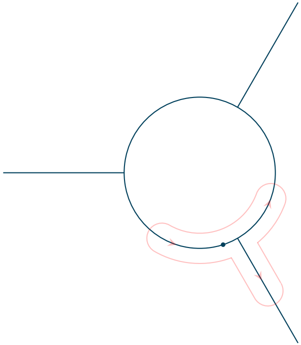


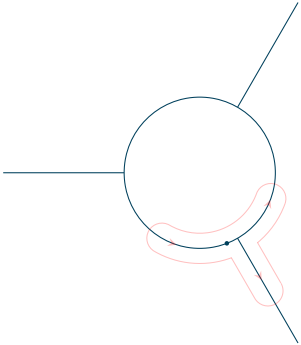


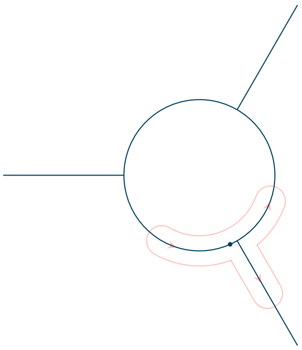


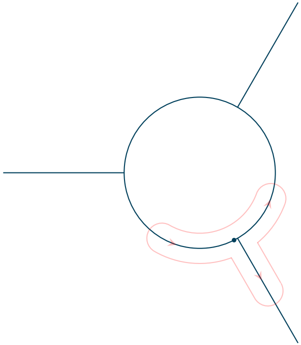


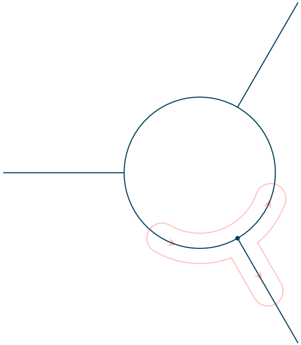


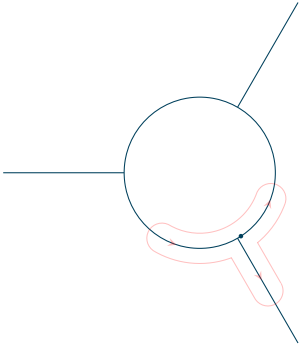


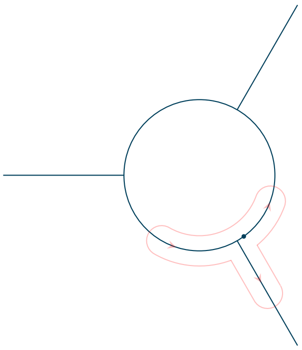


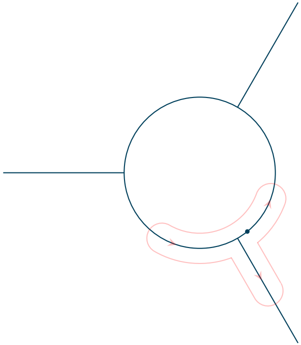


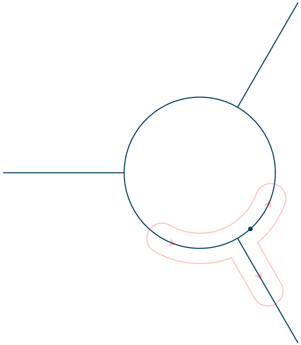


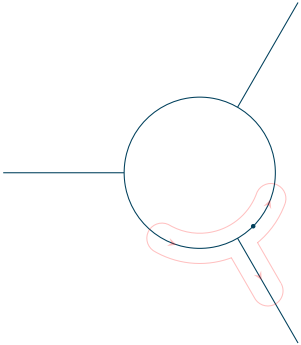


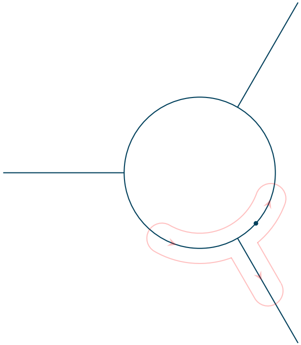


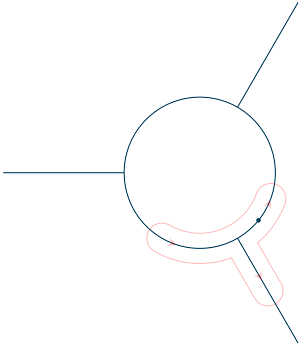


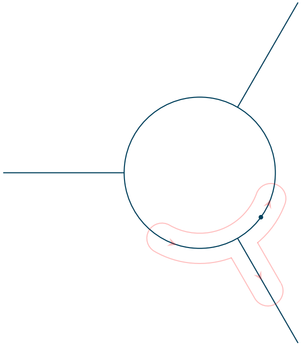


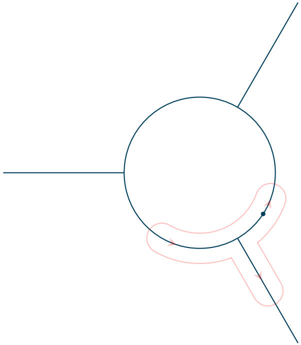


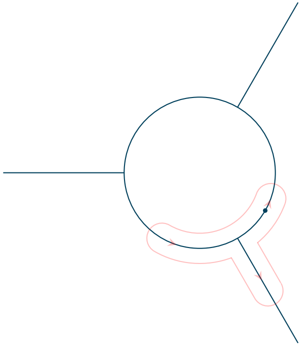


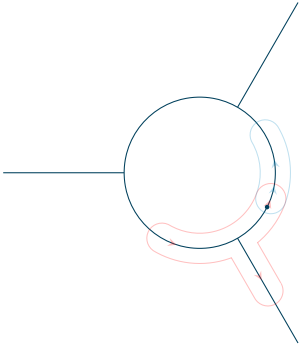


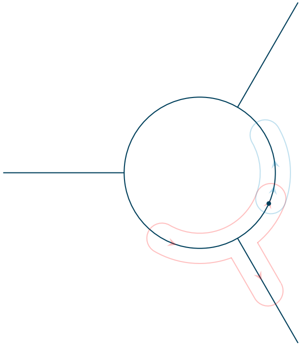


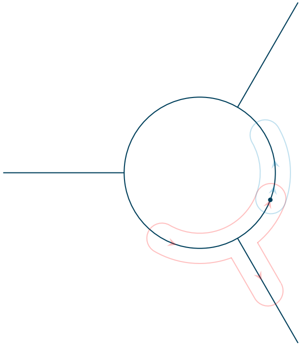


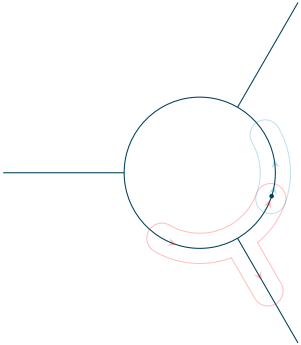


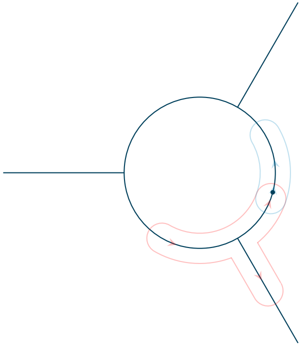


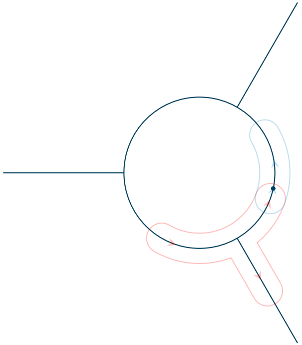


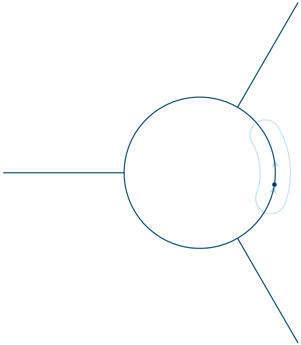


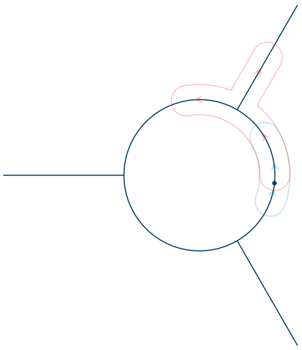


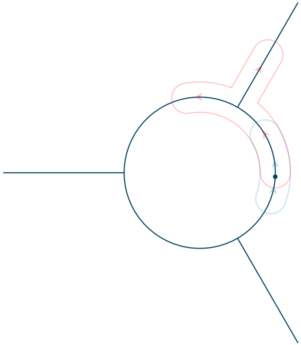


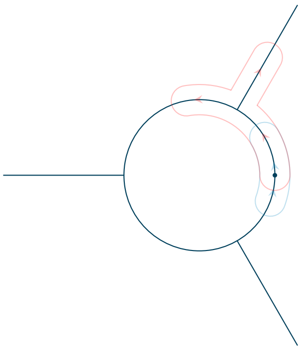


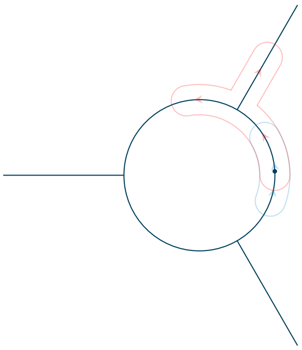


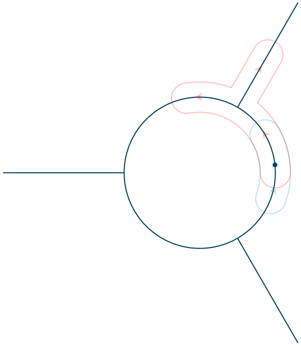


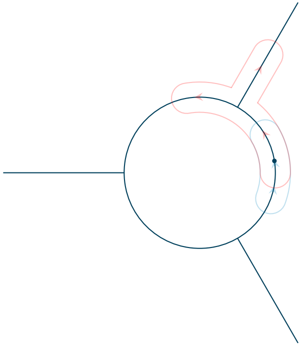


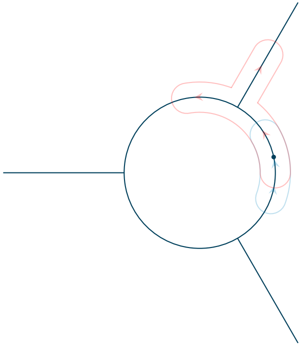


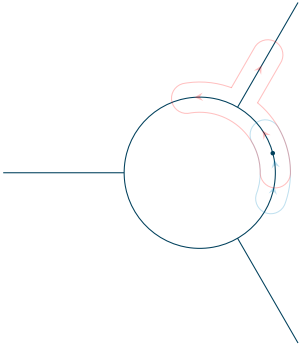


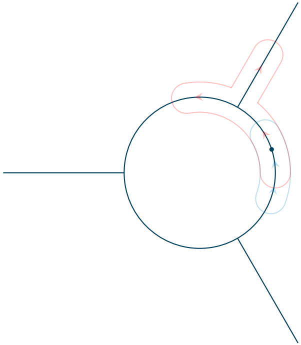


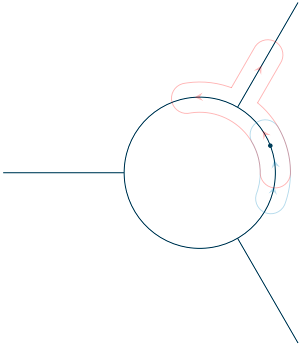


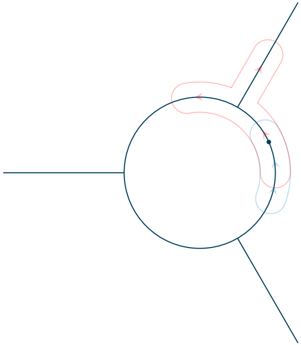


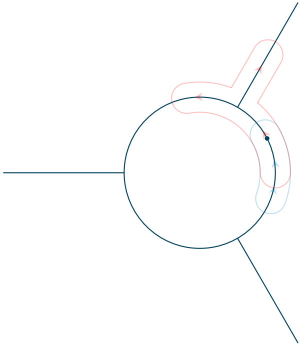


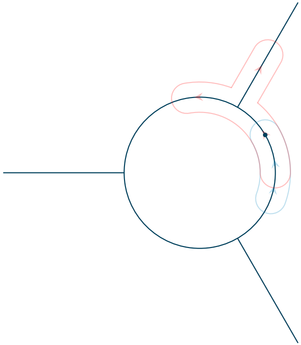


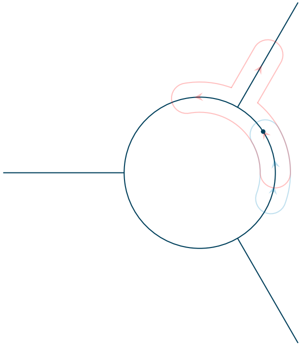


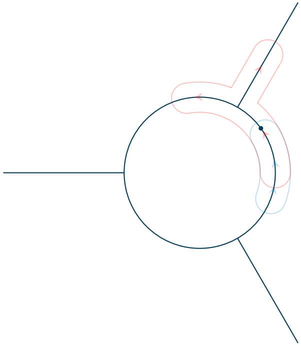


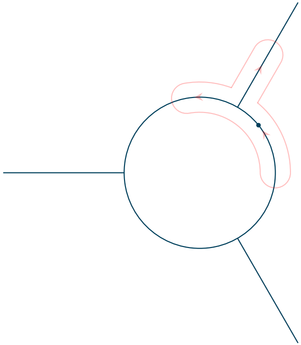


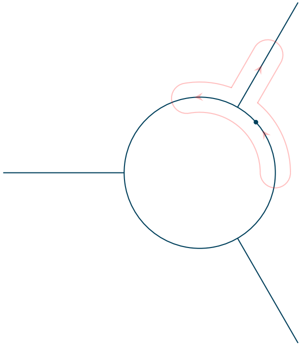


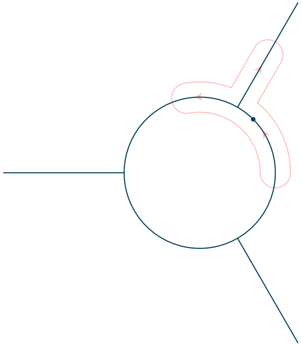


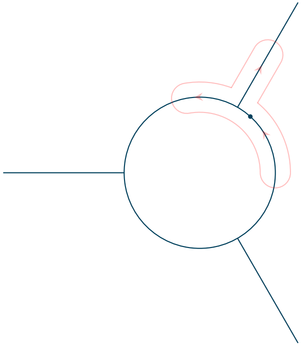


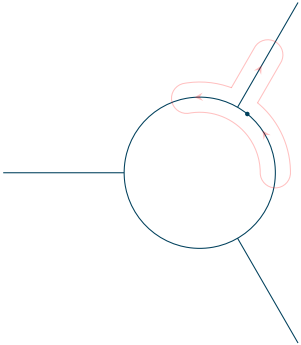


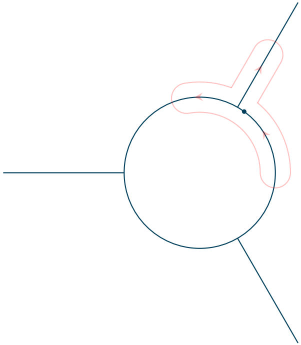


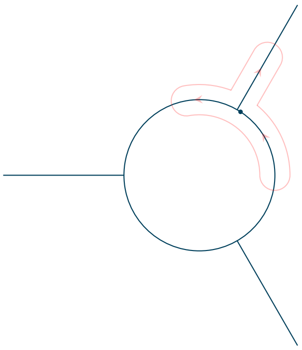


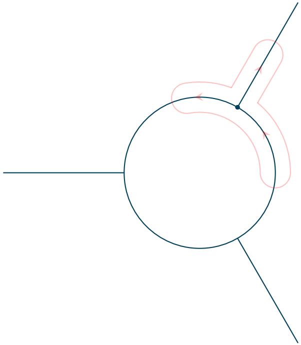


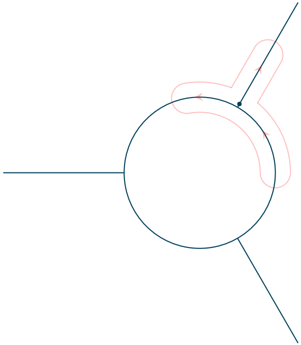


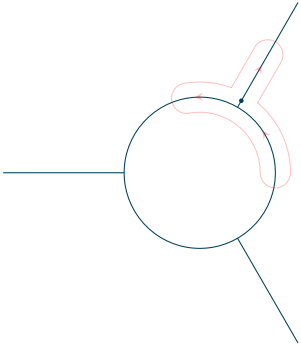


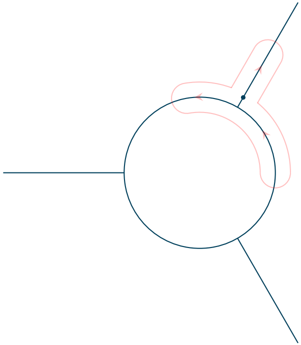


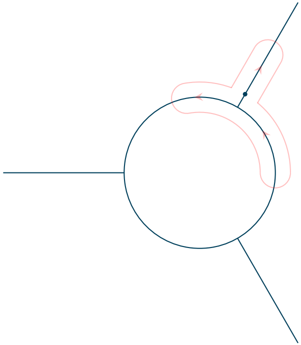


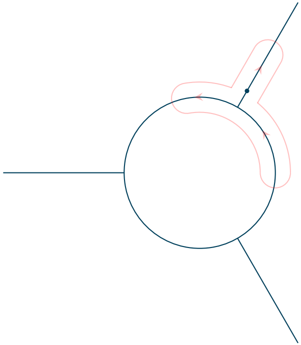


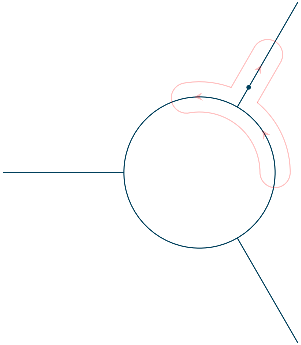


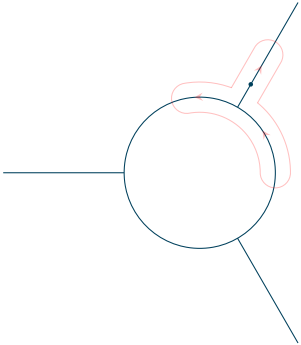


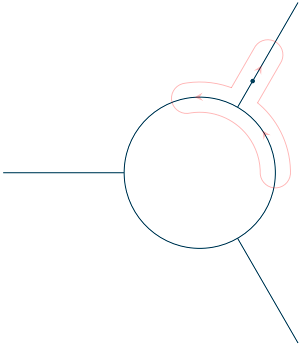


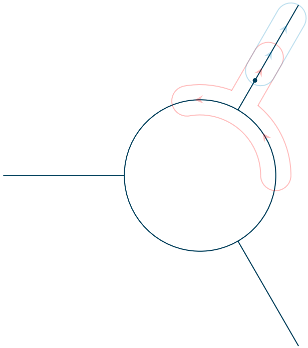


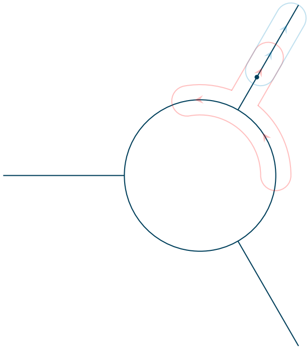


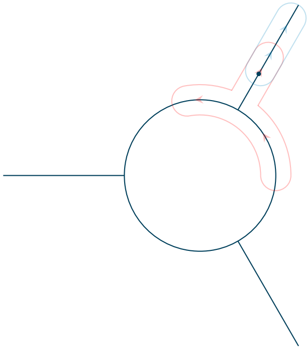


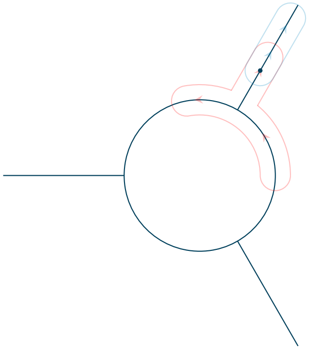


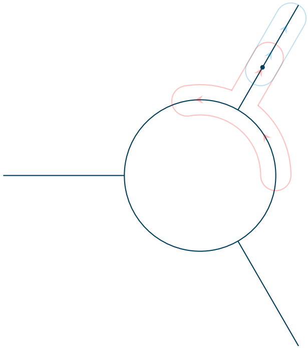


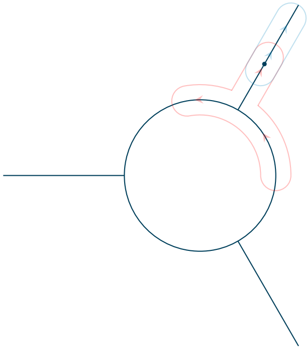


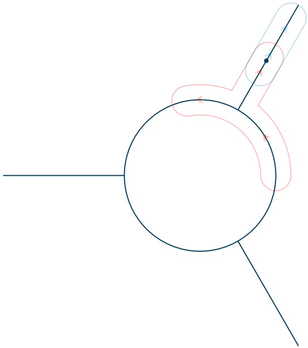


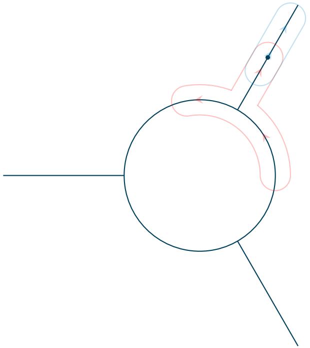


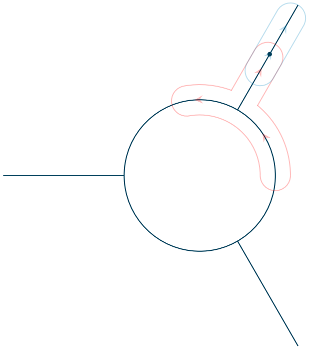


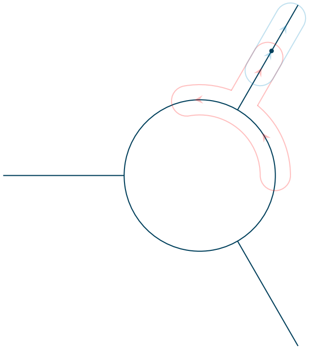


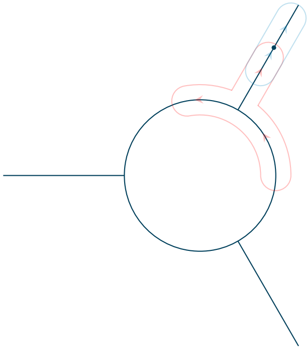


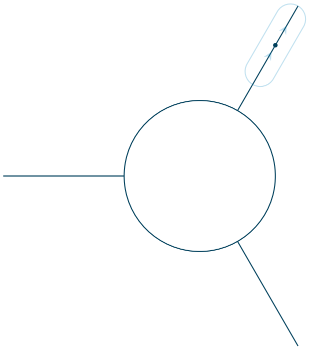


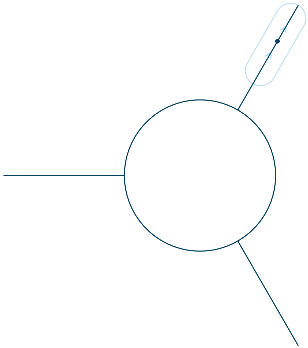


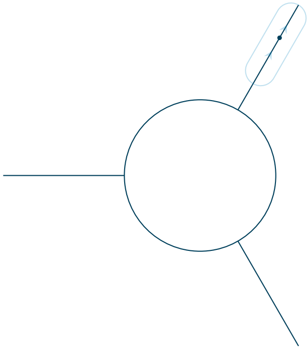


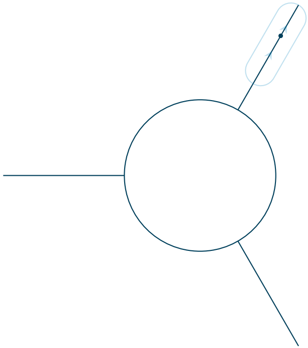


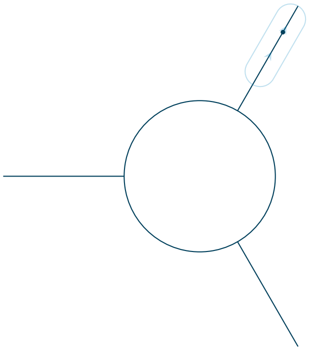


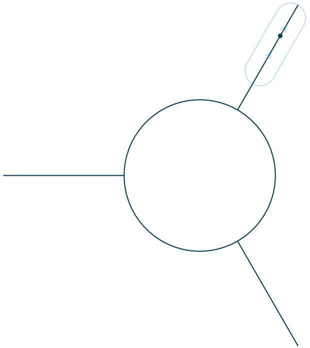


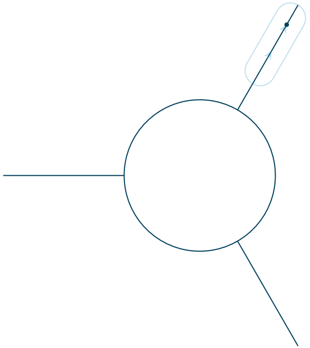


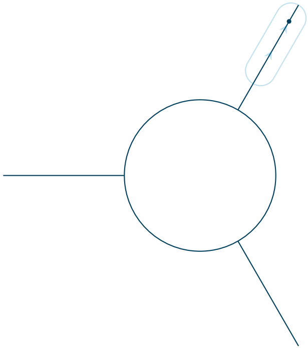


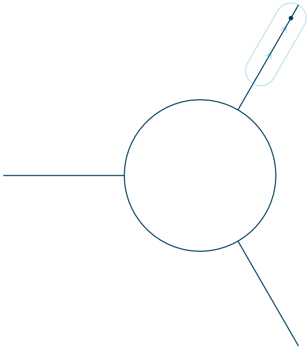


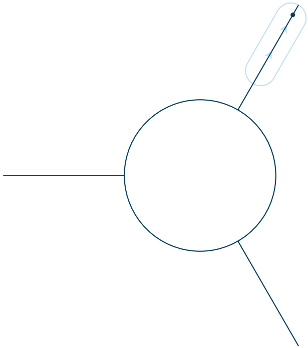


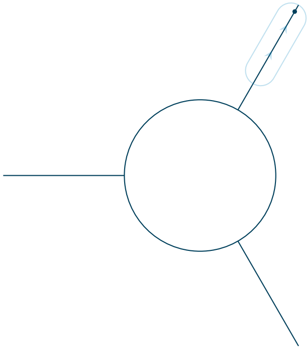


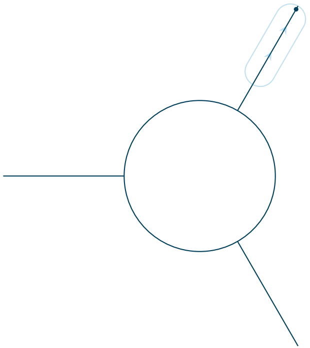


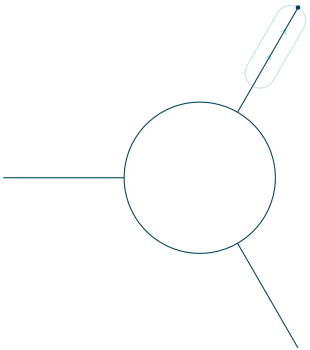




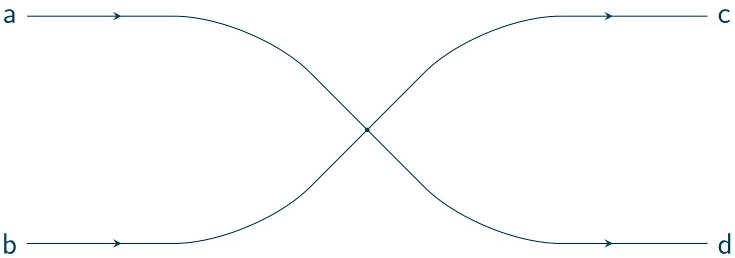








BLOWING UP SINGULARITIES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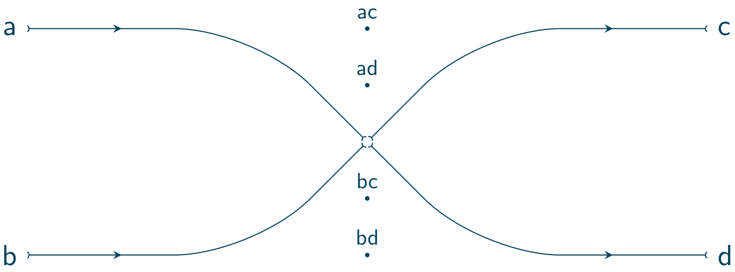
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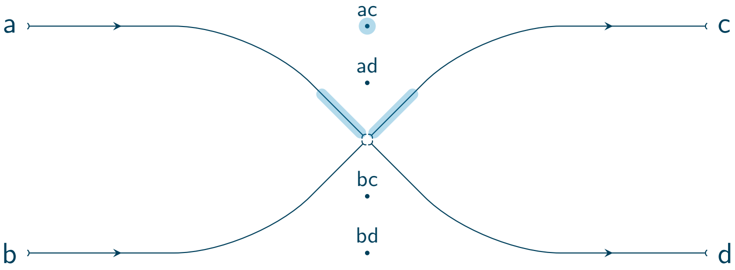
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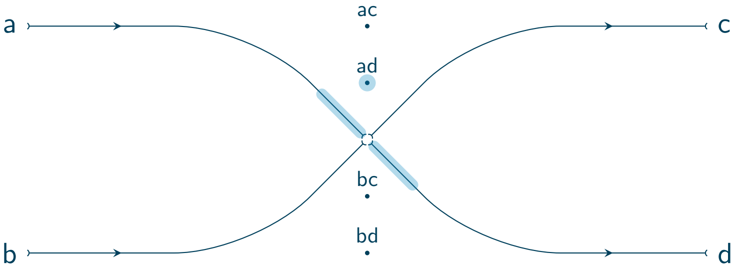
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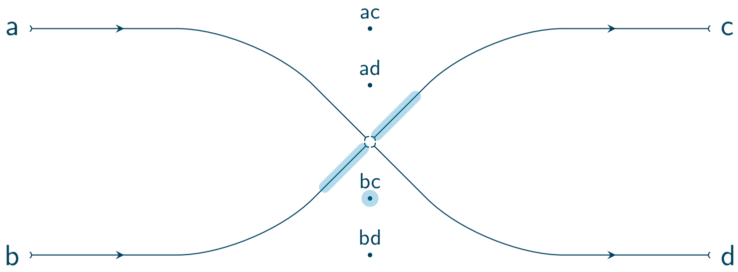
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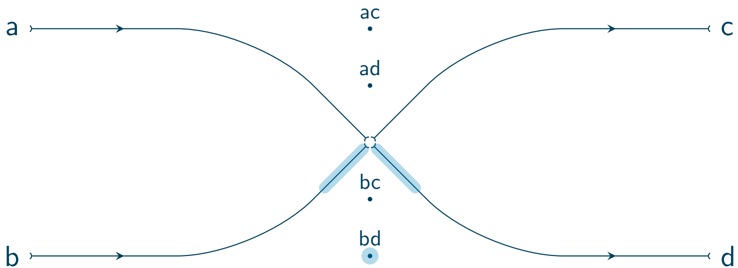

a
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$$G = \left(G^{(1)} \underset{src}{\overset{tgt}{\rightrightarrows}} G^{(0)} \right) \quad : \quad \text{graph}$$

$$G \;=\; \left(\; G^{(1)} \underset{src}{\overset{tgt}{\rightrightarrows}} G^{(0)} \; \right) \quad : \quad \text{graph}$$

$$\|G\| \;=\; \left(G^{(1)} \times]0,1[\right) \cup \left\{ (a,b) \in G^{(1)} \times G^{(1)} \mid \mathrm{tgt}(a) = \mathrm{src}(b) \right\} \quad : \quad \text{set}$$

$$G \;=\; \left(\; G^{(1)} \underset{src}{\overset{tgt}{\rightrightarrows}} G^{(0)} \; \right) \quad : \quad \text{graph}$$

$$\|G\| \;=\; \left(G^{(1)} \times]0,1[\right) \cup \left\{(a,b) \in G^{(1)} \times G^{(1)} \mid \mathrm{tgt}(a) = \mathrm{src}(b)\right\} \quad : \quad \text{set}$$

For small $\varepsilon > 0$, the ε-neighborhoods of (a, t) and (a, b) are

$$\begin{cases} \{a\} \times \,]t - \varepsilon, t + \varepsilon[& (\text{for } \varepsilon \leq \min\{t, 1 - t\}) \\ \{a\} \times \,]1 - \varepsilon, 1[\;\cup\; \{(a, b)\} \;\cup\; \{b\} \times \,]0, \varepsilon[& (\text{for } \varepsilon \leq \frac{1}{2}) \end{cases}$$

$$G = \left(G^{(1)} \underset{src}{\overset{tgt}{\rightrightarrows}} G^{(0)} \right) \quad : \quad \text{graph}$$

$$\|G\| = \left(G^{(1)} \times]0,1[\right) \cup \left\{ (a,b) \in G^{(1)} \times G^{(1)} \mid \mathrm{tgt}(a) = \mathrm{src}(b) \right\} \quad : \quad \text{set}$$

For small $\varepsilon > 0$, the ε-neighborhoods of (a,t) and (a,b) are

$$\begin{cases} \{a\} \times]t-\varepsilon, t+\varepsilon[& (\text{for } \varepsilon \leq \min\{t, 1-t\}) \\ \{a\} \times]1-\varepsilon, 1[\ \cup\ \{(a,b)\} \ \cup\ \{b\} \times]0,\varepsilon[& (\text{for } \varepsilon \leq \frac{1}{2}) \end{cases}$$

The *standard ordered base* $\mathcal{E}_G$ of G is the collection of ε-neighborhoods (each of them being equipped with the obvious total order); it is *euclidean*.

The *blowup* of G is the map

$$\begin{array}{rcrcl} \beta_G & : & \|G\| & \to & |G| \\ & & (a, b) & \mapsto & \mathrm{tgt}(a)(= \mathrm{src}(b)) \\ & & (a, t) & \mapsto & (a, t) \end{array}$$

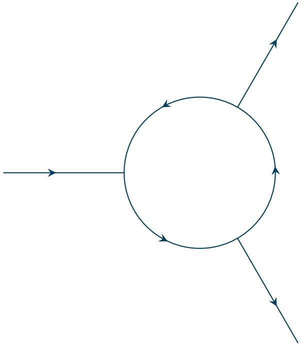
The *blowup* of G is the map

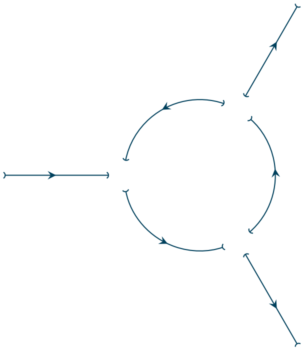
$$\begin{array}{rcrcl} \beta_G & : & \|G\| & \to & |G| \\ & & (a,b) & \mapsto & \mathrm{tgt}(a)(=\mathrm{src}(b)) \\ & & (a,t) & \mapsto & (a,t) \end{array}$$

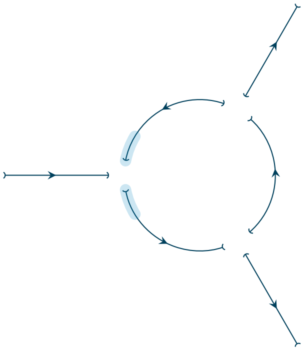
The map β_G induces a euclidean local embedding from $\mathcal{E}_G$ to $\mathcal{X}_G$.

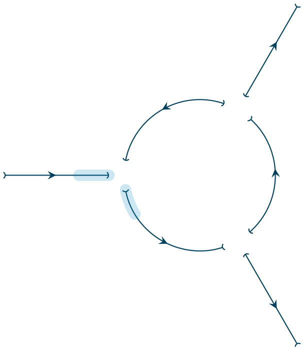
Theorem (Universal property of graph blowups)

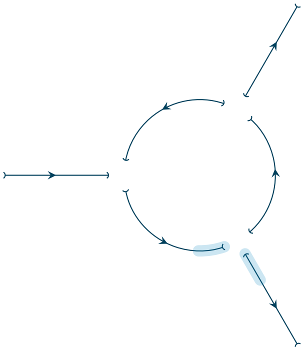
For every euclidean local embedding $f : \mathcal{E} \to \mathcal{X}_{G_1} \times \cdots \times \mathcal{X}_{G_n}$ *of dimension* n*, there is a unique* continuous *map* $g : \mathcal{E} \to \mathcal{E}_{G_1} \times \cdots \times \mathcal{E}_{G_n}$ *such that* $f = \bar{\beta} \circ g$ *with* $\bar{\beta} = \beta_{G_1} \times \cdots \times \beta_{G_n}$*; moreover* g *is a euclidean local dihomeomorphism.*

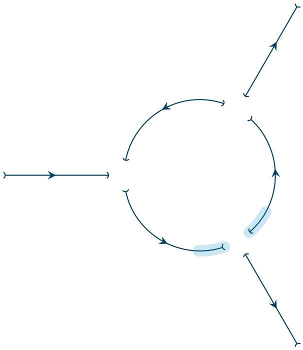


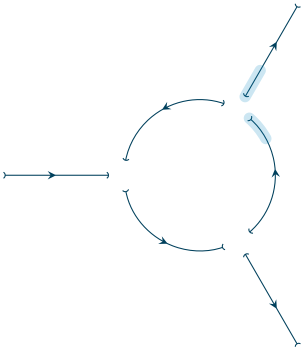


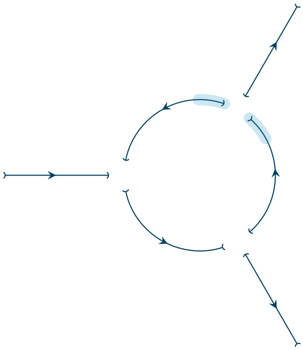












FROM CONTINUOUS TO SMOOTH

The *standard charts* of $\|G\|$ are the following bijections

$$
\begin{aligned}
\phi_a \quad &: \quad \{a\} \times \,]0,1[\;\to\;]0,1[\;, \quad \text{and} \\
\phi_{ab} \quad &: \quad \{a\} \times \,]\tfrac{1}{2},1[\;\cup\; \{(a,b)\} \;\cup\; \{b\} \times \,]0,\tfrac{1}{2}[\;\to\;]-\tfrac{1}{2},\tfrac{1}{2}[\\
\text{with} \quad & (a,t) \mapsto t-1\,, \quad (a,b) \mapsto 0\,, \quad (b,t) \mapsto t
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for all arrows a and all 2-tuples of arrows (a,b) such that $\mathrm{tgt}(a) = \mathrm{src}(b)$.

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The *standard atlas* $\mathcal{A}_G$ of G is the collection of its standard charts.

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The *standard atlas* $\mathcal{A}_G$ of G is the collection of its standard charts.

The *transition maps* are translations:

$$\begin{aligned} \phi_{ab} \circ \phi_a^{-1} \;:\; t \;\in]\tfrac{1}{2},1[\quad &\mapsto \quad t-1 \quad \in \quad]-\tfrac{1}{2},0[\\ \phi_{ab} \circ \phi_b^{-1} \;:\; t \;\in]0,\tfrac{1}{2}[\quad &\mapsto \quad t \qquad\;\; \in \quad]\;\;0,\tfrac{1}{2}[\end{aligned}$$

Given ϕ and ψ standard charts of G, we have $d(\psi \circ \phi^{-1})_{\phi(p)} = \mathrm{id}_{\mathbb{R}}$.
If u and v represent the same tangent vector in the standard charts ϕ and ψ, then $u = v$.

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$$T\mathcal{A}_G \quad \cong \quad \mathcal{A}_G \times \mathbb{R} \qquad \text{and} \qquad T_p\mathcal{A}_G \quad \cong \quad \{p\} \times \mathbb{R}$$

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The *standard vector field* on the standard atlas is

$$\begin{array}{rcl} \mathcal{A}_G & \to & T\mathcal{A}_G \\ p & \mapsto & (p, 1) \end{array}$$

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The tangent vector to γ at t is of the form $(\gamma(t), \gamma'(t))$; γ is locally order-preserving iff $\gamma'(t) \geqslant 0$ for every t.

Proposition (standard vector field vs standard ordered base)

For every $\phi \in \mathcal{A}_G$*, for all* $p, q \in \operatorname{dom}(\phi)$*, we have* $p \leqslant q$ *(with* $(\operatorname{dom}(\phi), \leqslant) \in \mathcal{E}_G$*) iff there exists a smooth path* γ *on* $\mathcal{A}_G$ *from* p *to* q *with* $\operatorname{im}(\gamma) \subseteq \operatorname{dom}(\phi)$ *and* $\gamma' \geqslant 0$*, i.e.* $\phi \circ \gamma$ *is a smooth map between open intervals of* $\mathbb{R}$ *with nonnegative derivative,* $\min(\phi \circ \gamma) = \phi(p)$*, and* $\max(\phi \circ \gamma) = \phi(q)$*.*

APPROXIMATION

From every norm $|\_|$ on $\mathbb{R}^n$ one defines the length of a smooth path $\gamma = (\gamma_1, \ldots, \gamma_n)$ on $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$ by

$$\mathcal{L}(\gamma) \quad = \quad \int_{t \in I} |\gamma'(t)| dt$$

with $\gamma'(t) = (\gamma_1'(t), \ldots, \gamma_n'(t))$ the coordinates of the tangent vector to γ at t in the standard base $((\gamma_1(t), 1), \ldots, (\gamma_n(t), 1))$ of the tangent space at $\gamma(t)$.

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We also define the distance between p, $q \in |G_1| \times \cdots \times |G_n|$ as $d(p, q) = |d_{G_1}(p_1, q_1), \ldots, d_{G_n}(p_n, q_n)|$ from which we deduce the length $L(\gamma)$ of any path γ on $|G_1| \times \cdots \times |G_n|$.

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If δ is a smooth path on $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$ then $\mathcal{L}(\delta) = L((\beta_{G_1} \times \cdots \times \beta_{G_n}) \circ \delta)$.

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$\lvert x_1, \dots, x_n \rvert_2$	$=$	$\sqrt{\sum_{i=1}^n x_i^2}$	Riemannian
$\lvert x_1, \dots, x_n \rvert_1$	$=$	$\sum_{i=1}^n \lvert x_i \rvert$	cumulative execution time
$\lvert x_1, \dots, x_n \rvert_\infty$	$=$	$\max\{x_1, \dots, x_n\}$	parallel execution time

A subset X of $|G_1| \times \cdots \times |G_n|$ is said to be *tile compatible* when for all $p, q \in |G_1| \times \cdots \times |G_n|$ such that $(\pi_{G_1}, \ldots, \pi_{G_n})(p) = (\pi_{G_1}, \ldots, \pi_{G_n})(q)$, we have $p \in X$ iff $q \in X$.

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The *standard cone* of $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$ at $p = (p_1, \ldots, p_n)$ is the cone $C_p = \{\sum_{i=1}^{n}(p_i, \lambda_i) \mid \lambda_i \geqslant 0\} \subseteq T_p(\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n})$.

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A *conal path* on a subset Y of $\|G_1\| \times \cdots \times \|G_n\|$ is a smooth path δ on $\mathcal{A}_{G_1} \times \cdots \times \mathcal{A}_{G_n}$ such that $\delta(t) \in Y$ and $T\delta(t) \in C_{\delta(t)}$ for every $t \in \mathrm{dom}(\delta)$.

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Theorem (Approximation)

For every directed path $\gamma = (\gamma_1, \ldots, \gamma_n)$ on a tile compatible subset X of $|G_1| \times \cdots \times |G_n|$, and every $\varepsilon > 0$, there exists a conal path $\delta = (\delta_1, \ldots, \delta_n)$ on $(\beta_{G_1} \times \cdots \times \beta_{G_n})^{-1}(X)$ such that:

- *γ and $(\beta_{G_1} \times \cdots \times \beta_{G_n}) \circ \delta$ start (resp. finish) at the same point,*
- $\max \left\{ d_i(\gamma_i(t), \beta_i(\delta_i(t))) \mid t \in \mathrm{dom}(\gamma);\ i \in \{1, \ldots, n\} \right\} < \varepsilon$, *and*
- $\mathcal{L}_\infty(\delta) < L_\infty(\gamma)$.