The Boolean Algebra of Cubical Areas as a Tensor Product in the Category of Semilattices with Zero

Nicolas Ninin
CEA, LIST and University Paris-Sud, France
nicolas.ninin@cea.fr

Emmanuel Haucourt
CEA, LIST
emmanuel.haucourt@cea.fr

In this paper we describe a model of concurrency together with an algebraic structure reflecting the parallel composition. For the sake of simplicity we restrict to linear concurrent programs i.e. the ones with no loops nor branchings. Such programs are given a semantics using cubital areas that we call geometric. The collection of all these cubical areas enjoys a structure of tensor product in the category of semi-lattice with zero. These results naturally extend to fully fledged concurrent programs up to some technical tricks.

1 Introduction

In the two last decades, many geometrical or topological models of concurrent programs have emerged [6, 5, 15, 16, 11, 7, 14]. We are especially interested in a simple geometrical one based on the so-called n-dimensional cubical areas which model the control flow for parallel composition of threads without loops nor branchings . Their collection actually forms a boolean algebra $\mathscr{B}_{\mathbb{R}^n}$ whose operations are crucial in [3]. The purpose of our paper is to formalize the fact these operations are actually deduced from their much simpler analog in $\mathscr{B}_{\mathbb{R}}$. Formally we prove the tensor product of two boolean algebras is still a boolean algebra when it is taken in the category of semilattice with zero ($\mathbf{SLat_0}$). We then show the boolean algebra $\mathscr{B}_{\mathbb{R}^n}$, which is in particular a semilattice with zero, can be seen as such a product.

The class of concurrent program we study arises from a toy language manipulating mutex. Using Djikstra's notation [6], we consider processes to be sequences of locking operations Pa on mutex a and unlocking operations Va. To each concurrent programs made of n processes we have a subset of $\mathbb{R}^n$ representing the consistent states. By construction, such subsets of $\mathbb{R}^n$ are finite union of n-cubes, they are called cubical areas. The points of this subset are to be considered as the states of the PV program. Holes in this subsets arise from synchronizations between processes. The set of increasing paths on it then overapproximate the collection of execution traces, and we have a natural equivalence relation upon increasing paths such that equivalent paths have the same effect over the system [7].

We provide a motivating example for the result to be developped in the paper. Consider the following program, written in PV language [6], that consists of two parallel processes $T_1 = Pa.Pb.Vb.Va$ and $T_2 = Pb.Pa.Va.Vb$ where a and are mutex. Any PV program can be given a geometric semantics [5], in our specific example it boils down to the so-called "Swiss flag", Fig 1, regarded as a subset of $\mathbb{R}^2$. The (interior of the) horizontal rectangle comprises global states that are such that $T1$ and $T2$ both hold a lock on a, which is not allowed by the very definition of a mutex. Similarly, the (interior of the) vertical rectangle consists of states violating the mutual exclusion property on b. Therefore both rectangles form the inconsistent states, which is the complement of $[\![T_1|T_2]\!]$ the cubical area of (consistent) states i.e. the model of the program. A cubical area (of dimension n) is a finite union of n-dimensional parallelepipeds (or n-cubes for short) i.e. n-fold cartesian products of intervals of $\mathbb{R}$. All geometric models of PV programs actually arise as cubical areas whose dimension is the number of processes the program is made

To appear in EPTCS.

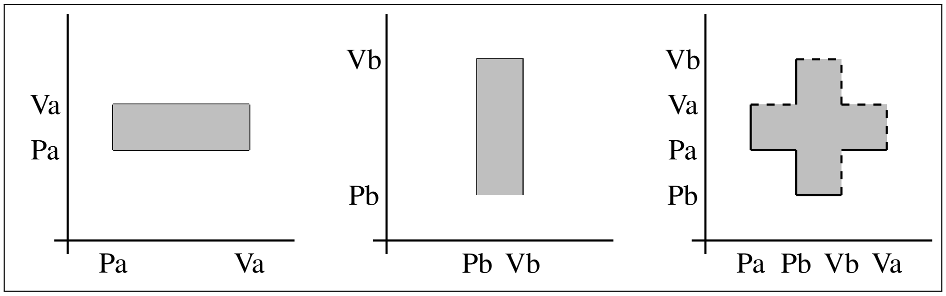


Figure 1: The Swiss flag; At the left the forbiden region of mutex a, at the center the forbiden region of mutex b, and the union of the two

of. More precisely the algorithm producing the geometric model of a PV program first returns the cubical area of its inconsistent states and then compute the set theoretic complement of the later to obtain the actual model of the program. For example the *deadlock attractor* of the program i.e. the subset of points of the geometric model from which all emerging paths can be extended to a path ending at a deadlock, is also a cubical area. In fact the collection of n-dimensional cubcical areas form a boolean subalgebra of the powerset $2^{\mathbb{R}^n}$. Moreover the cubical areas can be handled automatically which makes them suitable for implementation, this practical fact is at the origin of our interest for them. It is also worth to notice the boolean algebra of cubical sets actually provides the ground upon which the static analyzer ALCOOL is based. There is another crucial property of the geometric semantics of the PV language. Suppose we are given two groups of processes $P_1,\dots,P_n$ and $Q_1,\dots,Q_m$ so their sets of occuring resources are disjoint, then

$$[\![P_1|\cdots|P_n|Q_1|\cdots|Q_m]\!] = [\![P_1|\cdots|P_n]\!] \times [\![Q_1|\cdots|Q_m]\!]$$

from which one can (rather easily) deduces that

$$\mathscr{B}_{[\![P_1|\cdots|P_n|Q_1|\cdots|Q_m]\!]} = \mathscr{B}_{[\![P_1|\cdots|P_n]\!]} \otimes \mathscr{B}_{[\![Q_1|\cdots|Q_m]\!]}$$

where $\mathscr{B}_{[\![X]\!]}$ denotes the boolean algebra of subareas of the model $[\![X]\!]$ of a PV program X. Conversely one may ask whether a tensor decomposition of $\mathscr{B}_{[\![X]\!]}$ indicate a potential parallelization of X i.e. gatering its processes in groups that do not interact with each other; and even more theoretically whether $\mathscr{B}_{[\![X]\!]}$ admits a prime decomposition [3]. The purpose of this paper is to define and study the aforementioned tensor product.

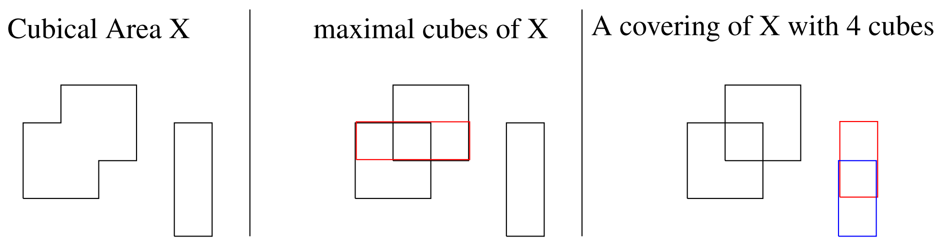
First remark the 1-dimensional cubical areas are the finite unions of intervals of the real line. Then our main goal is to prove the boolean structure of n-dimensional cubical areas is the n-fold tensor product of the boolean algebra 1-dimensional cubical areas. The main obstacle is that tensor product in the category of boolean algebras is degenerated and so inapropriate. Yet we have finally discovered the right category for our purpose is the one of semilattices with zero. It is worth to notice the zero hypothesis (the presence of a least element) cannot be dropped.

Outline of the paper.
Section 2 defines cubical areas, and provides details about their structure of boolean algebra. Section 3 introduces the notion of tensor product in a category, and show that the tensor product of two boolean algebras in $\mathbf{SLat_0}$ is still a boolean algebra. Section 4 relates the boolean algebra of cubical areas to the tensor product by proving $\mathscr{B}_{\mathbb{R}} \otimes \mathscr{B}_{\mathbb{R}} \simeq \mathscr{B}_{\mathbb{R}^2}$.

2 Cubical Area

A *cube* of dimension $n \in \mathbb{N}$ (or just *n- cube*) is the set product of a n-uple of (potentially unbounded) intervals of the real line $\mathbb{R}$. It is therefore a subset of $\mathbb{R}^n$. A *maximal subcube* of $X \subseteq \mathbb{R}^n$ is a cube $C \subseteq X$ such that $C = C'$ holds for all cubes C' such that $C \subseteq C' \subseteq X$. The union of any $\subseteq$-chain of n-cubes is a cube. As a consequence any subcube of X is contained in a maximal subcube of X. A *cubical cover* of X is a family of cubes whose union is X. Then define $\alpha(X)$ as the collection of all maximal subcubes of X. Given $\mathscr{C}$ and $\mathscr{C}'$ two families of n-cubes define $\gamma(\mathscr{C})$ as the union of all the elements of $\mathscr{C}$ and write $\mathscr{C} \preccurlyeq \mathscr{C}'$ when any element of $\mathscr{C}$ is contained in some element of $\mathscr{C}'$. We call a *cubical area* the subset of $\mathbb{R}^n$ admitting a finite cubical cover.

Example of a cubical area of $\mathbb{R}^2$



Lemma 2.1 *Let $\mathscr{C}$ and $\mathscr{C}'$ be families of n-cubes that contains all the maximal subcubes of their unions $\gamma(\mathscr{C})$ and $\gamma(\mathscr{C}')$. Then the family of n-cubes*

$$\{C \cap C' \mid C \in \mathscr{C} \text{ and } C' \in \mathscr{C}'\}$$

contains all the maximal subcubes of $\gamma(\mathscr{C}) \cap \gamma(\mathscr{C}')$.

Let C'' be a subcube of $\gamma(\mathscr{C}) \cap \gamma(\mathscr{C}')$ and let C and C' be subcubes of $\gamma(\mathscr{C})$ and $\gamma(\mathscr{C}')$ respectively such that $C'' \subseteq C$ and $C'' \subseteq C'$. Then $C \cap C'$ is a subcube of $\gamma(\mathscr{C}) \cap \gamma(\mathscr{C}')$ containing C''.

Lemma 2.2 *The complement of any n-cube admits at most* $2n$ *maximal subcubes*

Let $I_1 \times \cdots \times I_n$ be a cube, then any maximal subcube of its complement can be written as

$$\mathbb{R} \times \cdots \underbrace{\times J_k \times}_{k\text{th position}} \cdots \times \mathbb{R}$$

with J_k being a maximal subinterval of the complement of I_k in $\mathbb{R}$. Given $X \subseteq \mathbb{R}^n$ we denote the complement of X in $\mathbb{R}^n$ by X^c.

Proposition 2.1 *A subset of $\mathbb{R}^n$ is a cubical area iff it has finitely many maximal subcubes.*

Corollary 2.1 *The collection $\mathscr{B}_{\mathbb{R}^n}$ of all the n-cubical areas is a sub boolean algebra of the powerset of* $\mathbb{R}^n$.

The empty set and $\mathbb{R}^n$ are cubical areas. From what we saw above it is quite clear that $\mathscr{B}_{\mathbb{R}^n}$ is stable under complement and binary intersection. From De Morgan laws it is also stable under binary unions .

3 Tensor Product of Boolean Algebra

Tensor products of vector spaces are well-known, but it exists in many other categories equiped with a forgetful functor to **Set** [4]. Examples of such categories that matters to us are boolean algebra,distributive lattices, semilattices with zero etc .

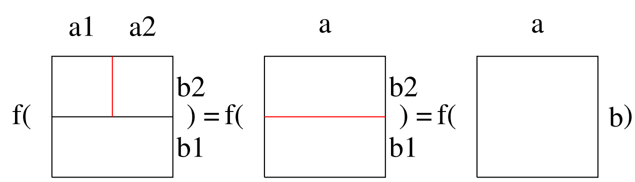
Given A, B and X three object of the same category, a *bimorphism* from A,B to X is a set theoretic map $f : A \times B \to X$ such that for all $a \in A$ and for all $b \in B$ the mappings $f(a, \_) : B \to X$ and $f(\_\,, b) : A \to X$ are morphisms. Being given an object X of the category and a bimorphism $i : A \times B \to X$ we say that X is a *tensor product* of $A \times B$ if for every object C and every bimorphism $f : A \times B \to C$ there exist a unique morphism $h : X \to C$ such that $f = h \circ i$. Tensor products are unique up to isomorphisms and they are denoted by $A \otimes B$. The bimorphism i is not surjective but its image generates $A \otimes B$, thus we call generating elements (of $A \otimes B$) those coming from $A \times B$ and we will write $i(a,b) = a \otimes b$.

Example of a bimorphism in $\mathbb{R}^2$

Let's take a bimorphism $f : \mathscr{B}_{\mathbb{R}} \times \mathscr{B}_{\mathbb{R}} \to C$ in $\mathbf{SLat_0}$. An element of $\mathscr{B}_R$ is simply a union of segment either open, close, or both (ie a cubical area) of $\mathbb{R}$. f is a bimorphism means that $f(0_{\mathscr{B}_{\mathbb{R}}}, b) = f(a, 0_{\mathscr{B}_{\mathbb{R}}}) = 0_C$, where $a,b \in \mathscr{B}_R$ and $0_{\mathscr{B}_{\mathbb{R}}}$ is the empty set (of $\mathbb{R}$), and also $f(a_1 \cup_{\mathscr{B}_{\mathbb{R}}} a_2, b) = f(a_1,b) \cup_C f(a_2,b)$. For example take $a_1 =]0,1]$,$a_2 = [1,2]$, $b_1 = [0,1]$,$b_2 = [1,2]$, let $a = a_1 \cup a_2 = [0,2] = b$, now you have

$$f(a_1,b_2) \cup f(a_2,b_2) \cup f(a,b_1) = f(a_1 \cup a_2, b_2) \cup f(a_2,b_2) = (a_1 \cup a_2, b_1 \cup b_2) = f(a,b)$$

Geometrically it means that f is constant on the cubical area $[0,2]^2$, even if you subdivide it.



Formally a *boolean algebra* is a distributive lattice together with a *complement* that is an involution $x \in X \mapsto x^c \in X$ satisfying $x \vee x^c = 0$ and $x \wedge x^c = 1$ for all $x \in X$, where 0 and 1 are the neutral elements for $\vee$ and $\wedge$ respectively. In particular any boolean algebra belongs to the categories of (bounded or not) distributive lattices, semi-lattices with zero etc, and all of these have their own tensor product. Among these categories we look for the one in which the n-fold tensor product of $\mathscr{B}_{\mathbb{R}}$ is isomorphic with $\mathscr{B}_{\mathbb{R}^n}$. This isomorphism will actually be an isomorphism of boolean algebra.

For example let f be a bimorphism of bounded lattices from A, B to X; given $a \in A$ and $b \in B$ we have $f(0_A, b) = 0_X$ and $f(a, 1_B) = 1_X$, thus $0_X = f(0_A, 1_B) = 1_X$. Hence the set of bimorphism from $A \times B \to X$ is a singleton if X is degenerated; empty otherwise. In other words $A \otimes B$ is degenerated. In particular the tensor product in **Bool** (resp. in bounded lattice and distributive bounded lattice) is irrelevant since we ultimately wants to recover $\mathscr{B}_{\mathbb{R}^n}$ from $\mathscr{B}_{\mathbb{R}}$.

Tensor products of semilattices and related structures have already been the source of many publications [2, 9, 10, 17, 12, 13]. In particular the next theorem has been proved in [8] for semilattices. Minor changes in the proof lead to the result for semilattices with zero.

Theorem 3.1 *The collection of distributive lattice with zero is stable under finitary tensor product in* $\mathbf{SLat_0}$. *Moreover let A,B be distributive lattices and ,a_i, b_i elements of A and B respectively, it satisfies :*

$$(a_1 \otimes b_1) \wedge (a_2 \otimes b_2) = (a_1 \wedge a_2) \otimes (b_1 \wedge b_2)$$

From now, unless otherwise stated, all the tensor products are understood in $\mathbf{SLat_0}$.

Proposition 3.1
The tensor product (in $\mathbf{SLat_0}$*) of a pair of boolean algebras is a boolean algebra*

The previous theorem give us solid ground to prove the proposition. Indeed a boolean algebra being a distributive lattice with complement, it now suffice to find a candidate for $(a\otimes b)^c$ for every element $a\in A$, $b\in B$ with $A,B\in\mathbf{Bool}$.

Lemma 3.1 *Given a pair of boolean algebras A, B and* $a\in A$*,* $b\in B$ *we have:*

$$(a\otimes b)\vee((1_A\otimes b^c)\vee(a^c\otimes 1_B))=1 \quad \textit{and} \quad (a\otimes b)\wedge((1_A\otimes b^c)\vee(a^c\otimes 1_B))=0$$

proof. First you need to expand the 1 as either $a\vee a^c$ or $b\vee b^c$

$$(a\otimes b)\vee\big((1_A\otimes b^c)\vee(a^c\otimes 1_B)\big)=(a\otimes b)\vee\big((a\vee a^c)\otimes b^c)\vee(a^c\otimes(b\vee b^c)\big)$$

Then by expanding or reducing using the fact that $(a\otimes b)\vee(a\otimes c)=a\otimes(b\vee c)$ and that $a\vee a^c=1$, we found this expression equal to 1. In quite the same way we deal with the second equality. We start by distributing the $\vee$ over the $\wedge$

$$(a\otimes b)\wedge\big((1_A\otimes b^c)\vee(a^c\otimes 1_B)\big)=(a\otimes b)\wedge(1_A\otimes b^c)\vee(a\otimes b)\wedge(a^c\otimes 1_B)$$

Similarly we prove that the preceding expression reduces to 0. □

Thus every generating element (i.e. of the form $a\otimes b$) has a complement. Any element is a finite union of such generating elements $x=\bigvee_{i\in I}(a_i\otimes b_i)$ where I is finite. The existence of a complement for any element then follows from the De Morgan's law:

$$((a_1\otimes b_1)\vee(a_2\otimes b_2))^c=(a_1\otimes b_1)^c\wedge(a_2\otimes b_2)^c$$

The later essentially derives from the relation, $(a_1\otimes b_1)\wedge(a_2\otimes b_2)=(a_1\wedge a_2)\otimes(b_1\wedge b_2)$ which is provided by Theorem 3.1.

4 The collection of cubical areas $\mathscr{B}_{\mathbb{R}\times\mathbb{R}}$ as a tensor product

Theorem 4.1 *The tensor product* $\mathscr{B}_{\mathbb{R}}\otimes\mathscr{B}_{\mathbb{R}}$ *in* $\mathbf{SLat_0}$ *is actually a boolean algebra isomorphic (as boolean algebras) with* $\mathscr{B}_{\mathbb{R}\times\mathbb{R}}$*.*

We will prove $\mathscr{B}_{\mathbb{R}\times\mathbb{R}}$ satisfies the universal property that characterizes the tensor product. Let $X\in\mathbf{SLat_0}$ and $f:\mathscr{B}_{\mathbb{R}}\times\mathscr{B}_{\mathbb{R}}\to X$ be a bimorphism in $\mathbf{SLat_0}$. We want to find a morphism $h:\mathscr{B}_{\mathbb{R}\times\mathbb{R}}\to X$ such that the diagram commutes :

$$\begin{tikzcd}
\mathscr{B}_{\mathbb{R}}\times\mathscr{B}_{\mathbb{R}} \arrow[r, hook, "i"] \arrow[dr, "f"'] & \mathscr{B}_{\mathbb{R}\times\mathbb{R}} \arrow[d, dashed, "h"] \\
& X
\end{tikzcd}$$

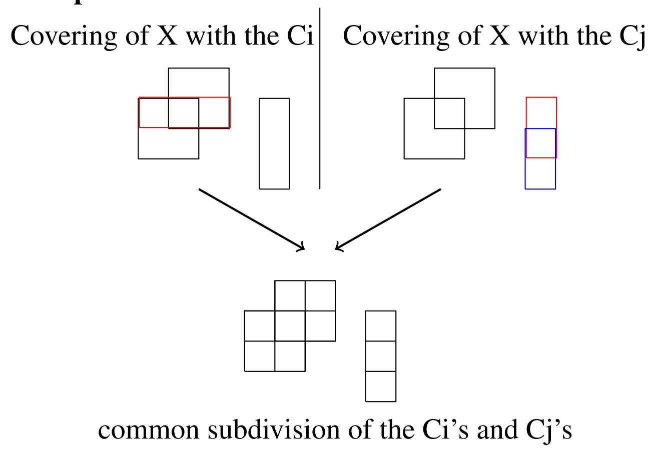
Where i is the inclusion. We define h on the image of i by $h(i(I_1,I_2))=f(I_1,I_2)$ with $I_1,I_2\in\mathscr{B}_{\mathbb{R}}$. Since h has to be a morphism this definition extends to all $\mathscr{B}_{\mathbb{R}\times\mathbb{R}}$ with $h(C_1\cup C_2)=h(C_1)\vee h(C_2)$ where the C_i are generating element of $\mathscr{B}_{\mathbb{R}\times\mathbb{R}}$ i.e. elementary cubes which we write $a\times b$. This mapping might however not be well defined since a cubical area of $\mathbb{R}^2$ can be covered by smaller cubes in infinitely many ways. So it remains to check the soundness of the definition.

Lemma 4.1 *Let h be defined as above, and let $X = \bigcup_{i \in I} C_i = \bigcup_{j \in J} C'_j$ be a cubical area describe as two finite unions of generating elements C_i and C_j then*

$$\bigvee_{i \in I} h(C_i) = \bigvee_{j \in J} h(C'_j)$$

and thus h is well defined .

Example in $\mathbb{R}^2$:



Let's take the first cubical area X seen in section 2. We can find a common subdivision of the C_i and the C_j, by cutting along every coordinate of the cubes. We know that h on a generating element $a \otimes b$ (a cube) is equal to $f(a,b)$. And since f is a bimorphism you can put together two cubes with one identical coordinate. By using this method inductively we get that the value of h is the same on those three families of cubes.

Perspectives.

These results extend to cartesian products of geometric realizations of graphs (instead of $\mathbb{R}^n$) so one can take programs with branchings and loops into account. It means we can substitute in this paper, connected subsets of the geometric realization of a graph to the intervals of $\mathbb{R}$. The graphs considered being the control flow graphs of threads [1].

References

[1] Frances E. Allen (1970): *Control flow analysis*. In: *ACM SIGPLAN Notices - Proceedings of a symposium on Compiler optimization*, 5, Association for Computing Machinery, Association for Computing Machinery, pp. 1–19.

[2] James A. Anderson & Naoki Kimura (1978): *The tensor product of semilattices*. *Semigroup Forum* 16, pp. 83–88.

[3] Thibaut Balabonski & Emmanuel Haucourt (2010): *A Geometric Approach to the problem of Unique Decomposition of Processes*. In: *Concurrency Theory 21th International Conference*, *Lecture Notes in Computer Science* 6269, Springer, pp. 132–146.

[4] Francis Borceux (1994): *Handbook of Categorical Algebra, II. Categories and Structures*. *Encyclopedia of Mathematics and its Applications* 51, Cambridge University Press.

[5] Scott David Carson & Paul F. Reynolds Jr. (1987): *The Geometry of Semaphore Programs*. *ACM Transactions on Programming Languages and Systems* 9(1), pp. 25–53.

[6] Edsger Wybe Dijkstra (1968): *Cooperating sequential processes*. In F. Genuys, editor: *Programming Languages: NATO Advanced Study Institute*, Academic Press, pp. 43–112.

[7] Lisbeth Fajstrup, Eric Goubault & Martin Raußen (2006): *Algebraic Topology and Concurrency*. *Theoretical Computer Science* 357(1), pp. 241–278. Presented at Mathematical Foundations of Computer Science in 1998 (London).

[8] Grant A. Fraser (1976): *The semilattice tensor product of distributive lattices*. Transactions of the American Mathematical Society 217.

[9] Grant A. Fraser (1976): *The tensor product of distributive lattices*. 20, Edinburgh Mathematical Society.

[10] Grant A. Fraser (1978): *The tensor product of semilattices*. Algebra Universalis 8, pp. 1–3.

[11] Marco Grandis (2003): *Directed Homotopy Theory, I. The Fundamental Category*. Cahiers de Topologie et Géométrie Différentielle Catégoriques 44(3), pp. 281–316.

[12] George Grätzer, H. Lakser & R. Quackenbush (1981): *The structure of tensor products of semilattices with zero*. Transactions of the American Mathematical Society 267(2), pp. 503–515.

[13] George Grätzer & Friedrich Wehrung (2000): *Tensor products of semilattices with zero, revisited*. Journal of Pure and Applied Algebra 147, pp. 273–301.

[14] Sanjeevi Krishnan (2009): *A Convenient Category of Locally Preordered Spaces*. Applied Categorical Structures 17(5), pp. 445–466.

[15] Vaughan Pratt (1991): *Modeling Concurrency with Geometry*. In: Proc. 18th Ann. ACM Symposium on Principles of Programming Languages, pp. 311–322.

[16] Vaughan Pratt (2000): *Higher dimensional automata revisited*. Mathematical Structures in Computer Science 10(4), pp. 525–548.

[17] Zahava Shmuely (1979): *The tensor product of distributive lattices* . Algebra Universalis 9, pp. 281–296.