∞-categorical models of linear logic

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Outline

1. Categorical semantics of linear logic
2. The theory of ∞-categories
3. Linear logic in ∞-categories

Categorical semantics

How to do denotational semantics in a category $\mathcal{C}$:

Syntax	Categorical semantics
Formulae A	Object $[\![A]\!]$ of $\mathcal{C}$
Proof π of $A \vdash B$	Morphism $[\![\pi]\!] : [\![A]\!] \to [\![B]\!]$ in $\mathcal{C}$
Cut elimination $\pi \rightsquigarrow \pi'$	Equality of morphisms $[\![\pi]\!] = [\![\pi]\!]'$
Additional syntactic constructions	Additional categorical structure

Intuitionistic linear logic

Formulas

$$F ::= A \mid B \mid \dots$$
$$\mid A \,\&\, B$$
$$\mid A \otimes B$$
$$\mid A \multimap B$$
$$\mid 1 \mid \top$$
$$\mid !A$$
$$\mid \dots$$

Contexts $\Gamma ::= A_1, \dots, A_n$

Judgements $\Gamma \vdash B$

Examples of rules

$$\frac{}{A \vdash A}\ (\text{ax}) \qquad \frac{\Gamma \vdash A \qquad \Delta, A \vdash C}{\Gamma, \Delta \vdash C}\ (\text{cut})$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\ (\otimes \text{L}) \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ (\otimes \text{R})$$

$$\frac{\Gamma, A_i \vdash B}{\Gamma, A_1 \,\&\, A_2 \vdash B}\ (\&\text{L}_i) \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B}\ (\&\text{R})$$

$$\frac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}\ (\multimap \text{L}) \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ (\multimap \text{R})$$

Categorical semantics of linear logic: $\otimes$

Formulas A interpreted as objects $[\![A]\!] \in \mathcal{C}$.
$[\![A \otimes B]\!] = ?$
Need a *(symmetric) monoidal structure* on $\mathcal{C}$:
A functor $- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and an object $1 \in \mathcal{C}$ with natural isomorphisms

$$X \otimes Y \simeq Y \otimes X,$$
$$(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z),$$
$$X \otimes 1 \simeq X \simeq 1 \otimes X$$

satisfying some axioms.
Due to $\dfrac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\,(\otimes\mathrm{L})$, can define $[\![A_1, \ldots, A_n]\!] := [\![A_1 \otimes \cdots \otimes A_n]\!] = [\![A_1]\!] \otimes \cdots \otimes [\![A_n]\!]$

Categorical semantics of linear logic: $\multimap$

We can use the rules

$$\frac{A, B \vdash C}{A \otimes B \vdash C}\,(\otimes\mathrm{L}) \qquad\qquad \frac{A, B \vdash C}{A \vdash B \multimap C}\,(\multimap\mathrm{R})$$

to show we need bijections

$$\mathrm{Hom}_{\mathcal{C}}([\![A]\!] \otimes [\![B]\!], [\![C]\!]) \simeq \mathrm{Hom}_{\mathcal{C}}([\![A]\!], [\![B \multimap C]\!])$$

Ask for $\mathcal{C}$ to be monoidal closed : $(X \otimes -) \dashv (X \multimap -)$.

$$\mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z) \simeq \mathrm{Hom}_{\mathcal{C}}(X, Y \multimap Z)$$

Categorical semantics of linear logic: &

The proofs

$$\dfrac{\dfrac{}{A_i \vdash A_i}\,(\mathrm{ax})}{A_1 \mathbin{\&} A_2 \vdash A_i}\,(\&\mathrm{L}_i)$$

will be interpreted as "projection" morphisms $\pi_i : [\![A_1 \mathbin{\&} A_2]\!] \to A_i$.
Thus we interpret & as the cartesian product in $\mathcal{C}$.

$$[\![A \mathbin{\&} B]\!] := [\![A]\!] \times [\![B]\!]$$

Linear and non-linear implications

Linear implication:

$$A \multimap B$$

Cannot duplicate or erase hypothesis A in proof

Non-linear (intuitionistic) implication:

$$!A \multimap B$$

Can duplicate or erase hypothesis A in proof.

Categorical semantics of linear logic: !

Rules for the exponential

$$\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\ (\text{der})$$

$$\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ (\text{prom})$$

$$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ (\text{contr})$$

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\ (\text{weak})$$

The exponential ! $\rightsquigarrow$ a functor $! : \mathcal{C} \to \mathcal{C}$.
Promotion and dereliction rules $\rightsquigarrow$! is a *comonad*.

$$\dfrac{\dfrac{}{!A \vdash !A}\ (\text{ax}) \qquad \dfrac{}{!A \vdash !A}\ (\text{ax})}{\dfrac{!A, !A \vdash !A \otimes !A}{!A \vdash !A \otimes !A}\ (\text{contr})}\ (\otimes\text{R})$$

Similarly, $!A \vdash 1$.
Cut elimination shows that this gives a *comonoid* structure on $[\![!A]\!]$.

Example: the relational model

The category Rel :

- Objects: sets $X, Y, \ldots$
- Morphisms: relations $R \subseteq X \times Y$
- Tensor product: cartesian product of underlying sets $X \times Y$
- Linear implication: also cartesian product of underlying sets, since

$$\mathrm{Rel}(X \times Y, Z) \simeq \mathrm{Rel}(X, Y \times Z)$$

- Cartesian product: disjoint union of underlying sets $X \sqcup Y$
- Exponential comonad: multisets $\mathrm{Mul}(X)$ on X (finite lists up to reordering, finite subsets with repetitions)

is a sound model of linear logic.

Example: the bicategorical model of species [Fio+08; FGH24]

- Objects: categories $\mathcal{C}, \mathcal{D}, \ldots$
- Morphisms: profunctors $F, G : \mathcal{C} \times \mathcal{D}^{\text{op}} \to \text{Set}$
- 2-morphisms: natural transformations $F \Rightarrow G$
- Tensor product: cartesian product of underlying categories $\mathcal{C} \times \mathcal{D}$
- Linear implication: $\mathcal{C}^{\text{op}} \times \mathcal{D}$
- Cartesian product: disjoint union of underlying categories $\mathcal{C} \sqcup \mathcal{D}$
- Exponential comonad: free symmetric monoidal category on underlying category $\text{Sym}(\mathcal{C})$

is a sound *bicategorical model of linear logic*.
Bicategory: hom-categories instead of hom-sets.

Example: the homotopical model of template games [Mel19a; Mel19b]

(with trivial template for simplicity)

- Objects: categories $\mathcal{C}, \mathcal{D}, \ldots$
- Morphisms: spans of *isofibrations* $\mathcal{C} \xleftarrow{F} \mathcal{X} \xrightarrow{G} \mathcal{D}$
- 2-morphisms: morphisms of spans
$$\begin{array}{ccccc} & & \mathcal{X} & & \\ & \overset{F}{\swarrow} & \big\downarrow{\scriptstyle \alpha} & \overset{G}{\searrow} & \\ \mathcal{C} & & & & \mathcal{D} \\ & \underset{F'}{\nwarrow} & & \underset{G'}{\nearrow} & \\ & & \mathcal{Y} & & \end{array}$$
- Tensor product: cartesian product of underlying categories $\mathcal{C} \times \mathcal{D}$
- Linear implication: same as tensor product $\mathcal{C} \times \mathcal{D}$
- Cartesian product: disjoint union of underlying categories $\mathcal{C} \sqcup \mathcal{D}$
- Exponential comonad: free symmetric monoidal category on underlying category $\mathrm{Sym}(\mathcal{C})$

is a sound "homotopical model of linear logic".
"Homotopical model": Quillen model structure on hom-categories.

The goal

Increasing interest in homotopical structures in models of linear logic.
$\to$ find a general framework to fit such new models ?

- Idea: work directly with ∞-categories.
- ∞-categories: the *language* of homotopy theory.
- Goal: find how to axiomatize models of linear logic in ∞-categories.

In ∞-categories, computational definitions don't work well: the **property** of a diagram commuting is replaced by the **data** of a higher isomorphism.
$\to$ need a way to package the categorical structure of models of LL in an *abstract*, "*unbiased*" way.
Multiple axiomatizations exist.

Seely categories

Definition ([See97])

A *Seely category* is a

1. symmetric monoidal closed category $(\mathcal{C}, \otimes, 1, \multimap)$
2. with finite products ($\&$ and $\top$),
3. a comonad $(!, \delta, \varepsilon) : \mathcal{C} \to \mathcal{C}$,
4. isomorphisms $m^2_{A,B} : !(A \mathbin{\&} B) \simeq !A \otimes !B$ and $m^0 : !\top \simeq 1$ so that $! : (\mathcal{C}, \&) \to (\mathcal{C}, \otimes)$ is a *symmetric monoidal functor*
5. such that the following diagram commutes

$$\begin{array}{ccccc} !A \otimes !B & & \xrightarrow{\delta_A \otimes \delta_B} & & !!A \otimes !!B \\ \downarrow {\scriptstyle m^2_{A,B}} & & & & \downarrow {\scriptstyle m^2_{!A,!B}} \\ !(A \mathbin{\&} B) & \xrightarrow[\delta_{A\&B}]{} & !!(A \mathbin{\&} B) & \xrightarrow[!\langle !\pi_1, !\pi_2 \rangle]{} & !(!A \mathbin{\&} !B) \end{array}$$

Point 5 is too ad hoc to have a natural ∞-categorical generalization.

Linear categories

Definition ([Ben+97])

A *linear category* is :

- a symmetric monoidal closed category $(\mathcal{L}, \otimes, 1)$,
- together with a *lax symmetric monoidal comonad* $((!, m), \delta, \varepsilon)$,
- and a natural commutative comonoid structure $d_A : !A \to !A \otimes !A$, $e_A : !A \to 1$,

such that d_A and e_A are coalgebra morphisms for $!$ and δ is a comonoid morphism.

Less ad hoc, but still a lot of structure.

Linear-non-linear adjunctions

Every linear category $(\mathcal{L}, \otimes, 1, !, \dots)$ induces $(\mathcal{L}^!, \times) \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} (\mathcal{L}, \otimes)$.
$\mathcal{L}^!$ category of coalgebras for the comonad !.
The morphisms in $\mathcal{L}^!$ represent the non-linear morphisms of linear logic $(!A \multimap B)$.

Definition ([Ben95])

A *linear-non-linear adjunction* is an adjunction

$$(\mathcal{M}, \times) \underset{M}{\overset{L}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} (\mathcal{L}, \otimes)$$

between a cartesian category $\mathcal{M}$ and a symmetric monoidal closed category $\mathcal{L}$, where the left adjoint $L : \mathcal{M} \to \mathcal{L}$ is strongly monoidal $L(X \times Y) \simeq LX \otimes LY$.

$\mathcal{L}$ "linear" category, $\mathcal{M}$ "multiplicative" (non-linear) category.

Linear-non-linear adjunctions

$$(\mathcal{M}, \times) \underset{M}{\overset{L}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} (\mathcal{L}, \otimes)$$

Induced comonad $LM : \mathcal{L} \to \mathcal{L}$ makes $\mathcal{L}$ into linear category.
Multiple choices of $\mathcal{M}$ may yield the same comonad : there is **more** structure than strictly needed.
But it is packaged in a more **minimalistic** way.
Only notions needed: monoidal functor, cartesian products, adjunctions.

A special case : Lafont categories

$!A$ must be a (commutative) comonoid.

Definition

$(\mathcal{L}, \otimes, !)$ is a *Lafont category* if $!A$ is the *cofree commutative comonoid on* A for every A.

Definition

Write $\mathrm{Comon}(\mathcal{L})$ for the category of commutative comonoids in $\mathcal{L}$.

Proposition

The category $\mathrm{Comon}(\mathcal{L})$ is cartesian. If $\mathcal{L}$ is Lafont, there is a linear-non-linear adjunction

$$(\mathrm{Comon}(\mathcal{L}), \times) \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} (\mathcal{L}, \otimes).$$

Example: the relational model

The relational model Rel is Lafont.

Proposition

$(\mathsf{Rel}, \times, \mathsf{Mul})$ is Lafont.
i.e. $\mathsf{Mul}(X)$ is the cofree commutative comonoid X in Rel.

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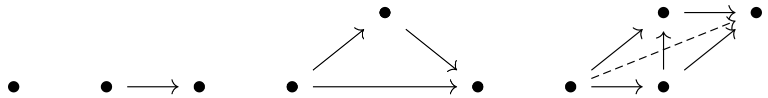
3 Linear logic in ∞-categories

Almost all results in this section are from Joyal and Lurie's work [Joy08; Lur09; Lur17; Lur18] or straightforward corollaries.

Shapes for higher morphisms

In categories, there is a unique way to compose morphisms.
In an ∞-category, various compositions may exist, and they are only related by higher isomorphisms between them.
To define ∞-categories, we need "shapes" for morphisms (cells) of arbitrary dimensions, and how they relate to one another.

Many possible choices, but the most developed one is that of **simplices**.



Simplicial sets

Definition (Simplex category)

$\mathbf{\Delta}$ denotes the category with objects the linear orders $[n] = \{0 < \cdots < n\}$ with $n \in \mathbb{N}$, and monotonous maps between them.

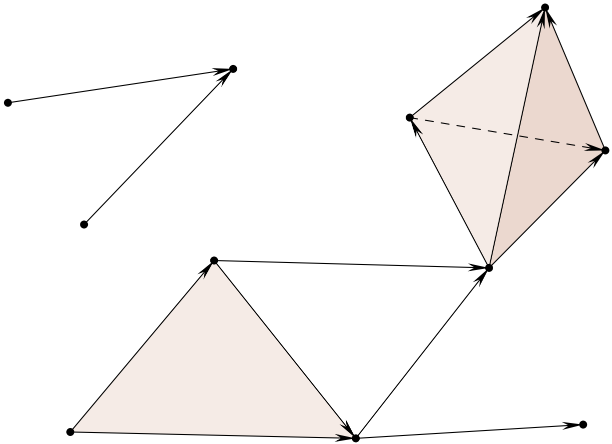
Definition

A *simplicial set* is a functor $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$. Their category is written sSet.

- elements of X_0 are thoughts of as vertices of X
- $f \in X_1$ is thought of as an edge. The inclusions $\{0\} \hookrightarrow \{0,1\}$ and $\{1\} \hookrightarrow \{0,1\}$ give F a source and target vertices $d_0 f$ and $d_1 f$.
- $\sigma \in X_2$ is thought of as a filled triangle witnessing that "h is a composition of f and g"

$$\begin{array}{ccccc} & & y & & \\ & \overset{f}{\nearrow} & \sigma & \overset{g}{\searrow} & \\ x & & \xrightarrow[h]{} & & z \end{array}$$

Simplicial sets: a drawing



Examples of simplicial sets

Definition

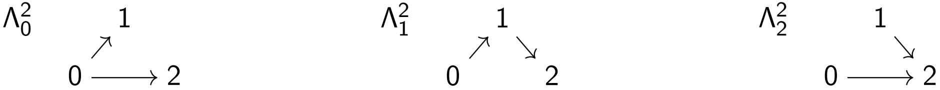
For every $n \in \mathbb{N}$ there is a simplicial set Δ^n such that

$$\forall X, \mathrm{Hom}(\Delta^n, X) \simeq X_n.$$

Δ^n is called the standard n-simplex.

Definition

Let $n > 1$, $0 \leq k \leq n$. The *horn* Λ^n_k is the subsimplicial set of Δ^n obtained by removing the unique cell of dimension n and the cell of dimension $(n-1)$ opposite to the vertex k.
The horn is an *inner horn* if $0 < k < n$, and an *outer horn* if $k = 0$ or $k = n$.



Categories as simplicial sets

Definition

Every category $\mathcal{C}$ determines a simplicial set $N\mathcal{C}$ called its **nerve**.
The n-simplices in $N\mathcal{C}$ are given by sequences of composable morphisms in $\mathcal{C}$.

$$d_0\sigma \longrightarrow d_1\sigma \cdots\cdots\to d_{n-1}\sigma \longrightarrow d_n\sigma$$

The action of morphisms in Δ is given by composition and discarding in $\mathcal{C}$.

Example

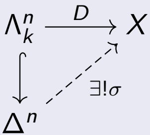
The inclusion $\{0,1,2\} \simeq \{1,2,4\} \hookrightarrow \{0,1,2,3,4\}$ gives the action

$$x_0 \xrightarrow{f} x_1 \xrightarrow{g} x_2 \xrightarrow{h} x_3 \xrightarrow{k} x_4 \qquad \mapsto \qquad x_1 \xrightarrow{g} x_2 \xrightarrow{h \circ k} x_4$$

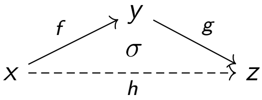
Categories as simplicial sets

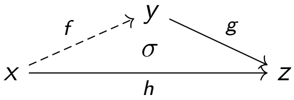
Proposition

A simplicial set X is isomorphic to the nerve of a category if and only if for every $0 < k < n$, $n > 1$, and morphism $D : \Lambda^n_k \to X$, there **exists a unique** cell $\sigma \in X_n$ making the following diagram commute.



It is the nerve of a groupoid if and only if this condition also applies when $0 \leq k \leq n$, $n > 0$.





$\Lambda^2_1 \qquad h = g \circ f$ $\qquad\qquad$ $\Lambda^2_0 \qquad f = g^{-1} \circ h$

∞-categories, ∞-groupoids

Definition

An ∞-category is a simplicial set X such that there **exists a (non-necessarily unique)** lift with respect to every inclusion of inner horn $\Lambda^n_k \hookrightarrow \Delta^n$:

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{D} & X \\ \big\downarrow & \nearrow_{\exists \sigma} & \\ \Delta^n & & \end{array}$$

It is an ∞-groupoid if it admits lifts also for outer horn inclusions.
The vertices of an ∞-category are called *objects*, its edges are called *morphisms*.

Example

The nerve of a category is an ∞-category, the nerve of a groupoid is an ∞-groupoid.
The nerve functor $N : \mathrm{Cat} \to \infty\mathrm{Cat}$ is fully faithful.

Composition of morphisms

Definition

In an ∞-category $\mathcal{C}$, given a triangle $\sigma \in \mathcal{C}_2$

$$\begin{array}{ccccc} & & y & & \\ & \overset{f}{\nearrow} & \sigma & \overset{g}{\searrow} & \\ x & & \xrightarrow[h]{\qquad} & & z \end{array}$$

we say that σ witnesses that **h is a composition of g and f.**

Proposition

In an ∞-category $\mathcal{C}$, composition of morphisms always exists, and is generally not unique.

Homotopy between morphisms

Let $\mathcal{C}$ be an ∞-category.

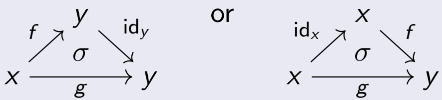
Definition

Let $x \in \mathcal{C}_0$. There is an identity morphism $\mathrm{id}_x : x \to x$ given by the action of $\mathcal{C}$ on the only map $\{0, 1\} \to \{0\}$.

Let $f, g : x \to y$ be morphisms in $\mathcal{C}$.

Definition

A homotopy between f and g is a 2-cell $\sigma \in \mathcal{C}_2$ of the following shape



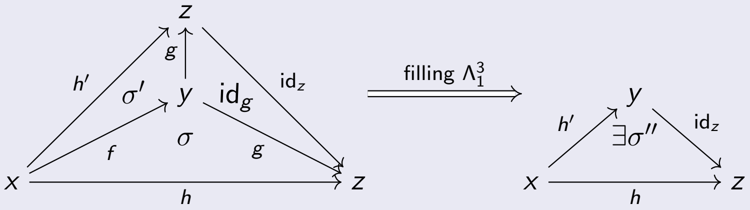
f and g are *homotopic* (written $f \sim g$) if there exists a homotopy between f and g.

Homotopies and composition

Proposition

Let $f : x \to y$, $g : y \to z$, and $h, h' : x \to z$ two compositions of g and f. Then $h \sim h'$.

Proof.



□

Homotopies and composition

Proposition

The relation $\sim$ is an equivalence relation.

Proposition

Composition is unique up to homotopy.

Proposition

Composition is associative and unital up to homotopy.

All proofs: playing with horn filling conditions

In particular, can define the **homotopy category** $h\mathcal{C}$ with same objects as $\mathcal{C}$, and morphisms are morphisms in $\mathcal{C}$ up to homotopy.

Functor ∞-categories

A *functor* between ∞-categories is just a morphism of simplicial sets.

Proposition

The category sSet is cartesian closed (as a presheaf category), with internal hom given by

$$Fun(X, Y)_n := \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n \times X, Y)$$

Proposition

If Y is an ∞-category (resp. ∞-groupoid), then $\mathrm{Fun}(X, Y)$ is an ∞-category (resp. ∞-groupoid).

The objects of $\mathrm{Fun}(X, Y)$ are exactly the morphisms of simplicial sets $X \to Y$.

Natural transformations, equivalences

Definition

Let $F, G : X \to Y$ be morphisms of simplicial sets, with Y an ∞-category.
A natural transformation is morphism $\alpha : F \to G$ in $\mathrm{Fun}(X, Y)$.
Equivalently, $\alpha : \Delta^1 \times X \to Y$ such that $\alpha_{|\{0\}\times X} = F$ and $\alpha_{|\{1\}\times X} = G$.

Definition

A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of ∞-categories if there exists $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $G \circ F \to \mathrm{id}_{\mathcal{C}}$, $F \circ G \to \mathrm{id}_{\mathcal{D}}$.

Hom ∞-groupoid

Let x, y be objects of an ∞-category $\mathcal{C}$. Write $\mathrm{Hom}_{\mathcal{C}}(x, y)$ or simply $\mathrm{Hom}(x, y)$ for the following pullback in sSet.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(x,y) & \longrightarrow & \mathrm{Fun}(\Delta^1, X) \\ \downarrow & \lrcorner & \downarrow \text{restriction to endpoints} \\ \Delta^0 & \xrightarrow[(x,y)]{} & \mathrm{Fun}(\Delta^0 \sqcup \Delta^0, X) \end{array}$$

Proposition

$\mathrm{Hom}_{\mathcal{C}}(x, y)$ is an ∞-groupoid whose objects are given by morphisms $f : x \to y$ in $\mathcal{C}$ and whose morphisms are given by homotopies.

Proposition

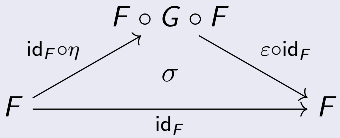
The existence of composite of morphisms in $\mathcal{C}$ can be enhanced to the choice of a functor $\mathrm{Hom}(x, y) \times \mathrm{Hom}(y, z) \to \mathrm{Hom}(x, z)$.

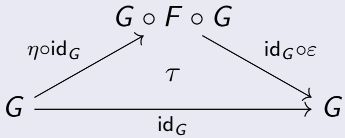
Adjunctions

Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ be functors between ∞-categories, and $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$, $\varepsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$ be natural transformations.

Definition

(η, ε) is a *unit-counit pair* for F and G if there exist compositions





in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ and $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$.

Definition

F is left adjoint to G (and G right adjoint to F) if there exists a unit-counit pair for F and G.

(Co)limits

Let $x, y \in \mathcal{C}$ and ∞-category. A product of x and y is a diagram

$$\begin{array}{ccc} & z & \\ {\scriptstyle \pi_1} \swarrow & & \searrow {\scriptstyle \pi_2} \\ x & & y \end{array}$$

such that for all $z' \in \mathcal{C}$, the induced map

$$\mathrm{Hom}(z', z) \to \mathrm{Hom}(z', x) \times \mathrm{Hom}(z', y)$$

is an equivalence of ∞-groupoids.

Remark

The universal property is up to equivalence, while in 1-categories it's up to isomorphism.

General limits and colimits can be defined along those lines.

Summary of ∞-category theory so far

- ∞-categories have objects, morphisms, homotopies
- existence of compositions
- uniqueness up to homotopy
- Hom-∞-groupoids instead of Hom-sets
- universal properties are up to equivalence
- adjunctions can be defined as usual

Monoids in categories

In a category $\mathcal{C}$ with finite products, a commutative monoid is an object M together with maps $\mu : M \times M \to M$, $\eta : 1 \to M$, such that the following commute.

associativity

$$\begin{array}{ccc} M \times M \times M & \xrightarrow{\mathrm{id}_M \times \mu} & M \times M \\ {\scriptstyle \mu \times \mathrm{id}_M}\downarrow & & \downarrow{\scriptstyle \mu} \\ M \times M & \xrightarrow[\mu]{} & M \end{array}$$

unitality

$$\begin{array}{ccccc} M & \xrightarrow{\mathrm{id}_M \times \eta} & M \times M & \xleftarrow{\eta \times \mathrm{id}_M} & M \\ & {\scriptstyle \mathrm{id}_M}\searrow & \downarrow & \swarrow{\scriptstyle \mathrm{id}_M} & \\ & & M & & \end{array}$$

commutativity

$$\begin{array}{ccc} M \times M & \xrightarrow{\langle \pi_2, \pi_1 \rangle} & M \times M \\ {\scriptstyle \mu}\downarrow & \swarrow{\scriptstyle \mu} & \\ M & & \end{array}$$

In an ∞-category, need further coherence conditions on the data of homotopies, in every dimension.

How to specify everything in a homogeneous way ?

Monoids in categories

The previous definition of commutative monoid is *biased* : many other operations than μ and η exist in monoids.

$$M^5 \to M^2$$
$$(x_1, x_2, x_3, x_4, x_5) \mapsto (\mu(x_3, x_1), \mu(x_2, x_5))$$

Every partial map $f : \{1, \dots, m\} \to \{1, \dots, n\}$ induces a map

$$M^m \to M^n$$
$$(x_i)_{1 \le i \le m} \mapsto \Big(\prod_{f(i)=j} x_i \Big)_{1 \le j \le n}$$

Write FinSet_* for the category of finite sets $\{1, \dots, n\}$ and partial maps.

Proposition

Commutative monoids in $\mathcal{C}$ correspond to functors $F : \mathrm{FinSet}_* \to \mathcal{C}$ such that $F(\{1, \dots, n\}) \simeq F(\{1\})^n$.

Monoids in ∞-categories

Definition

A commutative monoid in an ∞-category $\mathcal{C}$ is a functor $F : N\mathrm{FinSet}_* \to \mathcal{C}$ such that $F(\{1, \ldots, n\}) \simeq F(\{1\})^n$.

This is for monoids with respect to **cartesian products.**

Symmetric monoidal ∞-categories

Fun fact

A symmetric monoidal category is exactly a commutative (pseudo)monoid in the bicategory of categories.

Proposition

There is an ∞-category $\infty\mathrm{Cat}$ whose objects are ∞-categories, morphisms are functors, homotopies are natural isomorphisms, etc.
This ∞-category admits cartesian products, given by the cartesian product of the underlying simplicial sets.

Definition

A symmetric monoidal ∞-category is a commutative monoid M in $\infty\mathrm{Cat}$, i.e. $M : N\mathrm{FinSet}_* \to \infty\mathrm{Cat}$. Its underlying ∞-category is $M(1)$.

Monoidal functors and monoids

With more effort, possible to define :

- commutative monoids in symmetric monoidal ∞-categories
- strong monoidal functors ($F(x \otimes y) \simeq F(x) \otimes F(y)$ + higher structure)
- lax monoidal functors (with maps $F(x) \otimes F(y) \to F(x \otimes y)$ + higher structure)

and show (lax) monoidal functors preserve monoids.

Outline

1 Categorical semantics of linear logic

2 The theory of ∞-categories

3 Linear logic in ∞-categories

Content of our article [HM25]

∞-linear-non-linear adjunction

Definition

An LNL adjunction in ∞-categories is an adjunction

$$(\mathcal{M}, \times) \underset{M}{\overset{L}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} (\mathcal{L}, \otimes)$$

between a cartesian ∞-category $\mathcal{M}$ and a symmetric monoidal closed ∞-category $\mathcal{L}^{\otimes}$, such that the left adjoint L is strong monoidal.

Proposition

The right adjoint is lax monoidal.

Sanity check : comonoid structure on !A

Proposition

In a cartesian ∞-category, every object admits a unique commutative comonoid structure. (comultiplication is given by the diagonal map $X \to X \times X$)

Since strongly monoidal functors preserve commutative comonoids, we get

Corollary

In an LNL adjunction between ∞-categories,

$$(\mathcal{M}, \times) \underset{M}{\overset{L}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} (\mathcal{L}, \otimes)$$

For every object $x \in \mathcal{L}$, $!x := LMx$ inherits a canonical commutative comonoid structure.

Sanity check : Seely isomorphisms

Let

$$(\mathcal{M}, \times) \underset{M}{\overset{L}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} (\mathcal{L}, \otimes)$$

be an LNL adjunction between ∞-categories, where $\mathcal{C}$ has cartesian products.
Since right adjoints preserve limits, M is strongly monoidal from $(\mathcal{L}, \times)$ to $(\mathcal{M}, \times)$. Hence the composite $! = LM : \mathcal{L} \to \mathcal{L}$ is strongly monoidal $(\mathcal{L}, \times) \to (\mathcal{L}, \otimes)$.

In particular this gives Seely isomorphisms $!(A \times B) \simeq !A \otimes !B$, $!\top = 1$.

Lafont ∞-categories

A monoidal structure on an ∞-category $\mathcal{C}$ determines a monoidal structure on $\mathcal{C}^{\mathrm{op}}$ via the self-equivalence $\mathrm{op} : \infty\mathrm{Cat} \to \infty\mathrm{Cat}$.

Definition (Commutative comonoids)

Given a SM∞C $\mathcal{C}$, the ∞-category $\mathrm{Comon}(\mathcal{C})$ is defined as $\mathrm{Mon}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$.

Theorem

The ∞-category $\mathrm{Comon}(\mathcal{C})$ is cartesian and the forgetful functor $\mathrm{Comon}(\mathcal{C}) \to \mathcal{C}$ is strongly monoidal from the cartesian structure to the monoidal one.

Corollary

If $\mathrm{Comon}(\mathcal{C}) \to \mathcal{C}$ has a right adjoint, it induces an LNL adjunction $(\mathrm{Comon}(\mathcal{C}), \times) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} (\mathcal{C}, \otimes)$

Definition

In that case, we say that $\mathcal{C}$ is a *Lafont ∞-category*.

An explicit formula for cofree comonoids

The following has been shown in 1-category theory by [MTT].

Theorem

Let $(\mathcal{L}, \otimes)$ be a symmetric monoidal ∞-category, and $X \in \mathcal{C}$.
If for all $A \in \mathcal{C}$,

$$A \otimes \prod_{n \in \mathbb{N}} (X^{\otimes n})^{\mathfrak{S}_n} \to \prod_{n \in \mathbb{N}} (A \otimes X^{\otimes n})^{\mathfrak{S}_n}$$

is an equivalence, then

$$\prod_{n \in \mathbb{N}} (X^{\otimes n})^{\mathfrak{S}_n}$$

is the cofree commutative comonoid on X.

It follows easily from more general results of Lurie [Lur17].

Example : ∞-categorical generalized species

(∞)-category	Rel	Prof
Objects	Sets X, Y	∞-categories $\mathcal{C}, \mathcal{D}$
Linear morphisms	Relations $R : X \times Y \to \mathrm{Bool}$	∞-profunctors $\mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \infty\mathrm{Grpd}$
Lafont exponential	$\mathrm{Mul}(X)$ multisets on underlying set	$\mathrm{Sym}(\mathcal{C})$ free symmetric monoidal ∞-category
Non-linear morphisms	"multi-relations" $\mathrm{Mul}(X) \times Y \to \mathrm{Bool}$	"∞-generalized species" $\mathrm{Sym}(\mathcal{C}) \times \mathcal{D}^{\mathrm{op}} \to \infty\mathrm{Grpd}$
Extensional objects	Complete lattices $P(X) = \mathrm{Bool}^X$	Presheaf ∞-categories $\mathcal{P}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \infty\mathrm{Grpd})$
Extensional morphisms	Maps $P(X) \to P(Y)$ that preserve arbitrary joins	Functors $\mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{D})$ that preserve small colimits
Extensional non-linear morphisms	?	Analytic functors ?

Another criterion for existence of cofree comonoids

Definition

An ∞-category $\mathcal{C}$ is *presentable* if

- it is closed under small colimits
- there is a small set of objects $S \subset \mathcal{C}_0$ such that every object is a *filtered colimit* of objects of S

Theorem

Let $\mathcal{C}$ be a symmetric monoidal presentable ∞-category such that $\forall x \in \mathcal{C}$, the functor $x \otimes - : \mathcal{C} \to \mathcal{C}$ preserves small colimits. Then $\mathcal{C}$ is Lafont (it admits cofree comonoids).

But in general there is no nice formula in this context.

Example

Spectra (abelian groups), module spectra (modules).

Conclusion

- Building upon the heavy machinery of ∞-categories developed, we generalized two notions of models of linear logic to the ∞-categorical setting (Lafont categories and LNL adjunctions).
- We constructed new such models analogous to variants of the relational model and bicategorical models of species and polynomials.

Future work

- Give direct definitions of linear ∞-categories and Seely ∞-categories, and show they induce LNL adjunctions.
- Explicit comparison of our generalized ∞-species and analytic functors.
- Generalize Mellies' span model (template games) to this new setting (in connection with polynomial functors [HM24])
- Generalize to $(\infty, 2)$-categorical setting to model differential linear logic.
- Try to fit advanced homotopical constructions with linear flavour (Goodwillie calculus ?) into this new setting.

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A reminder on the 1-categorical story

Intensional	Extensional
category Rel	full subcat of SupLat on the $P(X)$, $X \in$ Set
category Porel	full subcat of SupLat on the $P(X)$, $X \in$ Poset
Mul(X) free commutative monoid on underlying (po)set	free commutative comonoid in SupLat
non-linear maps Mul(X) $\to Y$	?
FC(X) free poset with finite joins on X	$!_S P(X)$ exponential induced by LNL adjunction $\text{Scott} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \text{SupLat}$
non-linear maps FC(X) $\to Y$	Scott-continuous maps $P(X) \to P(Y)$

∞-categories with colimits

Let $\mathbb{K}$ be a class of simplicial sets. Write $\infty\mathrm{Cat}_{\mathbb{K}}$ for the sub-∞-category of $\infty\mathrm{Cat}$ on ∞-categories that admit colimits indexed by simplicial sets in $\mathbb{K}$, and functors that preserve such colimits.

Special cases : $\infty\mathrm{Cat}_{\mathrm{cc}}$ for $\mathbb{K} =$ all simplicial sets ("cc" for cocontinuous), $\infty\mathrm{Cat}_{\mathrm{filtr}}$ for filtered simplicial sets, $\infty\mathrm{Cat}_{\mathrm{sift}}$ for sifted simplicial sets.

Proposition

The ∞-category $\infty\mathrm{Cat}_{\mathbb{K}}$ admits a symmetric monoidal closed structure whose tensor products classifies functors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ that preserve $\mathbb{K}$-colimits *independently in both variables*. Moreover, if $\mathbb{K} \subseteq \mathbb{K}'$, the forgetful functor $\infty\mathrm{Cat}_{\mathbb{K}'} \to \infty\mathrm{Cat}_{\mathbb{K}}$ admits a strongly monoidal left adjoint.

Proposition

If $\mathbb{K}$ consists only of sifted simplicial sets, then the previous monoidal structure is cartesian. That is the case for $\infty\mathrm{Cat} = \infty\mathrm{Cat}_{\emptyset}$, $\infty\mathrm{Cat}_{\mathrm{filtr}}$ and $\infty\mathrm{Cat}_{\mathrm{sift}}$.

Cocompletion-based LNLs

There is a chain of strongly monoidal left adjoints

$$\mathcal{S} \underset{\bot}{\rightleftarrows} \infty\mathrm{Cat} \underset{\bot}{\rightleftarrows} \infty\mathrm{Cat}_{\mathrm{filtr}} \underset{\bot}{\rightleftarrows} \infty\mathrm{Cat}_{\mathrm{sift}} \underset{\bot}{\rightleftarrows} \infty\mathrm{Cat}_{\mathrm{cc}}$$

where the monoidal structures on all but $\infty\mathrm{Cat}_{\mathrm{cc}}$ are cartesian.
Moreover they are all monoidal closed, in particular we get 4 LNL adjunctions, and hence 4 exponential comonads on $\infty\mathrm{Cat}_{\mathrm{cc}}$.
Write $!_{\mathrm{f}}$ for the one induced by the adjunction with $\infty\mathrm{Cat}_{\mathrm{filtr}}$ and similarly for $!_{\mathrm{s}}$ and $\infty\mathrm{Cat}_{\mathrm{sift}}$.

Theorem

For a small ∞-category $\mathcal{C}$, $!_s\mathcal{P}(\mathcal{C}) = \mathcal{P}(\mathcal{C}^{\sqcup})$, where $\mathcal{C}^{\sqcup}$ is the free cocompletion of $\mathcal{C}$ under finite coproducts.

Theorem

For a small ∞-category $\mathcal{C}$, $!_f\mathcal{P}(\mathcal{C}) = \mathcal{P}(\mathcal{C}^{fin})$, where $\mathcal{C}^{fin}$ is the free cocompletion of $\mathcal{C}$ under finite colimits.

Intensional point of view

We defined $!_s$ and $!_f$ at the extensional level (cocomplete ∞-categories).

Intensional	Extensional
Profunctors $\mathcal{C} \times \mathcal{D}^{\mathsf{op}} \to \mathcal{S}$	Cocontinuous functor $\mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{D})$
Completion under finite coproducts comonad on Prof	$!_s$ comonad on $\infty\mathsf{Cat}_{\mathsf{cc}}$
Completion under finite colimits comonad on Prof	$!_f$ comonad on $\infty\mathsf{Cat}_{\mathsf{cc}}$

At the level of posets, finite coproducts and finite colimits coincide. Hence we have two generalizations of the comonad FC on Porel.

Free exponential on $\infty\mathrm{Cat}_{cc}$

Theorem

The full sub-∞-category of $\infty\mathrm{Cat}_{cc}$ on presheaf ∞-categories admits cofree commutative comonoids.
Moreover, the presheaf construction $\infty\mathrm{Cat} \to \infty\mathrm{Cat}_{cc}$ maps free commutative monoids to cofree commutative comonoids : $!\mathcal{P}(\mathcal{C}) = \mathcal{P}(\mathrm{Sym}(\mathcal{C}))$, where $\mathrm{Sym}(\mathcal{C}) := \coprod_{n\in\mathbb{N}} \mathcal{C}^n /\!/ \mathfrak{S}_n$ is the free symmetric monoidal ∞-category on $\mathcal{C}$.

Intensional	Extensional
Profunctors $\mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathcal{S}$	Cocontinuous functors $\mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{D})$
Free symmetric monoidal category comonad on Prof	Cofree commutative comonoid on $\infty\mathrm{Cat}_{cc}$
Non-linear morphisms	
Genralized ∞-species $\mathrm{Sym}(\mathcal{C}) \times \mathcal{D}^{\mathrm{op}} \to \mathcal{S}$	Analytic ∞-functors ?

0-∞ analogy

0-categories	∞-categories
set $X \in \mathrm{Set}$	∞-groupoid $X \in \mathcal{S}$
poset E	∞-category $\mathcal{C}$
Fibred relation $R \subseteq X \times Y$	Span $Z \to X \times Y$
Indexed relation $X \times Y \to \mathrm{Bool}$	Functor $X \times Y \to \mathcal{S}$
Monotonous relations $E \times F^{\mathrm{op}} \to \mathrm{Bool}$	Profunctor $\mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathcal{S}$
Free suplattice $P(E) := (E^{\mathrm{op}} \to \mathrm{Bool})$	Presheaf ∞-category $\mathcal{P}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$
Suplattice morphism	Small colimit-preserving functor
Scott-continuous map	Filtered-colimit preserving functor (or sifted-colimit preserving)