Open problems in applications of Fourier learning to the Diffie-Hellman problem in finite fields

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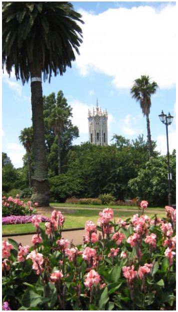

Outline

- Hardcore bits
- Goldreich-Levin algorithm
- Decoding linear codes
- Fourier learning
- Applications revisited
- Open questions

Big thanks to Barak Shani and Joel Laity.

Some papers on elliptic curve bit security

- Dan Boneh and Igor Shparlinski. On the unpredictability of bits of elliptic curve Diffie- Hellman scheme. CRYPTO 2001.
- David Jao, Dimitar Jetchev and Ramarathnam Venkatesan: On the Bits of Elliptic Curve Diffie-Hellman Keys. INDOCRYPT 2007.
- Dimitar Jetchev and Ramarathnam Venkatesan: Bits Security of the Elliptic Curve Diffie-Hellman Secret Keys. CRYPTO 2008.

Applications of Fourier Learning in Cryptography

- Daniel Bleichenbacher. On The Generation of One-Time Keys in DL Signature Schemes. Talk, 2000.
- Adi Akavia, Shafi Goldwasser and Shmuel Safra: Proving Hard-Core Predicates Using List Decoding. FOCS 2003.
- Adi Akavia: Solving Hidden Number Problem with One Bit Oracle and Advice. CRYPTO 2009.
- Paz Morillo and Carla Ràfols: The Security of All Bits Using List Decoding. PKC 2009.
- Daniele Micciancio and Petros Mol: Pseudorandom Knapsacks and the Sample Complexity of LWE Search-to-Decision Reductions. CRYPTO 2011.

Applications of Fourier Learning in Cryptography

- Alexandre Duc and Dimitar Jetchev: Hardness of Computing Individual Bits for One-Way Functions on Elliptic Curves. CRYPTO 2012.
- Nelly Fazio, Rosario Gennaro, Irippuge Milinda Perera and William E. Skeith III: Hard-Core Predicates for a Diffie-Hellman Problem over Finite Fields. CRYPTO (2) 2013.
- Elke De Mulder, Michael Hutter, Mark E. Marson and Peter Pearson. Using Bleichenbacher's Solution to the Hidden Number Problem to Attack Nonce Leaks in 384-Bit ECDSA, CHES 2013.
- Diego F. Aranha, Pierre-Alain Fouque, Benot Grard, Jean-Gabriel Kammerer, Mehdi Tibouchi and Jean-Christophe Zapalowicz: GLV/GLS Decomposition, Power Analysis, and Attacks on ECDSA Signatures with Single-Bit Nonce Bias. Asiacrypt 2014.

Can these works tell us anything interesting about the DLP in finite fields?

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Executive Summary:

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Executive Summary:

No

So what is this all about?

Hardcore bits

- Let $f : X \to Y$ be a one-way function.
- This means: Given x one can efficiently compute $y = f(x)$, but given y it is computationally infeasible to find $x \in X$ such that $f(x) = y$.
- By definition, given $f(x)$ one cannot efficiently deduce x. But, given $f(x)$ it might be possible to compute some **partial information** about x.
- Let $X = \{0, 1\}^n$ and $\underline{x} = (x_1, \ldots, x_n) \in X$. Define $\mathrm{bit}_i(\underline{x}) = x_i$.

Hardcore bits

- Let $f : X \to Y$ be a one-way function.
- We say the i-th bit is **easy** for f if, given $f(\underline{x})$, one can compute the value $\mathrm{bit}_i(\underline{x})$ with probability significantly better than guessing.
  (I assume the values $\underline{x}$ are sampled uniformly in the game.)
- It is immediate that a one-way function has many bits that are not easy.
  We call such bits **hardcore bits** for f.
- The problem is to prove that a specific bit is hardcore.

Proof technique

- To show $\text{bit}_i(x)$ is hardcore one argues as follows:
- Suppose one has an algorithm/oracle that on input any $y \in Y$ outputs with high probability the correct value of $\text{bit}_i(x)$ where $y = f(x)$.
- Suppose one is also given a challenge $y^* = f(x^*)$.
- Then one constructs an efficient algorithm that computes x^* by making oracle queries to the algorithm.

Proof technique

- When f is "algebraic" and the oracle is always correct then this is easy.
- For example, suppose $f(x) = g^x \pmod{p}$ where g has odd prime order r.
- Let O be an oracle such that $O(y)$ returns $\mathrm{bit}_0(x)$.
- Let $y^* = g^{x^*}$ be the challenge.
- Calling $O(y^*)$ gives x_0^*.
- Now set

$$y = (y^* g^{-x_0^*})^{2^{-1} \pmod{r}}$$

- Calling $O(y)$ gives x_1^*.
- The process repeats in the obvious way.

Hardcore bits

- This approach generalises to functions other than bits.
- Let $g : X \to Z$ be some function.
- Then one can talk about whether $g(x)$ is hardcore.
- For example $g(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$ for some $1 < k < n$.
- One is arguing that it is hard to compute the first k bits of x.
- This is a weaker notion: It might be hard to compute the first k bits of x, but that does not imply it is hard to compute the first bit of x.

Ancient history

- The big challenge is to handle oracles that are not correct all the time.
- First case of interest was hardcore bits for RSA.
- Goldwasser, Micali and Tong (FOCS 1982): least significant bit of RSA is hardcore if oracle is correct with probability $1 - 1/\log_2(N)$.
  In other words, make about $\log_2(N)$ oracle queries and only one of them is wrong.
- Ben Or, Chor and Shamir (STOC 1983): least significant bit of RSA is hardcore if oracle is correct with probability $3/4 + \epsilon$.
- Alexi, Chor, Goldreich and Schnorr (1988): handles oracle that is correct with probability $1/2 + \epsilon$.
- None of these papers discuss Fourier analysis.

A general solution

- Goldreich, Levin (1989) gave a construction to derive a hardcore bit for any one-way function.
- Idea: Let $X = \mathbb{F}_2^n$. Given $\underline{x} \in X$ one chooses $\underline{t} \in X$ and sends $(\underline{t}, f(\underline{x}))$.
- The hardcore bit is

$$\underline{t} \cdot \underline{x} = \sum_{j=1}^{n} t_j x_j.$$

- Idea: Let O be an oracle that, on input $f(\underline{x})$, outputs $\underline{t} \cdot \underline{x}$ non-negligibly better than guessing.
  Using O and choosing $\underline{t}$ one can compute $\underline{x}$.
- It was pointed out by Rackoff and Wigderson that the Goldreich-Levin algorithm is based on the Walsh transform (fourier analysis in the group $\mathbb{F}_2^n$).

Elementary approach to Goldreich-Levin

- Let O be an oracle that, for a secret value $\underline{x}$, on input $\underline{t}$ outputs $\underline{t} \cdot \underline{x}$ non-negligibly better than guessing.
- If there are no errors: query $O(\underline{e}_i)$ for unit vectors $\underline{e}_i$.
- Basic trick: Choose random $\underline{a}$ and query $O(\underline{a} + \underline{e}_i) - O(\underline{a})$.
- If both oracle outputs correct then have

$$(\underline{a} + \underline{e}_i) \cdot \underline{x} - \underline{a} \cdot \underline{x} \;=\; x_i.$$

- Repeat for many random $\underline{a}$ and take majority vote.
- **Note:** It is essential to be able to choose the inputs to O.

Connection with decoding linear codes

- We are getting a lot of values $\underline{t} \cdot \underline{x}$, for various $\underline{t} \in X$, some of them with errors.
- Putting all the rows $\underline{t}$ together as an $m \times n$ matrix we have measurements $T\underline{x} + \underline{e}$ where $\underline{e}$ is a length m column vector of low weight.
- Hence, can re-phrase computing $\underline{x}$ in terms of T being the "generator matrix" of a code and $T\underline{x} + \underline{e}$ being the received "code word".

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- Hence, can re-phrase computing $\underline{x}$ in terms of T being the "generator matrix" of a code and $T\underline{x} + \underline{e}$ being the received "code word".
- This is of course a bit of a cheat: We are essentially choosing the generator matrix to have a special structure.

Fourier Analysis on Finite Groups

- Consider $G = \mathbb{F}_2^n$, a finite additive group of order 2^n.
- The set of functions $f : G \to \mathbb{C}$ is a $\mathbb{C}$-vector space of dimension 2^n.
- There is an inner product

$$\langle f, g \rangle \;=\; \tfrac{1}{2^n} \sum_{\underline{x} \in G} f(\underline{x}) \overline{g(\underline{x})}$$

- An orthonormal basis for this set of functions is

$$\chi_{\underline{a}}(\underline{x}) = (-1)^{\underline{a} \cdot \underline{x}}$$

where $\underline{a}$ runs over all elements of $\mathbb{F}_2^n$.

Fourier analysis on finite groups

- Let $f : G \to \mathbb{C}$ be given, $G = \mathbb{F}_2^n$.
- Then f has a Fourier expansion

$$f(\underline{x}) \;=\; \sum_{\underline{a} \in \mathbb{F}_2^n} \hat{f}(\underline{a}) \chi_{\underline{a}}(\underline{x})$$

  where the Fourier coefficients are $\hat{f}(\underline{a}) = \langle f, \chi_{\underline{a}} \rangle$.
- Parseval's identity: $\langle f, f \rangle = \sum_{\underline{a} \in G} \hat{f}(\underline{a})^2$.
- We call a character $\chi_{\underline{a}}$ **heavy** if $|\hat{f}(\underline{a})|$ is relatively large with respect to $\langle f, f \rangle$.
- Parseval implies there cannot be many heavy Fourier coefficients.
- $f(\underline{x})$ is called **concentrated** if it has some heavy Fourier coefficients.

Lemma

- Let $f : \mathbb{F}_2^n \to \{1, -1\}$ be such that

$$f(\underline{x}) = (-1)^{\underline{x}\cdot\underline{s}} = \chi_{\underline{s}}(\underline{x})$$

for all $\underline{x} \in X \subseteq \mathbb{F}_2^n$, and

$$f(\underline{x}) = (-1)^{\underline{x}\cdot\underline{s}+1} = -\chi_{\underline{s}}(\underline{x})$$

for all $\underline{x} \in \overline{X} = \mathbb{F}_2^n \setminus X$.
- Let $|X| = \delta 2^n$.
- Then $\langle f, f \rangle = 1$ and $\hat{f}(\underline{s}) = 2\delta - 1$.

Fourier approach to Goldreich-Levin

- Let $G = \mathbb{F}_2^n$ and fix $\underline{s} \in G$.
- Let $f(\underline{t}) : G \to \{-1, 1\}$ be such that on $1/2 + \delta$ of the inputs $\underline{t} \in G$ we have $f(\underline{t}) = (-1)^{\underline{t} \cdot \underline{s}}$.
- Consider the Fourier series for $f(\underline{t})$.
- By the previous Lemma, $f(\underline{t})$ is concentrated and $\chi_{\underline{s}}$ is a heavy Fourier character.
  In other words, the coefficient $|\hat{f}(\underline{s})|$ is large.
- One extends the algorithmic ideas from learning one secret $\underline{s}$ to learning all the heavy Fourier characters.

List decoding connection

- Suppose $f(\underline{x})$ is a function on $G = \mathbb{F}_2^n$ which has several heavy characters $\underline{s}_1, \ldots, \underline{s}_k$.
- It means $f(\underline{x})$ and $(-1)^{\underline{x} \cdot \underline{s}_j}$ agree on a large set of inputs for each $1 \leq j \leq k$.
- We can write T again for the matrix corresponding to the function queries we will make, and represent the outputs of f (turned back from $\{-1, 1\}$ to $\{0, 1\}$) as a codeword.
- An algorithm that computes a list of heavy coefficients $\underline{s}_1, \ldots, \underline{s}_k$ can be viewed as a list decoding algorithm.

Further work

- There are general algorithms to compute all heavy Fourier coefficients of any concentrated function on a finite abelian group G.
- Best paper to read to get the main ideas is Kushilevitz and Mansour (STOC 1991).
- There are improved algrithms, see recent survey paper by Gilbert, Indyk, Iwen and Schmidt.
- All these works require chosen queries to the function.

Bit security of CDH

- Given $g, g^a, g^b \in \mathbb{F}_p^*$ want to know what bits of g^{ab} are secure.
- So suppose one has an oracle $O(g, g^a, g^b)$ that outputs some bits of g^{ab}.
- We want to use O to compute all of $s = g^{ab}$.
- Idea is to choose random r and call $O(g, g^a, g^b g^r)$, which gives bits of $g^{a(b+r)} = s(g^a)^r$.
- Hidden number problem: Fix $s \in \mathbb{F}_p^*$ and let O be an oracle such that $O(t) = \mathrm{LSB}(st)$. Goal is to compute s given access to O.

Boneh-Venkatesan

- Hidden number problem: Given $(t, \text{bits}(ts))$ to compute $s \in \mathbb{F}_p^*$.
- BV consider oracle that computes the $\sqrt{\log(p)}$ most significant bits of the Diffie-Hellman secret $s = g^{ab}$.
- Oracle is always correct.
- Multipliers t chosen randomly and non-adaptively.
- BV use lattice method.
- Lots of following work.
- Challenge: one bit, unreliable oracle.

Fourier analysis in $\mathbb{F}_p^*$

- Let $G = \mathbb{F}_p^*$ and consider functions $f : G \to \mathbb{C}$.
- Let $\zeta = \exp(2\pi i/p)$.
- For $a \in \{1, \ldots, p-1\}$ define the character

$$\chi_a(x) = \zeta^{ax}.$$

- Then, for $a, b \in \{1, \ldots, p-1\}$

$$\langle \chi_a, \chi_b \rangle = \frac{1}{p} \sum_{x=1}^{p-1} \chi_a(x) \overline{\chi_b(x)} = \frac{1}{p} \sum_{x=1}^{p-1} \zeta^{(a-b)x}.$$

This is 0 if $a \neq b$ and 1 if $a = b$.

Application: Bit security of CDH

- Given $g, g^a, g^b \in \mathbb{F}_p^*$ want to prove that a single bit of g^{ab} is secure.
- Akavia et al considered function $f(x) : \mathbb{F}_p \to \{-1, 1\}$ by $f(x) = (-1)^{\mathrm{LSB}(x)}$.
  One can show that this is concentrated.
  [Also $(-1)^{\mathrm{MSB}(x)}$.]
- Big result from CRYPTO 2009: "Solving Hidden Number Problem with One Bit Oracle and Advice".
- Idea: Translation of Fourier coefficients by s.
- Recall that we choose random r and call $O(g, g^a, g^b g^r)$ to get a bit of $g^{a(b+r)} = s(g^a)^r$.

Applications to elliptic curves

- Would like to prove certain bits are hardcore for ECDH.
- Given P, aP, bP want to show that if can compute some bit of abP then can compute all abP.
- Boneh and Shparlinski developed an approach.
- Suppose O is an oracle that takes (E, P, aP, bP) such that $P \in E$ and computes the most significant bits of the x-coordinate of abP.

Boneh and Shparlinski

- Trick is to consider isomorphism $\phi : E \to E'$ of Weierstrass curves given by
$$\phi(x, y) = (\lambda^2 x, \lambda^3 y).$$
- Then call $O(E', \phi(P), \phi(aP), \phi(bP))$.
- The secret is $s = x(abP)$ and, for chosen λ, get bits of $\lambda^2 s$, which is more-or-less back to hidden number problem with chosen multipliers.
- Boneh and Shparlinski used lattice methods.
- Duc and Jetchev used Fourier methods.

Elliptic curve Diffie-Hellman

- Changing Weierstrass models does not actually prove a hardcore bit for a fixed representation.
- I call it the "BS" trick.
- Hence, the problem of hardcore bits for ECDH is still open.
- Any ideas?

Obstruction

- "Beurling-Helson theorem".
- $f(x)$ concentrated and $f(g(x))$ concentrated implies that $g(x)$ is an affine map.
- If $g(x)$ comes from elliptic curve operations then it is a non-trivial rational function.
- Hence, elliptic curve operations do not preserve concentrated.

CDH in non-prime finite fields

- Fazio, Gennaro, Perera and Skeith (CRYPTO 2013) showed bit security for CDH in $\mathbb{F}_{p^2}$.
- Their model involves field isomorphisms (a version of the "BS trick").
- The paper "The Multivariate Hidden Number Problem" (ICITS 2015) written with my student Barak Shani treats general fields $\mathbb{F}_{p^n}$.
- Also uses a "BS trick".
- It is still open to prove bit security of Diffie-Hellman in finite fields.

Open Questions

- Do these ideas have anything to do with DLP?
- Can one actually prove bit security results for single bits in realistic models?
- Are Fourier learning algorithms optimal from a concrete point of view?

Thank you for your attention

See you at Asiacrypt!