

## Cas où on sait calculer $\vec{\pi}_1(X)$

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CEA Saclay & PPS

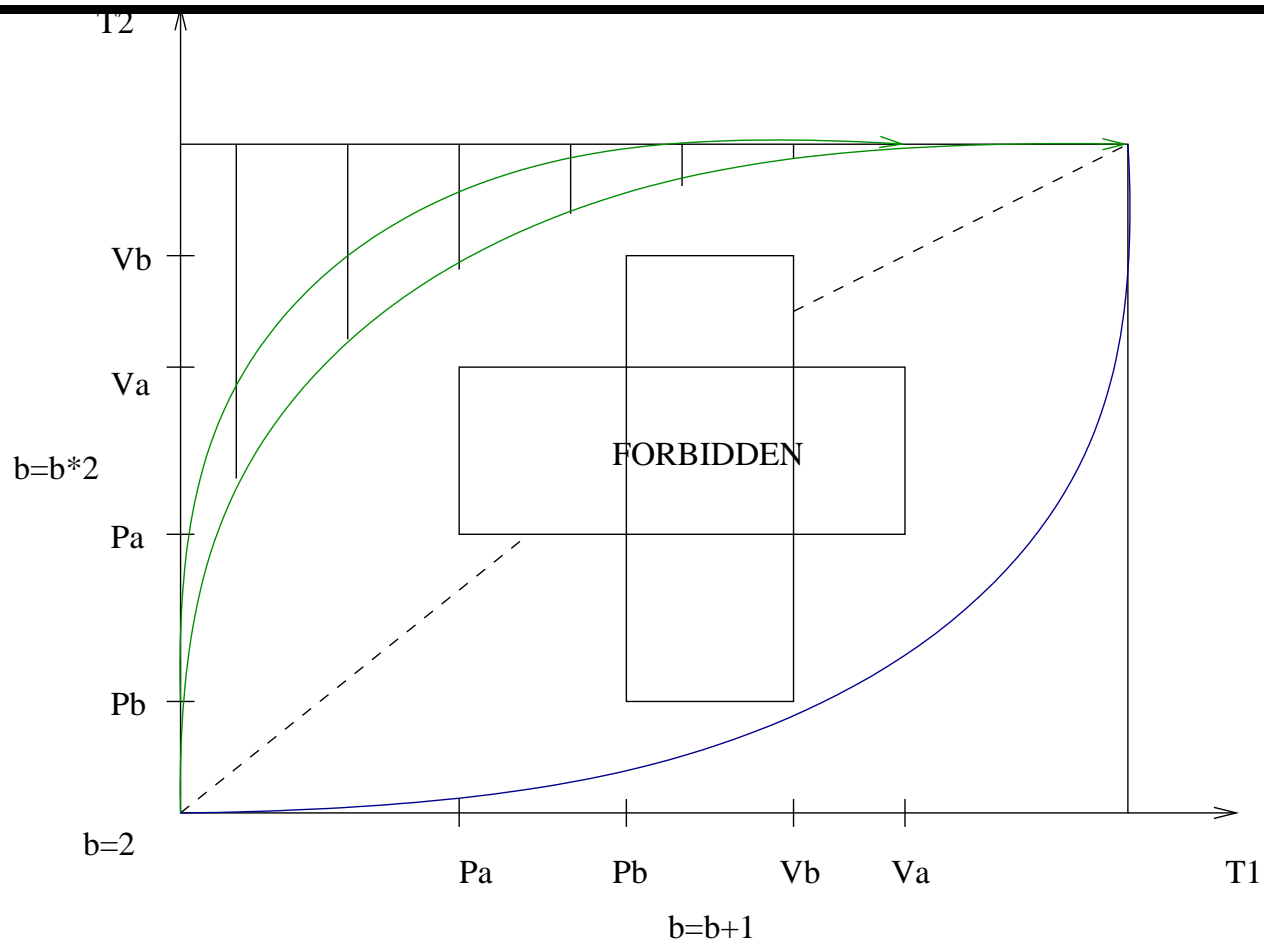
Le 27/11/03, AS “Topologie Algébrique...”

## Road map

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- The fundamental category, and the set of “diconnected components” in the topological and combinatorial cases
- “Generators and relations” of such categories, in the abstract
- A very particular case, which allows for computations
- Examples

# Directed homotopy and Concurrency



T2 gets a and b before T1 does:  $b=5!$

## The fundamental category

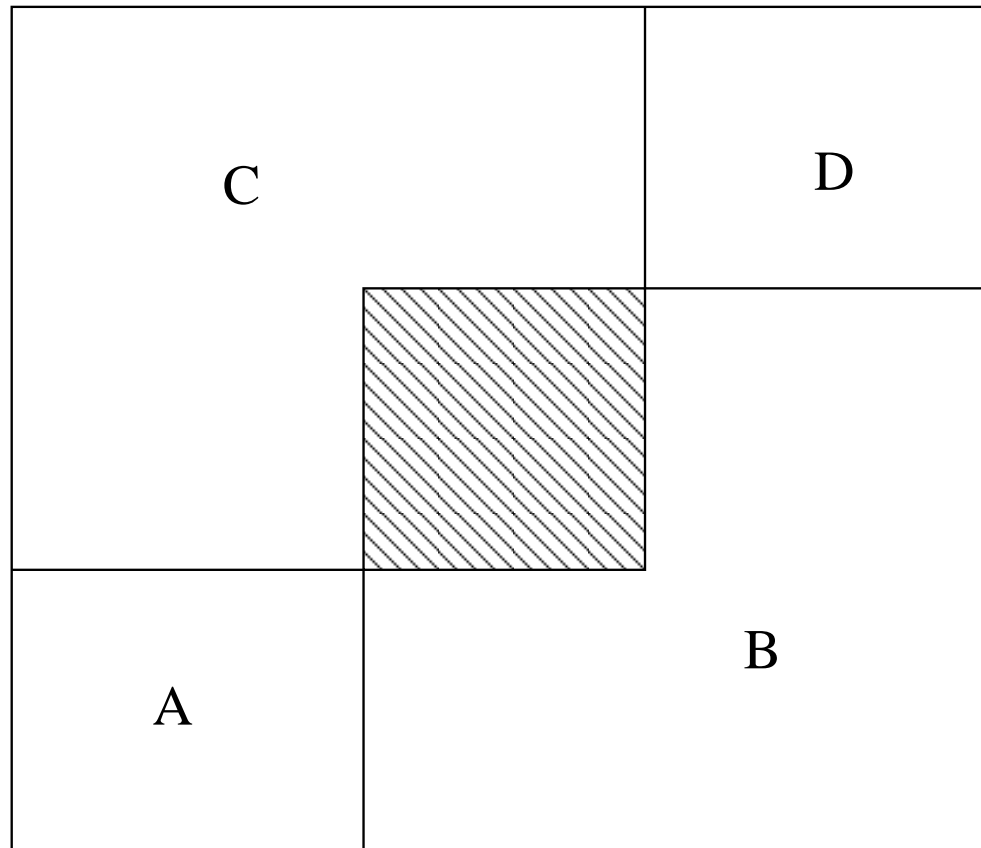
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Given is a notion of directed deformation (dihomotopy) of paths on po-spaces, flows, cubical sets  $X$  etc. Then we construct a category  $\vec{\pi}_1(X)$ :

- objects are “points” of the original space
- morphisms are dihomotopy classes of dipaths of the original space

## Example

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## A lifting property...(APCS'03)

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When we have a maximal pure left and right calculus of fractions of weakly invertibles  $\Sigma$  in  $\mathcal{C}$  be a category in which all endomorphisms are identities...

Let  $C_1, C_2 \subset Ob(\mathcal{C})$  denote two components such that the set of morphisms (in  $\mathcal{C}/\Sigma$ ) is *finite*. Then, for every  $x_1 \in C_1$  there exists  $x_2 \in C_2$  such that the quotient map

$$\mathcal{C}(x_1, x_2) \rightarrow \mathcal{C}/\Sigma(C_1, C_2), \ ; \ f \mapsto [f]$$

is *bijective*.

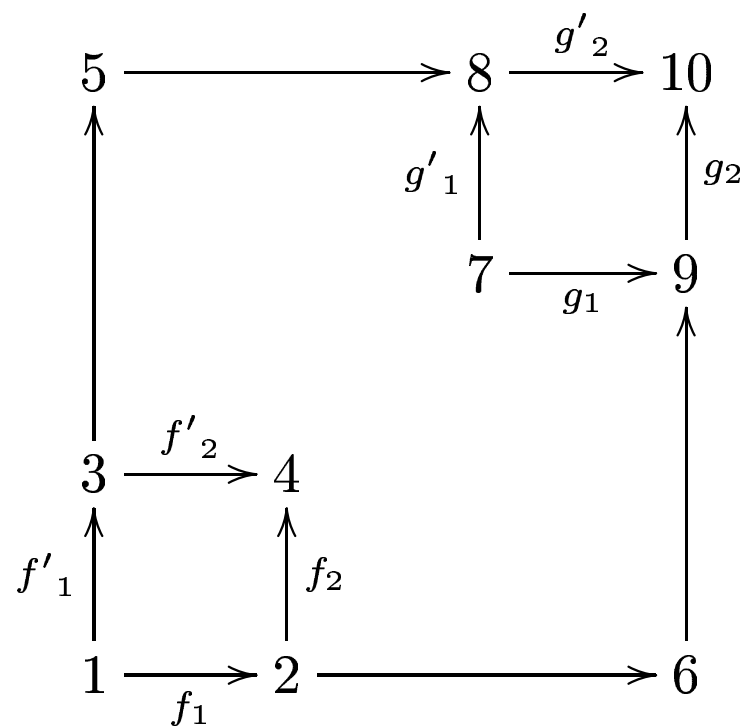
## Generators and relations?

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5			8	10
			7	9
			6	
3	4			
1	2			

## Generators and relations:

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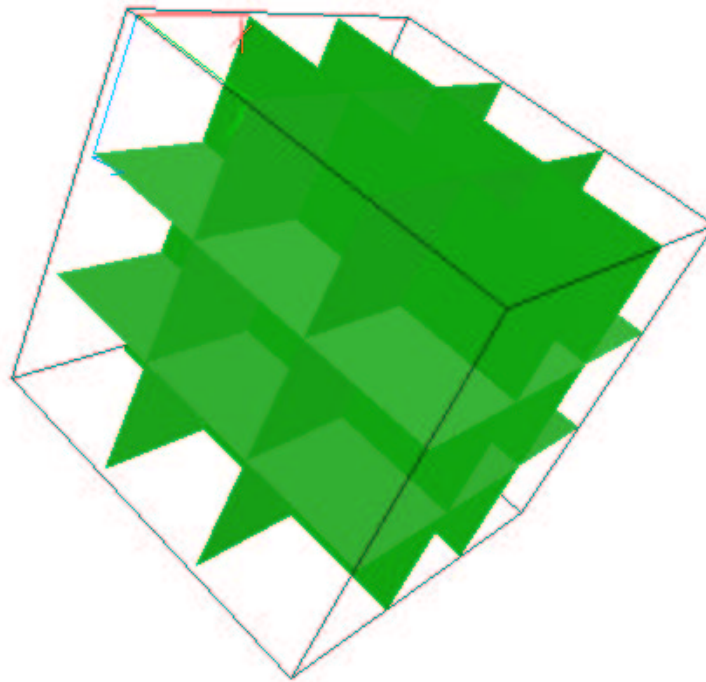


together with relations  $g'_2 \circ g'_1 = g_2 \circ g_1$  and  $f'_2 \circ f'_1 = f_2 \circ f_1$



## Example in dimension 3

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$$\vec{\pi}_0(X)$$

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- Difficult to define a priori... (reachability is not an equivalence relation)
- Do as with the fundamental groupoid: take the components of the fundamental groupoid, as chunks in which base points can vary
- $\vec{\pi}_0(X)$  will be the set of pairs  $(A, B)$  of components (in our new sense) such that there exists a morphism from  $A$  to  $B$  in  $\pi_1(X)$

$$\vec{\pi}_0 (X)$$

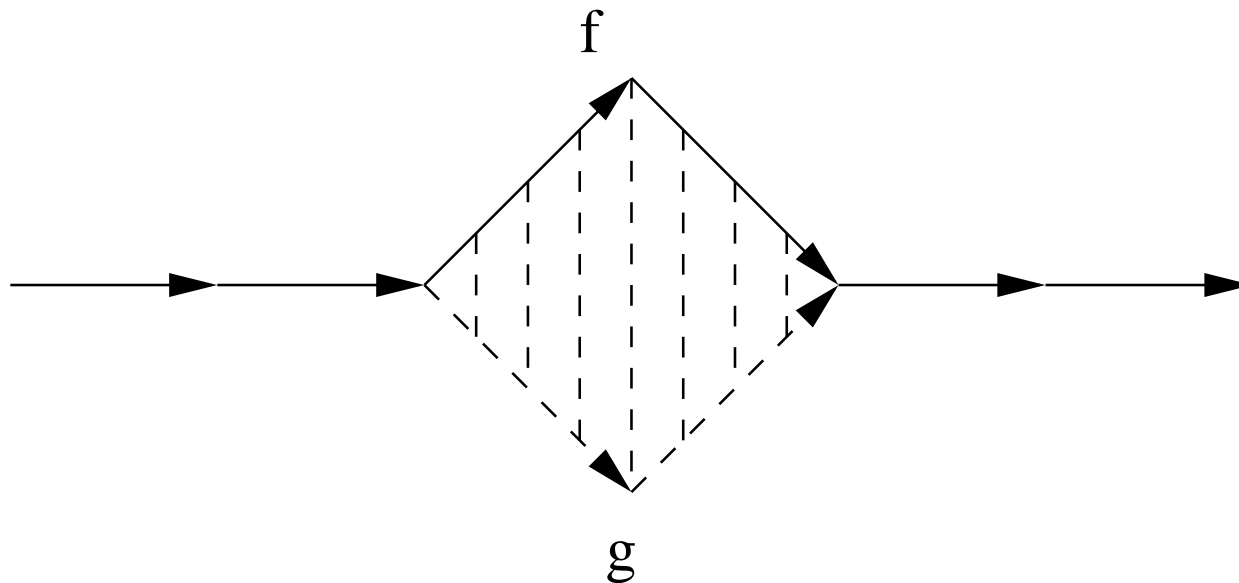

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- There is a “bi-base point” map  $\vec{\pi}_1 \rightarrow \vec{\pi}_0$
- The lifting property ensures that one can choose any bi-base point  $(p, q)$  in  $(A, B) \in \vec{\pi}_0 (X)$  without changing (canonically...)  $\vec{\pi}_1 (X)(p, q)$
- Dihomeomorphism implies bijection between  $\vec{\pi}_0$  and isomorphism of categories between the categories of components of  $\vec{\pi}_1 (X)$

## In the combinatorial case (cubical set)

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Corresponding to the fundamental category is the edge-path category. Deformations are generated by 2-cells:



## Morally the same...for geometric cubical sets

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- There is a directed version of a cubical approximation theorem
- Let  $X$  be the geometric realization of a geometric cubical set, then
  - $\vec{\pi}_1(X)(p, q)$  when  $p$  and  $q$  are two vertices, is isomorphic to the set of morphisms from  $p$  to  $q$  in the edge-path category
  - $\vec{\pi}_1(X)(p, q)$  is isomorphic to  $\vec{\pi}_1(X)(p, x)$  and to  $\vec{\pi}_1(X)(y, q)$  when  $p$  and  $q$  are two vertices and  $q$  is the lower vertex of the carrier of  $x$  and  $p$  is the upper vertex of the carrier of  $y$ .

## Future components

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Let  $\Sigma$  be a subcategory of  $\mathcal{C}$ .  $\Sigma$  is a subcategory of  $f$ -WE in short if:

- $\Sigma$  contains all right retracts i.e. all  $\sigma$  such that there is  $g$  in  $\mathcal{C}$  with  $g \circ \sigma = Id$
- all  $\sigma$  in  $\Sigma$  are epis in  $\mathcal{C}$ , meaning that for all  $u$  and  $v$  in  $\mathcal{C}$ , if the following diagram commutes

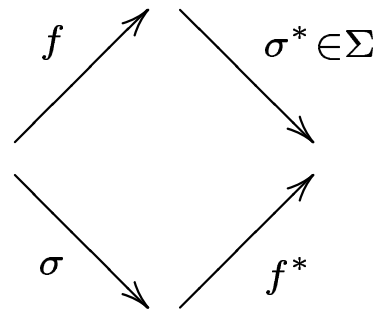
$$\xrightarrow{\sigma} \begin{array}{c} \xrightarrow{u} \\ \xleftarrow{v} \end{array}$$

then  $u = v$ .

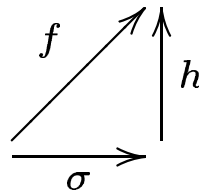
and...

## Future components

- $\Sigma$  is stable under pushout (with any morphism in  $\mathcal{C}$ ):



- If there is  $u : \beta \rightarrow \gamma$  in  $\mathcal{C}$ , then for all  $\sigma : \alpha \rightarrow \beta$  in  $\Sigma$ , and all  $f : \alpha \rightarrow \gamma$  in  $\mathcal{C}$ ,  $f$  factors through  $\sigma$ , that is, there exists  $h : \beta \rightarrow \gamma$  such that the following diagram commutes



## Past WE and WE

~~$\Sigma$  is a subcategory of  $p$ -WE if  $\Sigma^{op}$  is a subcategory of future weak equivalences in  $\mathcal{C}^{op}$ .~~

$\Sigma$  is a subcategory of weak equivalences (of WE in short) if it is both a subcategory of past and of future weak equivalences. This is the same as asking the following:

- (1)  $\Sigma$  contains all right and left retracts of  $\mathcal{C}$
- (2) all  $\sigma$  in  $\Sigma$  are monos and epis
- (3)  $\Sigma$  is stable under pushouts and pullbacks (with any morphism in  $\mathcal{C}$ )
- (4) If there is a  $h : \beta \rightarrow \gamma$ , then any  $f : \alpha \rightarrow \gamma$  in  $\mathcal{C}$  factors through any  $\sigma : \alpha \rightarrow \beta$  and dually, any  $g : \beta \rightarrow \alpha$  in  $\mathcal{C}$  factors through any  $\tau : \gamma \rightarrow \alpha$  in  $\Sigma$



## Existence?

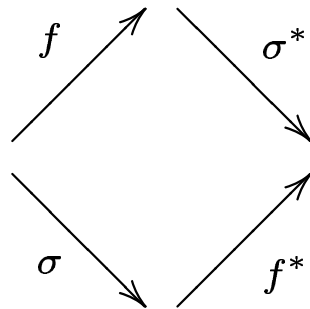
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- Suppose that in  $\mathcal{C}$ , all right (resp. left) retracts are epis (resp. monos). Then there exists a subcategory of future (resp. past) WE.
- If  $\mathcal{C}$  is such that all left retracts and all right retracts are isos then there exists a subcategory of WE.

## Proof

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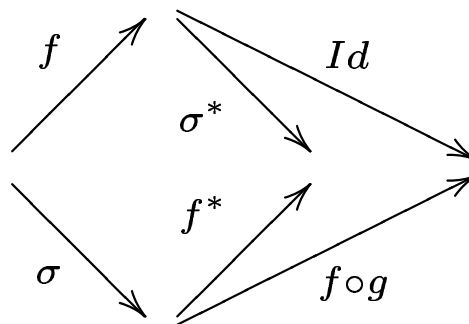
- The subset of all right (resp. left) retracts form a subcategory of  $\mathcal{C}$ .
- We have supposed that all right retracts are epis.
- Consider the following commutative diagram, where we have taken the pushout of  $\sigma$ , a right retract, with any  $f$  in  $\mathcal{C}$ :



## Proof (continued)

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Now consider the diagram:

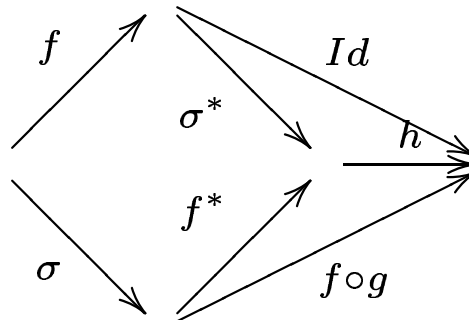


where  $g$  is such that  $g \circ \sigma = Id$ .

## Proof (continued)

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Thus  $f \circ g \circ \sigma = f$  which is equal to  $Id \circ f$ . Hence by the universal property of the pushout, there exists  $h$  such that the following diagram commutes:

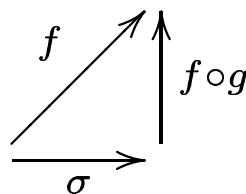


which means in particular that  $h \circ \sigma^* = Id$ , i.e.,  $\sigma^*$  is a right retract.

## Proof (continued)

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Suppose we have  $h : \beta \rightarrow \gamma$ ,  $f : \alpha \rightarrow \gamma$  and  $\sigma$ , a right retract from  $\alpha$  to  $\beta$ . Then there is a  $g$  with  $g \circ \sigma = Id$ , therefore, the following diagram is commutative:



## Examples

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- $\mathcal{C} = \vec{\pi}_1(X)$  where  $X$  is a po-space. Then  $h \circ \sigma = Id$  implies  $h = Id$  and  $\sigma = Id$ , implying the condition of existence of  $f$ -WE,  $p$ -WE and WE.
- A topos has WE iff left retract and right retracts are isos. In that case, WE=isos.
- ARS (Paul-André)...

## ARS

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- 2-dimensional rewriting system with “shape axiom” which should ensure the representation as a nice 2-dimensional cubical set
- The Levy permutation equivalence is exactly combinatorial dihomotopy
- In Paul-André’s residual theory, the fundamental category of this cubical set has all pushouts, and all morphisms are epis
- What about the extension property?

I’d like  $f$ -WE to exist and to have the entire fundamental category to be a  $f$ -WE (“sequential”)

## Connection to WI

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We say that a morphism  $\sigma : x \rightarrow y$  in  $\mathcal{C}$  is *weakly invertible* on the left (respectively on the right) if for all objects  $z$ ,  $\mathcal{Y}_{\mathcal{C}}(\sigma)$  (respectively  $\mathcal{Y}_{\mathcal{C}^{op}}(\sigma)$ ) is a natural isomorphism when restricted to  $\mathcal{C}_{\rightarrow x}$  (respectively on  $\mathcal{C}_{\rightarrow y}^{op} = \mathcal{C}_{y \rightarrow}$ ).

We say that  $\sigma$  is weakly invertible if  $\sigma$  is weakly invertible both on the left and on the right<sup>a</sup>.

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<sup>a</sup>The fact, that we look only at *restrictions* of the Yoneda functor on  $\mathcal{C}_{\rightarrow x}$  and  $\mathcal{C}_{y \rightarrow}$  is of primary importance: otherwise we would define the weakly invertible morphism to be the isomorphisms in the *original* category, by Yoneda's Lemma.

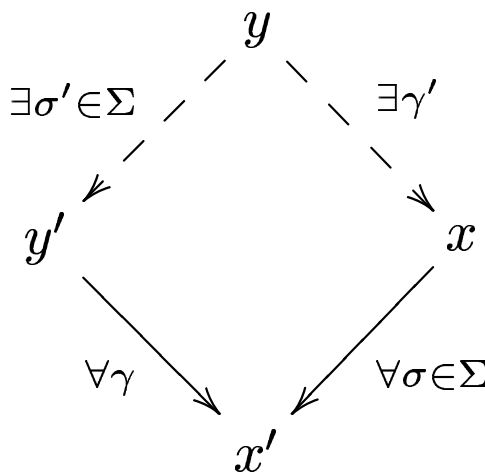


## Connection to left and right calculi

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Let  $\mathcal{C}$  be a category. A subcategory  $\Sigma$  in  $\mathcal{C}$  is said to *admit a right calculus of fractions* (for short: is an *r*-system) if it satisfies

- (iii)  $\forall \gamma : y' \longrightarrow x', \forall \sigma : x \longrightarrow x' \in \Sigma, \exists \sigma' : y \longrightarrow y', \exists \gamma' : y \longrightarrow x$  such that  $\sigma \circ \gamma' = \gamma \circ \sigma'$ , i.e. the following diagram is commutative:



## Connection to left and right calculi of fractions

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- (iv)  $\forall \gamma_1, \gamma_2 : x \longrightarrow y, \forall \sigma : y \longrightarrow y' \in \Sigma$  such that  $\sigma \circ \gamma_1 = \sigma \circ \gamma_2$ ,  
 $\exists \sigma' : x' \longrightarrow x \in \Sigma$  such that  $\gamma_1 \circ \sigma' = \gamma_2 \circ \sigma'$

$$x' \xrightarrow[\exists \sigma' \in \Sigma_X]{\exists \sigma' \in \Sigma_X} x \begin{array}{c} \xrightarrow{\forall \gamma_2} \\ \xrightarrow{\forall \gamma_1} \end{array} y \xrightarrow{\forall \sigma \in \Sigma_X} y'$$

## Connection to “purity”

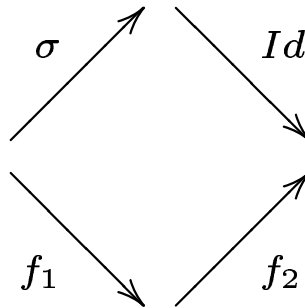
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Let  $\Sigma$  be a subcategory of WE in  $\mathcal{C}$ , then  $\Sigma$  is pure (as well as provides a left and right calculus of fractions of weak invertibles)

## Proof

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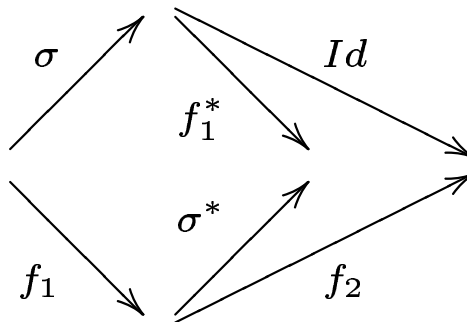
Suppose  $\sigma$  in  $\Sigma$  is equal to  $f_2 \circ f_1$  with  $f_1$  and  $f_2$  in  $\mathcal{C}$ . We rewrite this equality as the commutative diagram



## Proof (continued)

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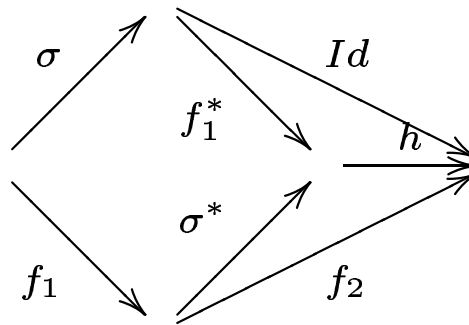
As  $\sigma$  is in  $\Sigma$ , we have a pushout between  $\sigma$  and  $f_1$ , hence we have the commutative diagram



## Proof (continued)

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Hence, by the universality of the pushout, we have a unique  $h$  in  $\mathcal{C}$  such that the diagram below commutes

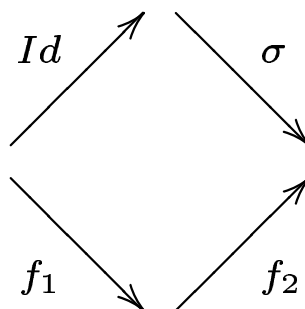


This implies that  $Id = h \circ f_1^*$ . Hence  $h$  is a left inverse in  $\mathcal{C}$  hence is in  $\Sigma$ . This also implies that  $f_2 = h \circ \sigma^*$ , a composite of two elements of  $\Sigma$ , hence is in  $\Sigma$ .

## Proof (continued)

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Now we can rewrite  $\sigma = f_2 \circ f_1$  as the commutative diagram



and use the co-universal property of the pullback of  $\sigma$  along  $f_2$ , to conclude in the same manner that  $f_1$  is in  $\Sigma$ .

## Theorem

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Let  $\mathcal{C}$  be a category such that all left retracts (resp. right retracts) are isomorphisms. There exists a maximal subcategory of WE in  $\mathcal{C}$ .



## Sketch of proof

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There exists one, as we saw earlier.

Now, it suffices to show that if  $\tau$  is a morphism of  $\mathcal{C}$  satisfying all axioms of a subcategory of WE (as the category generated by  $\tau$ ) and if  $\Sigma$  is a subcategory of WE, then all composites  $\tau \circ \sigma$  and  $\sigma \circ \tau$ , together with  $\Sigma$ , is a subcategory of WE.

By induction, we see that being WE is inductive. Using Zorn's lemma, we find the maximal WE.

## In a restricted case

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- Only in the “mutual exclusion model”: i.e. po-spaces are  $I^n$  minus isothetic hyperrectangles
- We are trying to see what we would like to prove about components in these systems...

## Two nice (and strong) properties

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Let  $X_1$  and  $X_2$  be two po-spaces, and  $X = X_1 \cap X_2$ . We call  $\sim_1$  (resp.  $\sim_2, \sim$ ) the dihomotopy equivalence of dipaths in  $X_1$  (resp.  $X_2, X$ ). The inclusion maps  $i_1 : X \rightarrow X_1$  and  $i_2 : X \rightarrow X_2$  induce maps  $i_1^*$  and  $i_2^*$  between the respective fundamental categories.

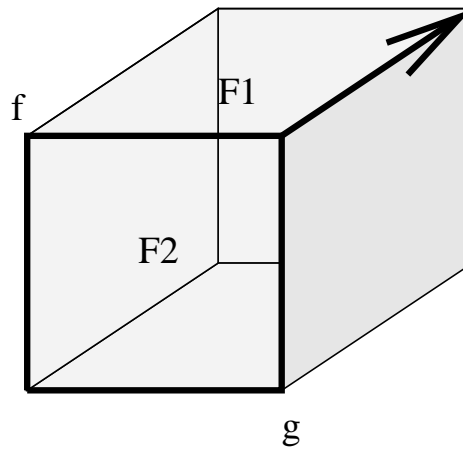
- (P1):  $f \sim g$  if and only if  $i_1^*(f) = i_1^*(g)$  and  $i_2^*(f) = i_2^*(g)$ .
- (P2): for all  $u, v, u', v'$  in  $X$ , for all  $f_1, g_1$  in  $X_1$ , for all  $f_2, g_2$  in  $X_2$ ,  $u' \circ f_1 \circ u \sim_1 v' \circ g_1 \circ v$  and  $u' \circ f_2 \circ u \sim_2 v' \circ g_2 \circ v$  implies that there exist  $f$  and  $g$  in  $X$  such that  $f_1 \sim_1 f$ ,  $f_2 \sim_2 f$ ,  $g_1 \sim_1 g$  and  $g_2 \sim g$ .

## Conjecture

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(P1) and (P2) hold for “PV-systems”

Certainly not for even geometric cubical sets:



## Proposition

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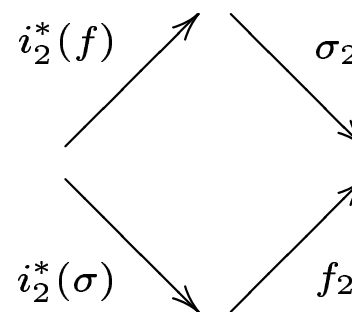
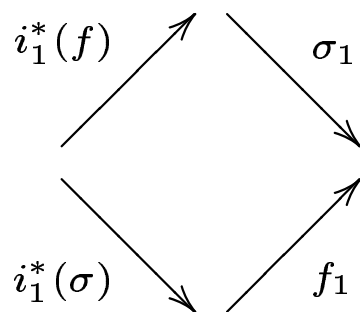
Suppose we have (P1), (P2).

Then  $i_1^*(\sigma) \in \Sigma_1$  (the maximal WE for  $X_1$ ) and  $i_2^*(\sigma) \in \Sigma_2$  (the maximal WE for  $X_2$ ) implies that  $\sigma \in \Sigma$  (the maximal WE for  $X$ ).

## Sketch of proof

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We prove that the pushout of  $\sigma$  with any  $f$  in  $\vec{\pi}_1(X)$  exists in  $\vec{\pi}_1(X)$ . First, we take the pushout in  $\vec{\pi}_1(X_1)$  (resp. in  $\vec{\pi}_1(X_2)$ ) of  $i_1^*(f)$  with  $i_1^*(\sigma)$  (resp. of  $i_2^*(f)$  with  $i_2^*(\sigma)$ ), thus we have commutative diagrams, with  $f_1$  and  $\sigma_1$  in  $\vec{\pi}_1(X_1)$  (resp. with  $f_2$  and  $\sigma_2$  in  $\vec{\pi}_1(X_2)$ ):



## Proof (continued)

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This means that we can apply (P2) with  $u' = Id$ ,  $v' = Id$ ,  $u = f$ ,  $v = \sigma$  and  $f_1 = \sigma_1$ ,  $g_1 = f_1$ ,  $f_2 = \sigma_2$  and  $g_2 = f_2$ . Hence we have  $f^*$  and  $\sigma^*$  in  $\vec{\pi}_1(X)$  with  $f^* \sim_1 f_1$ ,  $f^* \sim_2 f_2$  and  $\sigma^* \sim_1 \sigma_1$ ,  $\sigma^* \sim_2 \sigma_2$ . This implies that we have the two following commutative diagrams, in  $\vec{\pi}_1(X_1)$  (resp. in  $\vec{\pi}_1(X_2)$ ):

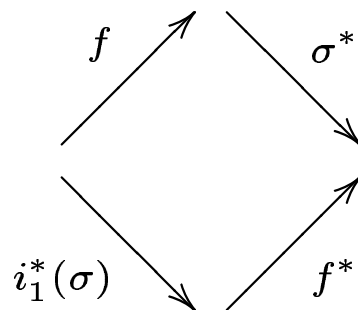
$$\begin{array}{ccc} & i_1^*(f) & \\ \nearrow & & \searrow \\ & i_1^*(\sigma) & \\ & \nearrow & \searrow \\ & i_1^*(f^*) & \end{array}$$

$$\begin{array}{ccc} & i_2^*(f) & \\ \nearrow & & \searrow \\ & i_2^*(\sigma) & \\ & \nearrow & \searrow \\ & i_2(f^*) & \end{array}$$

## Proof (continued)

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By (P1) this implies that the following is also a commutative diagram, in  $\vec{\pi}_1(X)$ :

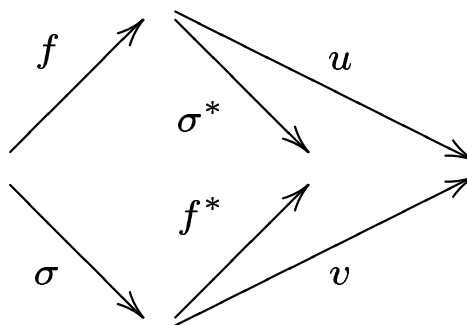




## Proof (continued)

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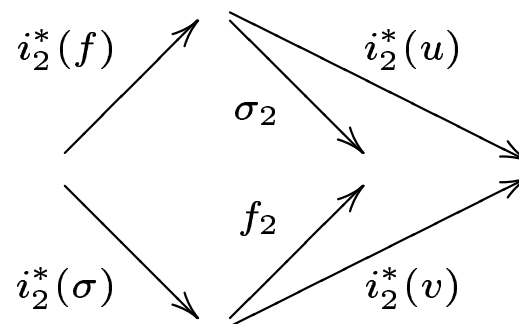
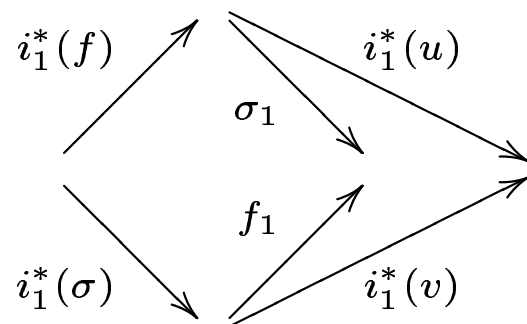
Now, let  $u, v$  in  $\vec{\pi}_1(X)$  such that the following diagram commutes:



## Proof (continued)

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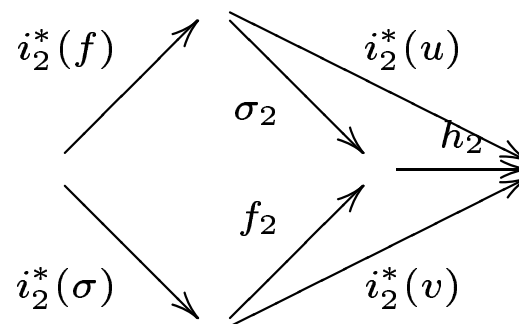
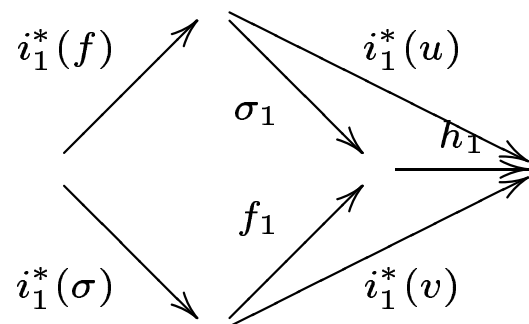
Applying functors  $i_1^*$  and  $i_2^*$  to this diagram gives us two commutative diagrams:



## Proof (continued)

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Hence, we have  $h_1$  in  $\vec{\pi}_1(X_1)$  (resp.  $h_2$  in  $\vec{\pi}_1(X_2)$ ) such that the diagrams below commute:



## Proof (continued)

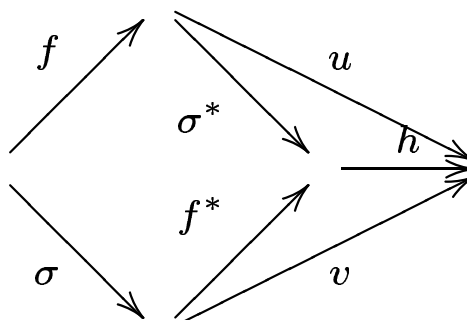
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We apply (P2) with  $u' = Id$ ,  $v' = Id$ ,  $u = f^*$  (which has the same dihomotopy class as  $f_1$  in  $\vec{\pi}_1(X_1)$  and as  $f_2$  in  $\vec{\pi}_1(X_2)$ ),  $v$  being  $v$ ,  $f_1 = h_1$  and  $f_2 = h_2$ . Hence we find a  $h$  in  $\vec{\pi}_1(X)$  with  $h \sim_1 h_1$  and  $h \sim_2 h_2$ . This means that the two diagrams below commute:



## Proof (continued)

Looking at the three equations in  $\vec{\pi}_1(X_1)$ , pairing them with the corresponding three equations in  $\vec{\pi}_1(X_2)$  and applying (P1) shows that the following diagram in  $\vec{\pi}_1(X_1)$  commutes:



What remains to be proven is that this  $h$  is unique. Suppose we have another such  $h$ , we call  $h'$ . We look at the corresponding diagrams, images by  $i_1^*$  and  $i_2^*$ . By the pushout properties in  $\vec{\pi}_1(X_1)$  and  $\vec{\pi}_1(X_2)$  we find that  $i_1^*(h') = i_1^*(h)$  and  $i_2^*(h') = i_2^*(h)$ . By application of (P1) we find  $h = h'$ .

## A theorem, as a consequence

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The component categories in the PV case are in fact “2-dimensional precubical sets” in the following sense:

## Definition

Let  $\Upsilon_2$  be the category of 2-dimensional precubical sets and  $\mathcal{C}$  be the category of categories. There is a functor:

$$F : \Upsilon_2 \rightarrow \mathcal{C}$$

defined by  $F(X) = C$  with:

- let  $D$  be the free category generated by the graph  $(X_0, X_1)$  underlying the 2-dimensional precubical set  $X$ .
- $C$  is the quotient of  $D$  by relations:

$$[d_0^1(A)] \circ [d_1^0(A)] = [d_1^1(A)] \circ [d_0^0(A)]$$

where  $A \in X_2$  is a 2-cell and  $[x]$  (where  $x \in X_1$ ) denotes the corresponding morphism in  $D$ .

## A duality principle?

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It is interesting to notice that in all cases above of component categories of po-spaces  $U \subseteq [0, 1]^n$ , the corresponding 2-dimensional precubical sets  $X$  are such that:

- $X_0$  is the set of components (objects in the category of components of  $U$ ), i.e. contains regions in  $[0, 1]^n$ ; in fact all the topological closure of these regions are traces of isothetic (closed) hypercubes of dimension  $n$  on  $U$ ,



## A duality principle?

- 
- $X_1$  contains unique morphisms from a component (in  $X_0$ ) to a neighbouring component, and we identify them with the intersection of the closures of these components. These are hyperplanes (isothetic hyperrectangles of dimension  $n - 1$ ), and in some sense, the morphisms of the component category are “orthogonal” to these separating hyperplanes.
  - $X_2$  is the set of “(cubical) relations”. Having a relation implies having as boundaries 4 morphisms which commute. Geometrically, this implies that two orthogonal hyperplanes intersect. We identify a relation with this intersection, which is geometrically an isothetic hyperrectangle of dimension  $n - 2$  (we will call this in the following a hyperline).

This can be seen as a duality principle.

## Inductive computations of components

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- we have the component category of such a po-space  $U$  which is generated by a 2-dimensional precubical set  $(X_0, X_1, X_2, d^0, d^1)$ ,
- we want to compute the component category of  $U \setminus R \subseteq [0, 1]^n$ , where  $R$  is an isothetic hyperrectangle.

## Inductive computations of components

---

- We already know that the component category of  $[0, 1]^n \setminus R$  is generated by a 2-dimensional precubical set, that we write as  $(Y_0, Y_1, Y_2, \delta^0, \delta^1)$ .
- We now define a new structure  $(Z_0, Z_1, Z_2, \partial^0, \partial^1)$  as follows, which we will prove to be a 2-dimensional precubical set, and which will generate (an “approximation” of) the component category of  $U \setminus R$ :

## Definition of $Z$

---

- $Z_0 = \{A \cap B \mid A \in X_0, B \in Y_0, A \cap B \neq \emptyset\}$
- $Z_1 = \begin{cases} \{A \cap f \mid A \in X_0, B \in Y_1, A \cap f \neq \emptyset\} \\ \cup \{e \cap B \mid e \in X_1, B \in Y_0, e \cap B \neq \emptyset\} \end{cases}$
- $Z_2 = \begin{cases} \{e \cap f \mid e \in X_1, f \in Y_1, e \cap f \neq \emptyset\} \\ \cup \{R \cap B \mid R \in X_2, B \in Y_0, R \cap B \neq \emptyset\} \\ \cup \{A \cap S \mid A \in X_0, S \in Y_2, A \cap S \neq \emptyset\} \end{cases}$

## Boundaries

- 
- $\partial_*^* : Z_1 \rightarrow Z_0$  are defined by:
    - $\partial_0^0(A \cap f) = A \cap \delta_0^0(f),$
    - $\partial_0^1(A \cap f) = A \cap \delta_0^1(f),$
    - $\partial_0^0(e \cap B) = d_0^0(e) \cap B,$
    - $\partial_0^1(e \cap B) = d_0^1(e) \cap B.$
  - $\partial_*^* : Z_2 \rightarrow Z_1$  are defined by:
    - $\partial_0^0(e \cap f) = d_0^0(e) \cap f,$
    - $\partial_1^0(e \cap f) = e \cap \delta_0^0(f),$
    - $\partial_0^1(e \cap f) = d_0^1(e) \cap f,$
    - $\partial_1^1(e \cap f) = e \cap \delta_0^1(f),$
    - $\partial_l^k(R \cap B) = d_l^k(R) \cap B$  (for  $k, l = 0, 1$ ),
    - $\partial_l^k(A \cap S) = A \cap \delta_l^k(S).$

## A lemma

---

$(Z_0, Z_1, Z_2, \partial^0, \partial^1)$  is a 2-dimensional precubical set.

We check the following relations:

- $$\begin{aligned}\partial_0^0(\partial_0^0(e \cap f)) &= \partial_0^0(d_0^0(e) \cap f) \\ &= d_0^0(e) \cap \delta_0^0(f)\end{aligned}$$
- $$\begin{aligned}\partial_0^0(\partial_1^0(e \cap f)) &= \partial_0^0(e \cap \delta_0^0(f)) \\ &= d_0^0(e) \cap \delta_0^0(f) \quad .\end{aligned}$$

## Proof (continued)

---

- This shows that  $\partial_0^0(\partial_0^0(e \cap f)) = \partial_0^0(\partial_1^0(e \cap f))$ .

- $$\begin{aligned}\partial_0^0(\partial_0^1(e \cap f)) &= \partial_0^0(d_0^1(e) \cap f) \\ &= d_0^1(e) \cap \delta_0^0(f)\end{aligned}$$

- $$\begin{aligned}\partial_0^1(\partial_1^0(e \cap f)) &= \partial_0^1(e \cap \delta_0^0(f)) \\ &= d_0^1(e) \cap \delta_0^0(f)\end{aligned}$$

This shows that  $\partial_0^0(\partial_0^1(e \cap f)) = \partial_0^1(\partial_1^0(e \cap f))$ .

etc.

## Simple cases

---

We know from APCS that the po-space and the component category corresponding to the following PV program (where  $a$  is a binary semaphore):

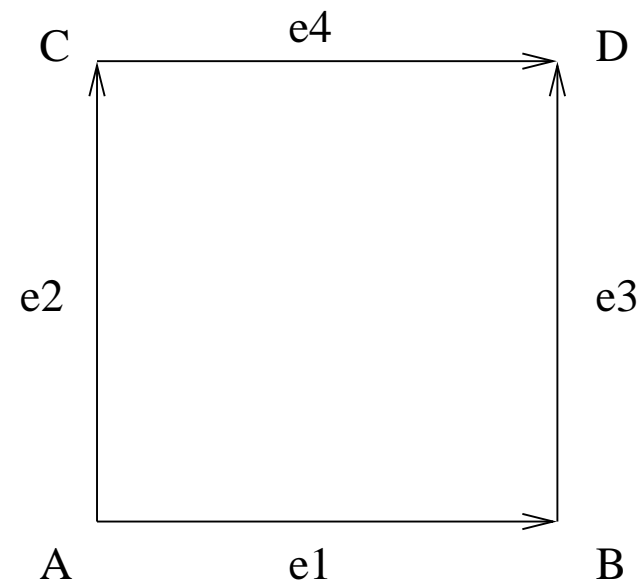
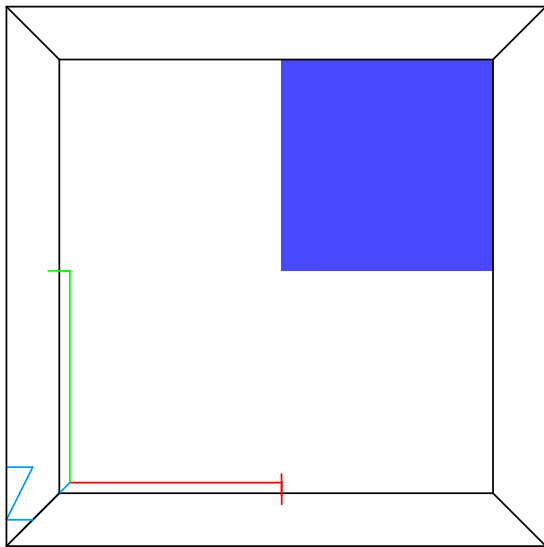
$$A = Pa . Va$$
$$B = Pa . Va$$
$$\text{PROG} = A \mid B$$

are those pictured in next slide (left: po-space; right: component category)



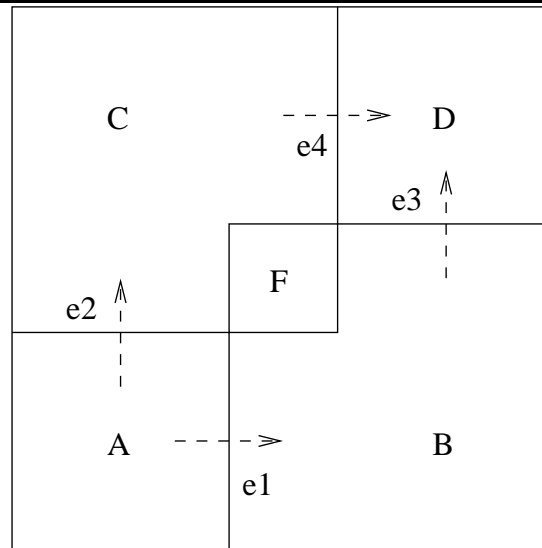
# Examples

---



## The component category, geometrically

---



The idea is that the components  $A$ ,  $B$ ,  $C$  and  $D$  correspond to the squares delineated by the horizontal and vertical lines from the min and max points of the forbidden region  $F$ . In fact, regions,  $B$  and  $C$  are not quite squares, since we should in fact consider  $B \setminus F$  and  $C \setminus F$ , but we will come back to that later.

## The component category, geometrically

---

- Also, the “unique” morphism (see APCS) from  $A$  to  $B$ , which is  $e_1$ , is in fact the “orthogonal” of the vertical line descending from the min point of  $F$ . So we identify  $e_1$  with the codimension 1 isothetic linear variety which is this vertical segment.
- Similarly,  $e_2$  is identified with the horizontal line coming left from the min point of  $F$  etc. There is here no codimension 2 variety of interest here, hence no relation between morphisms.

## Digging a new hole

---

Now, what if we dig in a new hole in the po-space we had before? We get the po-space pictured in next slide and should obtain the component category (where solid squares represent relations) pictured at the right hand side of next slide. This po-space corresponds to the PV program:

$$A = Pa . Va . Pb . Vb$$

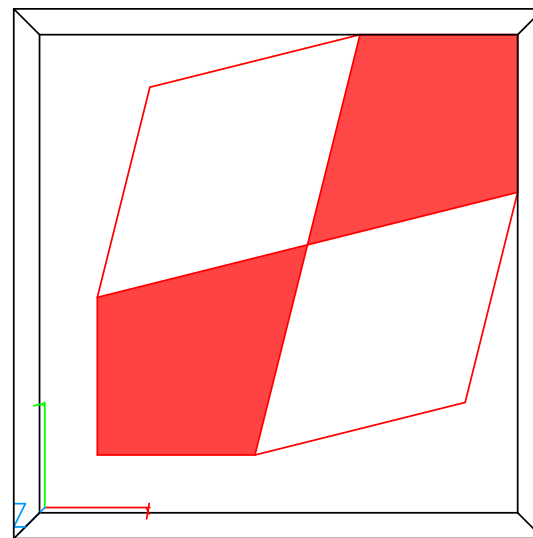
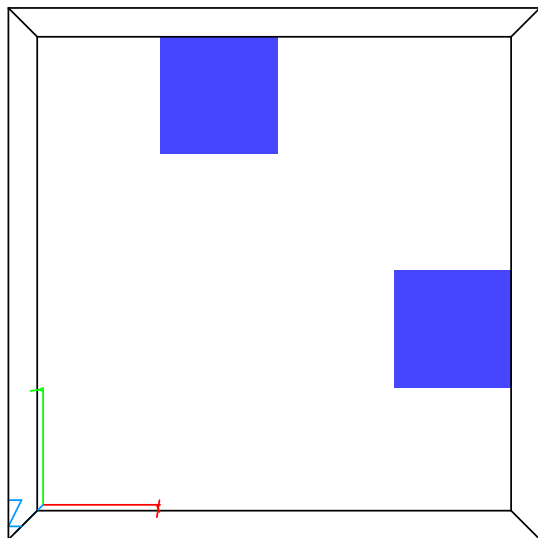
$$B = Pb . Vb . Pa . Va$$

$$PROG = A \mid B$$

And the component category corresponds to the precubical set of dimension 2, pictured “geometrically” in next slide.

# Pictures

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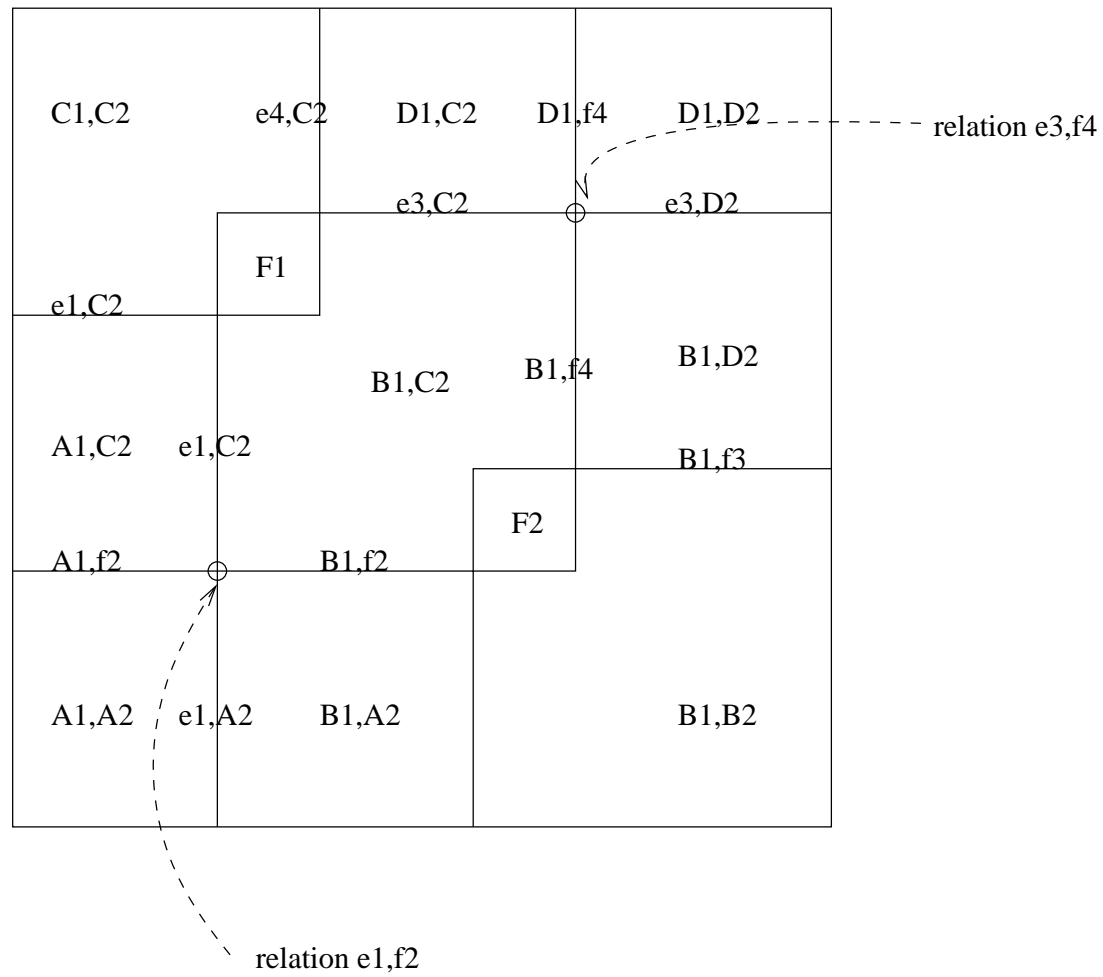


## Using the inductive calculus of components

---

- We just apply the inductive definition of the previous section, where  $F_2$  is the new hole, and where the morphisms of its corresponding category are denoted by  $f_1, f_2, f_3$  and  $f_4$  respectively.
- We see that we have two codimension 2 varieties of interest, namely the two intersections  $e_1 \cap f_2$  and  $e_3 \cap f_4$  which give the two relations, hence the two 2-cells of the component category, pictured in the next slide.

# Picture



## Yet another hole

---

What about digging another hole, inductively (at the top left corner of the picture)? This would correspond to program:

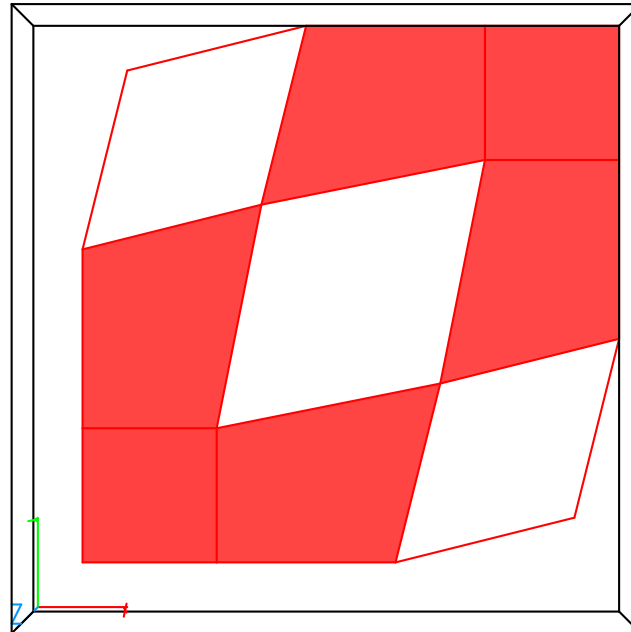
$$A = Pa . Va . Pb . Vb . Pc . Vcg$$
$$B = Pc . Vc . Pb . Vb . Pa . Va$$
$$PROG = A \mid B$$

And would give component category pictured in next slide.



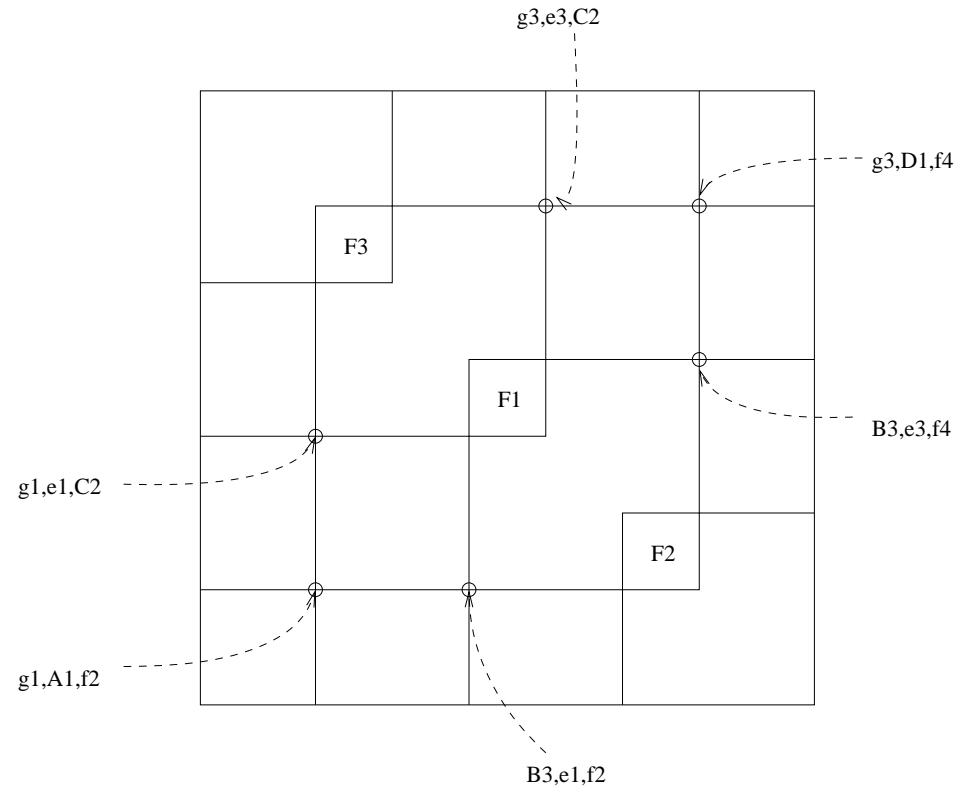
## Component category

---



## “Explanation”

The explanation of the six different relations that are found is given below ( $g_1, g_2, g_3$  and  $g_4$  are the morphisms of the component category of  $F_3$  alone).



## Yet another hole

---

With yet another hole, sideways, we would get left figure in next slide, corresponding to:

$$A = P_a . V_a . P_b . V_b . P_c . V_c . P_d . V_d$$

$$B = P_d . V_d . P_c . V_c . P_b . V_b . P_a . V_a$$

$$\text{PROG} = A \mid B$$

whereas, for two holes in a row, corresponding to:

$$A = P_a . V_a . P_b . V_b$$

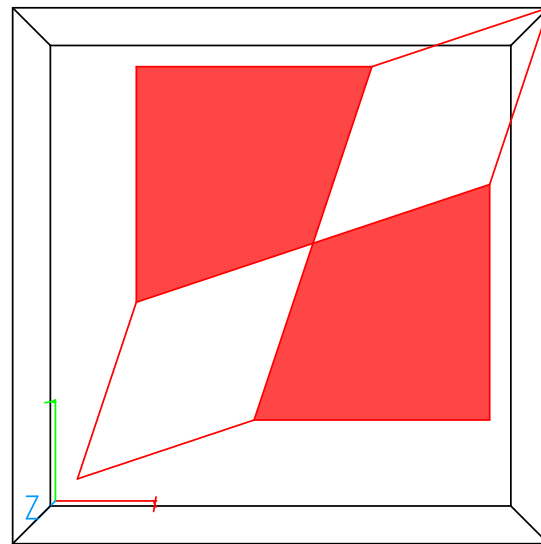
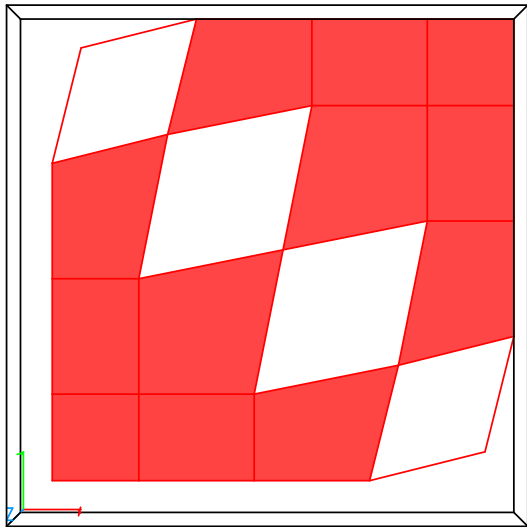
$$B = P_a . V_a . P_b . V_b$$

$$\text{PROG} = A \mid B$$

we would get right figure in next slide.

# Pictures

---



## The 3 philosophers' problem

---

$A = P_a . P_b . V_a . V_b$

$B = P_b . P_c . V_b . V_c$

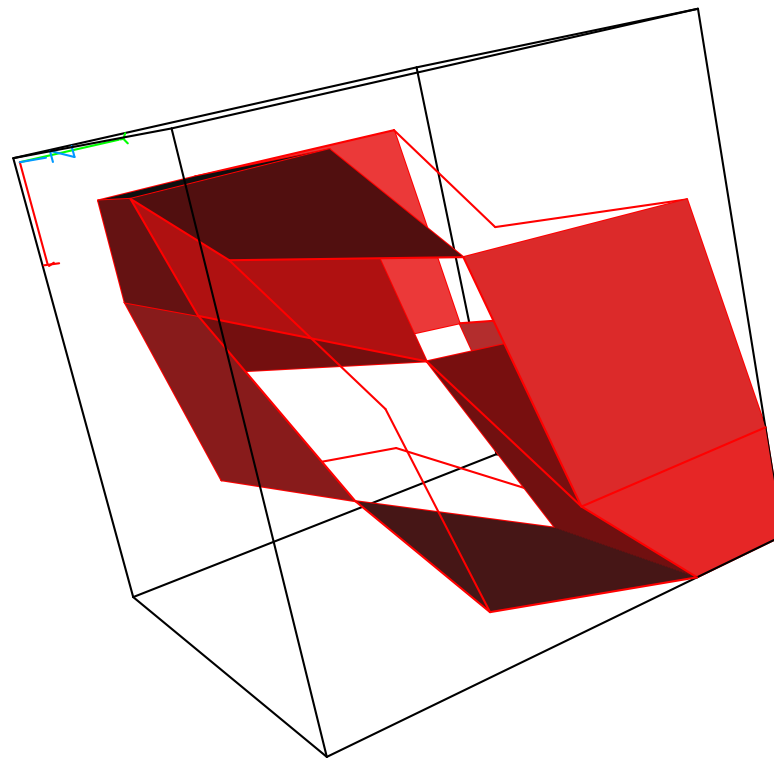
$C = P_c . P_a . V_c . V_a$

$PROG = A \mid B \mid C$

we get the very nice component category pictured in next slide,  
where the central point in fact included in fact the deadlock and the  
unreachable (discussion of this to follow).

# Pictures

---



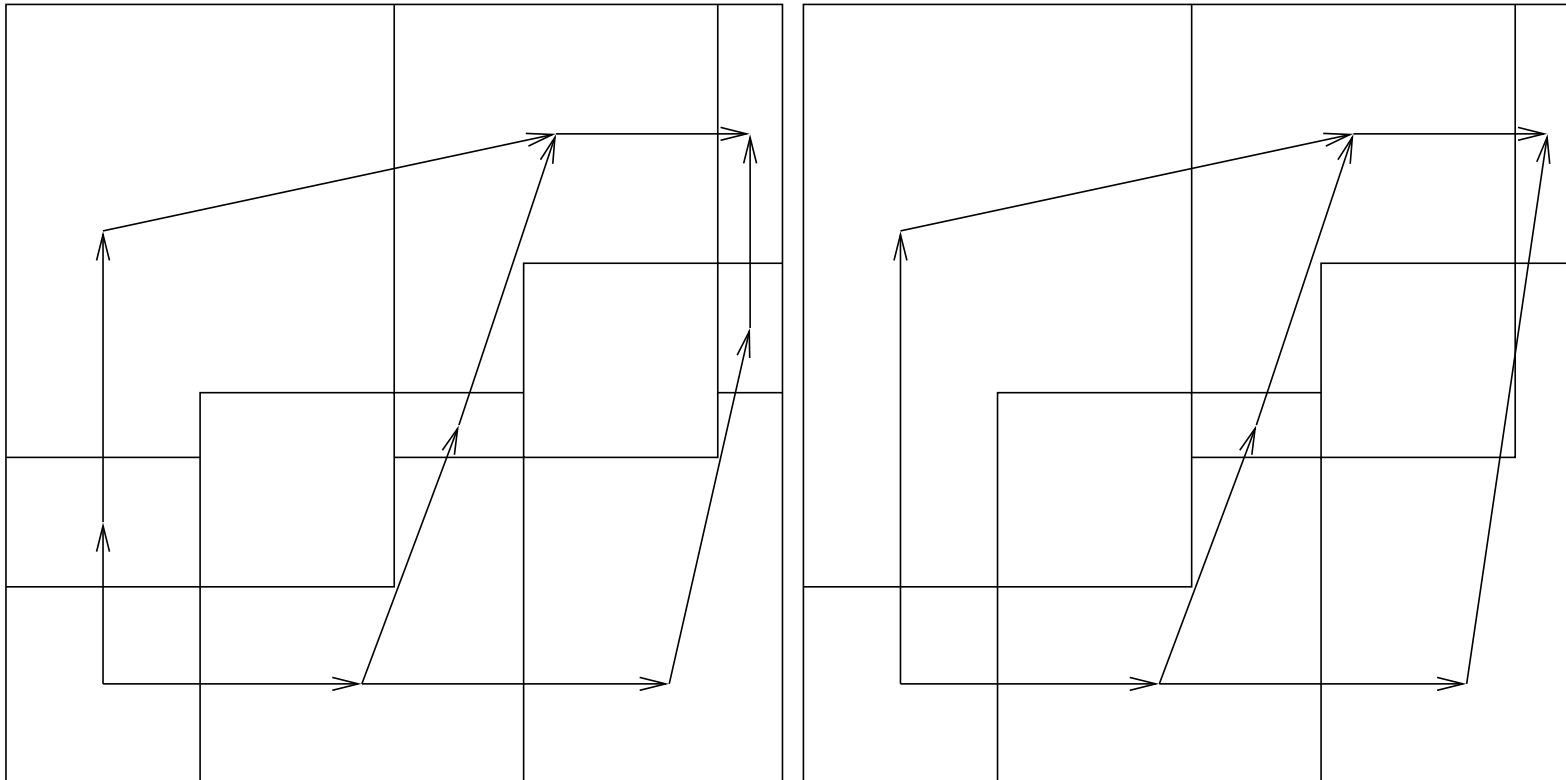
## “Over-approximation”

---

One problem is that we may get more components than there are really... For instance, one could have the following situation, with two holes, where we would find the component category (“geometrically”) pictured in the left hand place of next picture, whereas what we should find should be what is pictured in the right hand place of next picture.

## “Over-approximation”

---



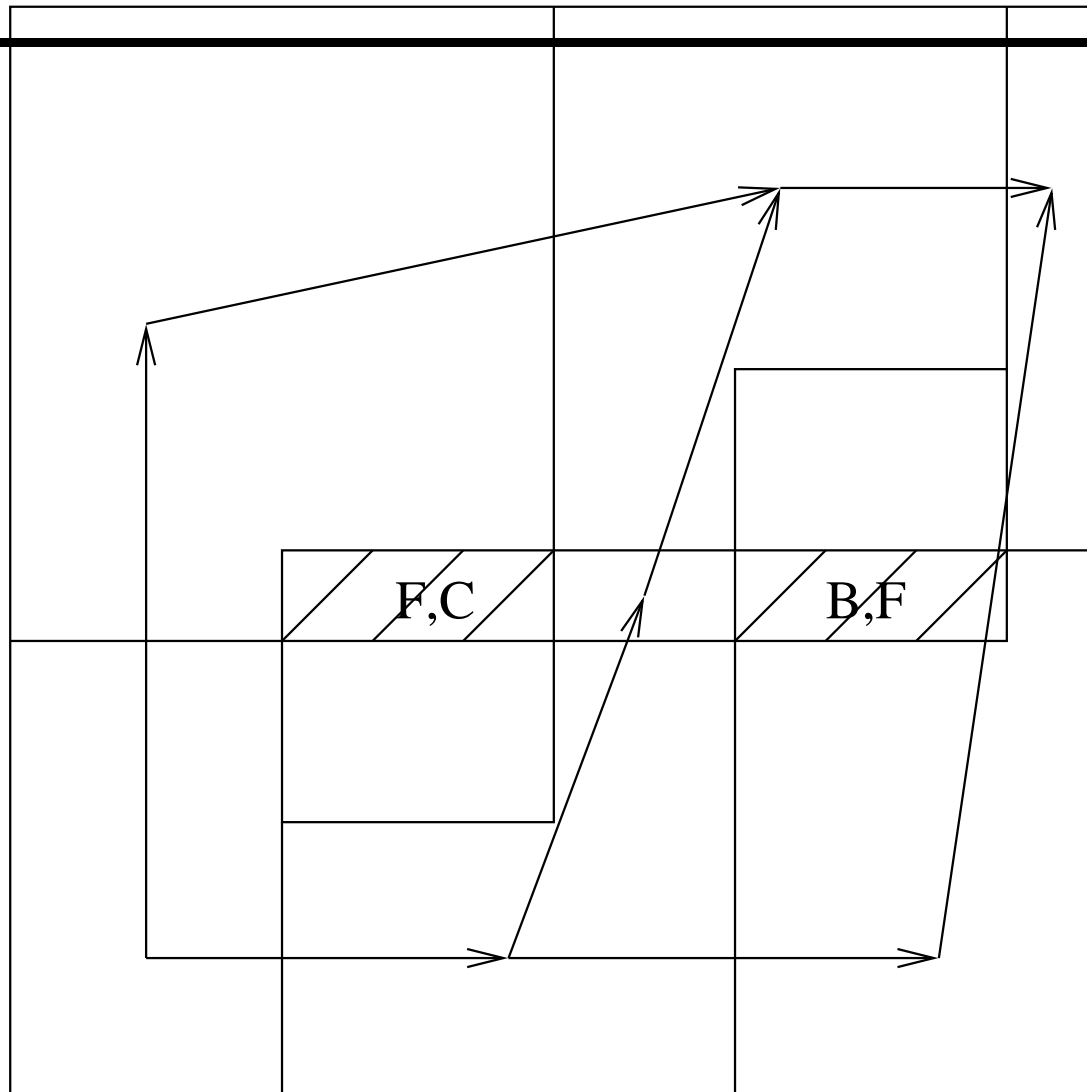


## Reason?

---

- The “reason” I see is that we should only consider the part of the hole which is in the B or C type of region. So we should consider only the dashed part of the holes, as in next picture.
- Ingredient of the proof: the two pictures at the right and at the left have the same component category since there is a dihomeomorphism between them (a retraction of the holes...).

# Pictures



## “Genericity” problem

---

- The problem is then that the 2 holes are not in a “generic” position. For instance we would find an intersection of 2 morphisms, being still of codimension 1.
- This is not a relation (which should be of codimension 2) and should be discarded of the inductive computation of the component category (in some ways it is a degenerate 2-cell, but we are looking only at a precubical formulation).

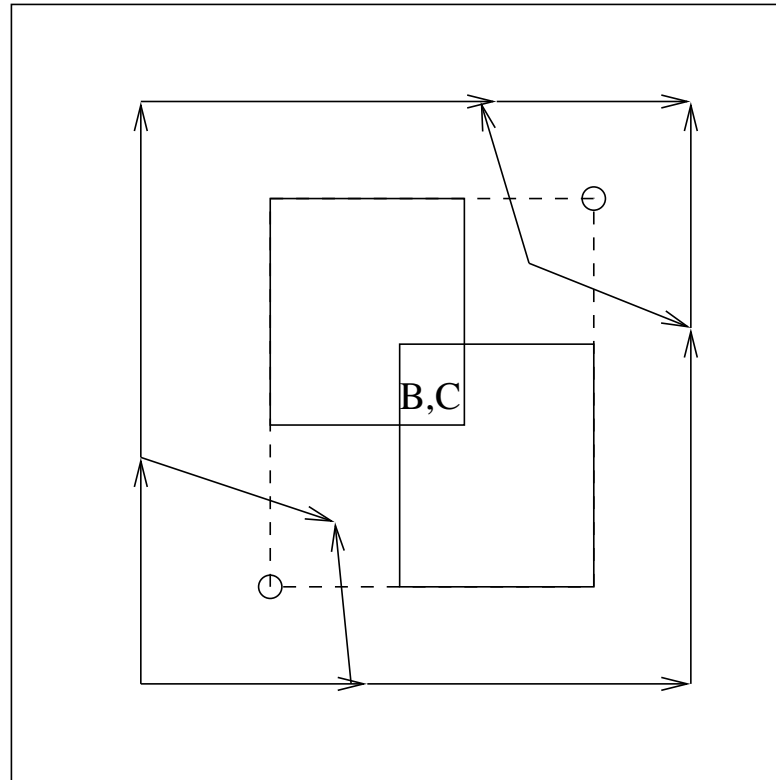
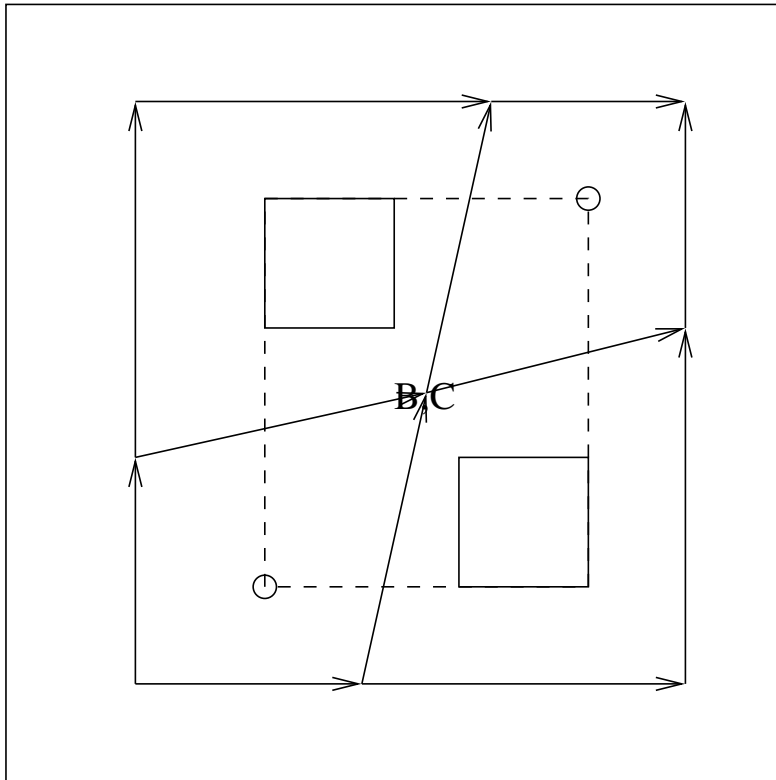
## Deadlocks and Unreachables

---

- The inductive computation we made does not distinguish between the situations left and right of the picture in next slide.
- Whereas the real components categories, pictured as well, are fairly different.

# Pictures

---



## Reason?

---

- Intersections of  $n$  B/C type of components are very peculiar
- They might include unsafe regions (containing their min) and/or unreachable regions (containing their max)
- In that case, we should “split” this kind of components, making two unlinked components...

## More general semaphores

---

I think this behaves accordingly, using the 26 objects component category for instance of the 3-cube minus an inner 3-cube (see APCS). The only trouble is to see what is the B/C type of components... (I think, only the 6 components sideways in each of the 3 directions, glued along a 2-face to the central forbidden region).

## Algorithm for breadth-first traversal of 2-dimensional precubical sets

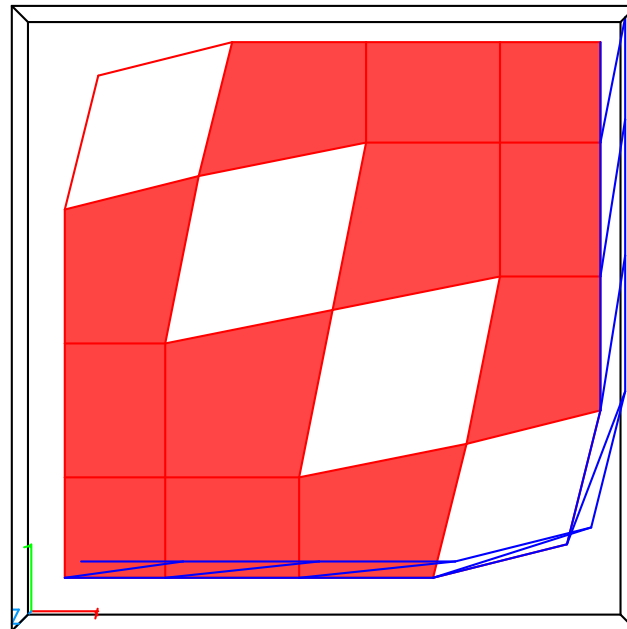
---

- A not very subtle breadth-first traversal modulo 2-cells of such precubical sets has been implemented.
- It is not subtle in the way it is exploring constructing all paths, and then take modulo relations (implemented through “coboundary relations” from 1-cells to 2-cells).
- For instance, the 5 paths modulo dihomotopy found in the 5 next pictures are found among the 70 maximal paths in the component category.



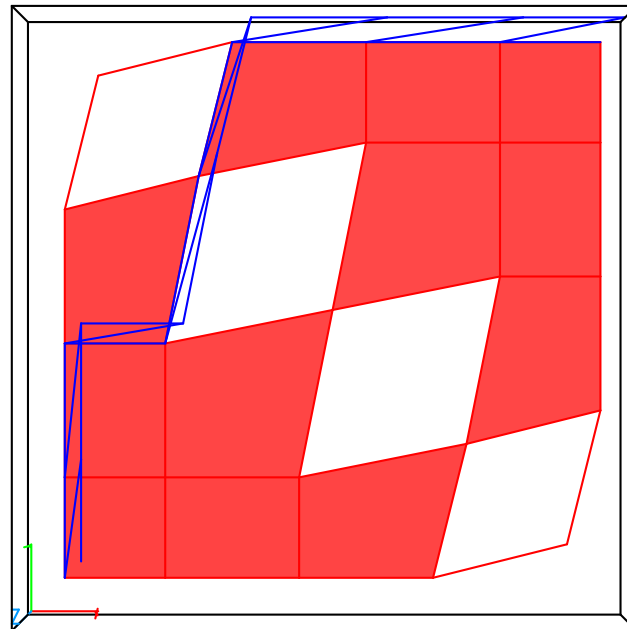
## Path 1

---



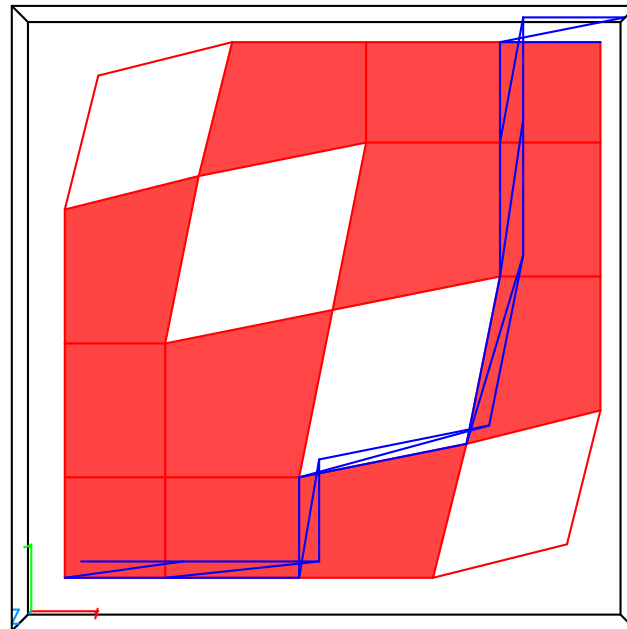
## Path 2

---



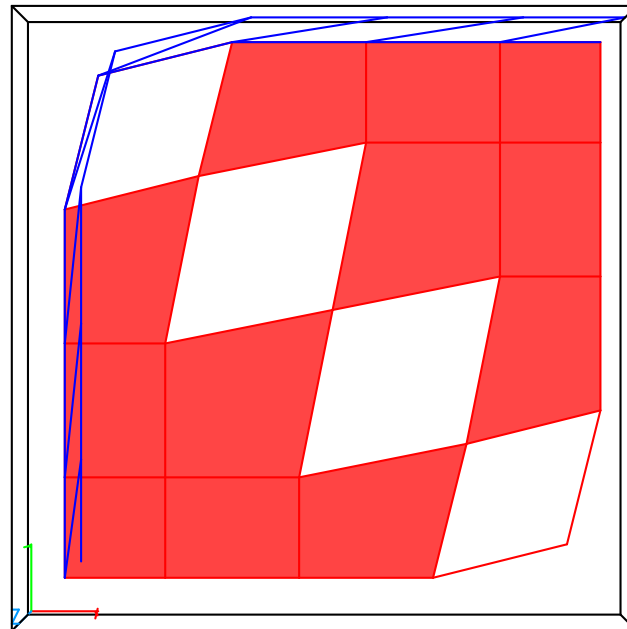
## Path 3

---



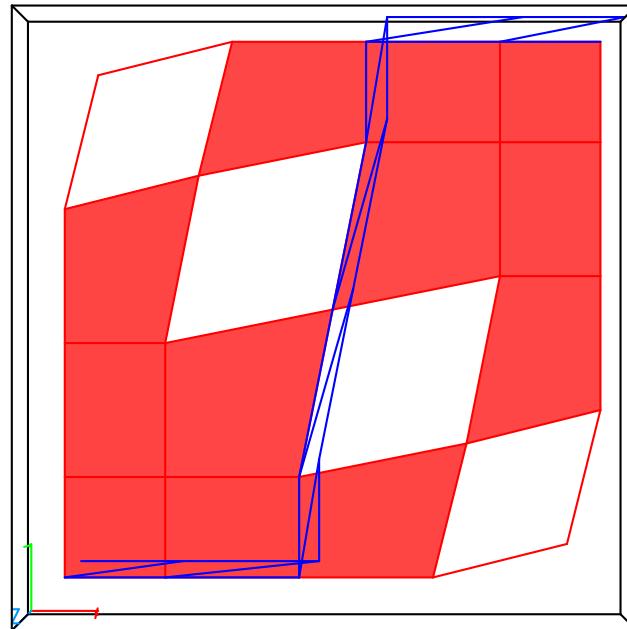
## Path 4

---



## Path 5

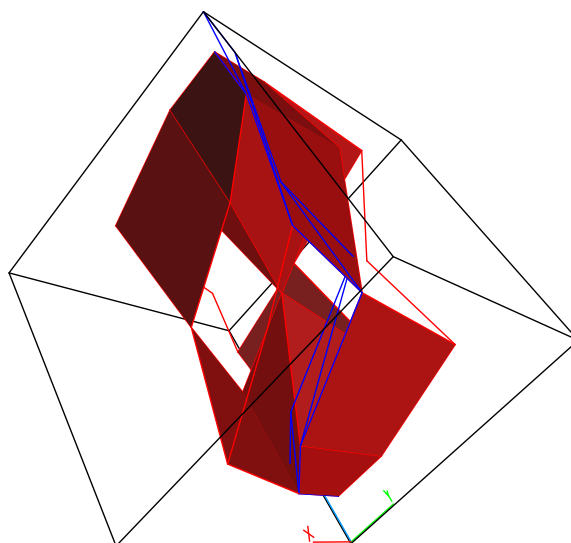
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## Example

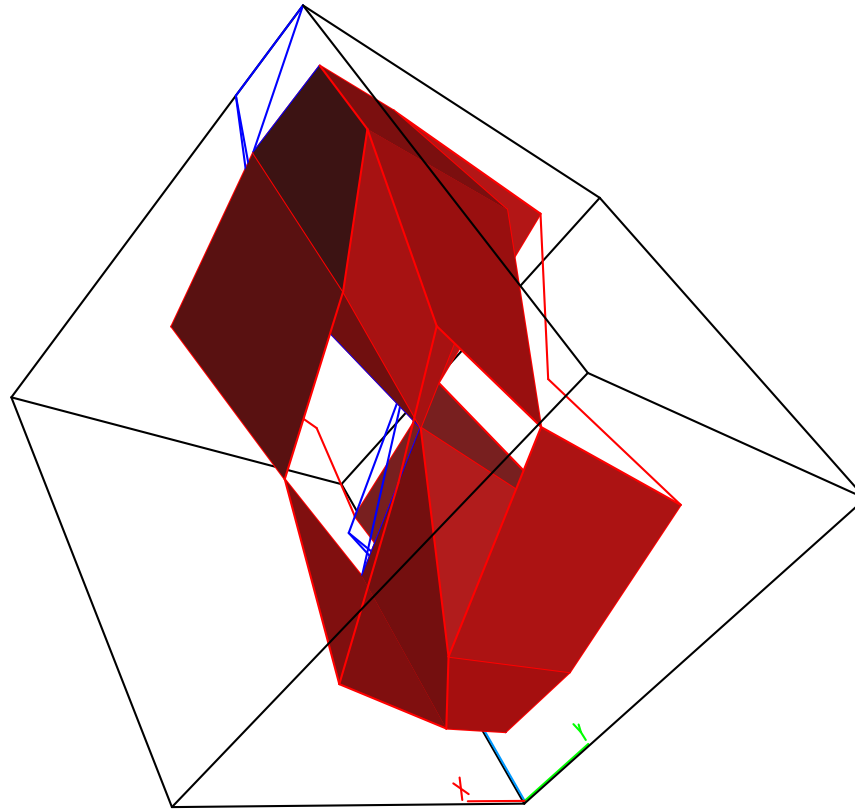
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In the case of the maximal dipaths modulo dihomotopy for the 3 philosophers, we find 7 paths, in the next 7 pictures.



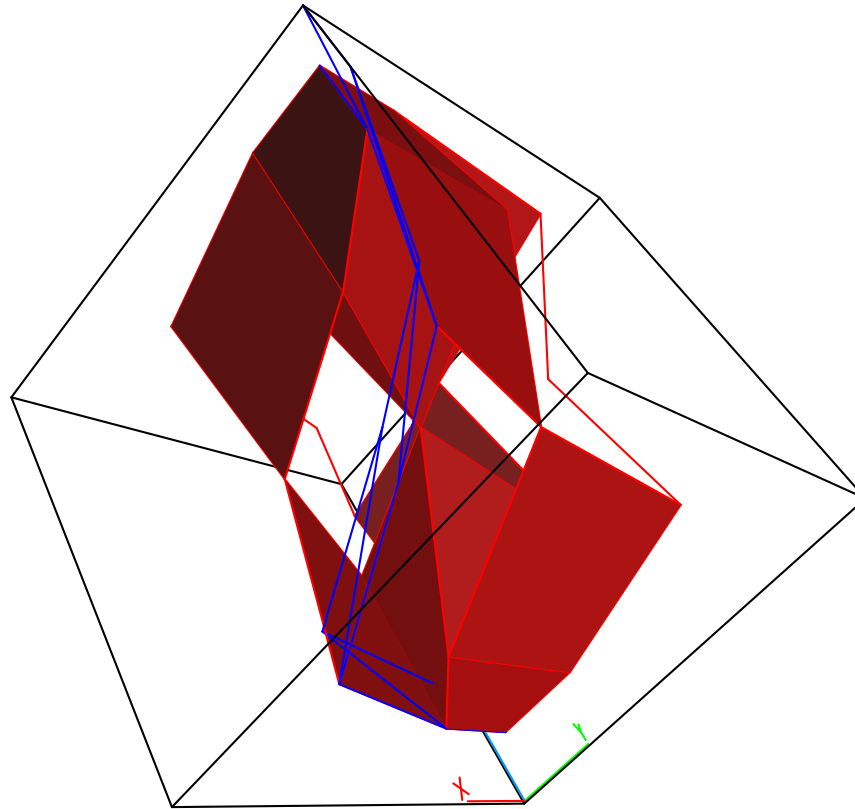
## Path 2

---



## Path 3

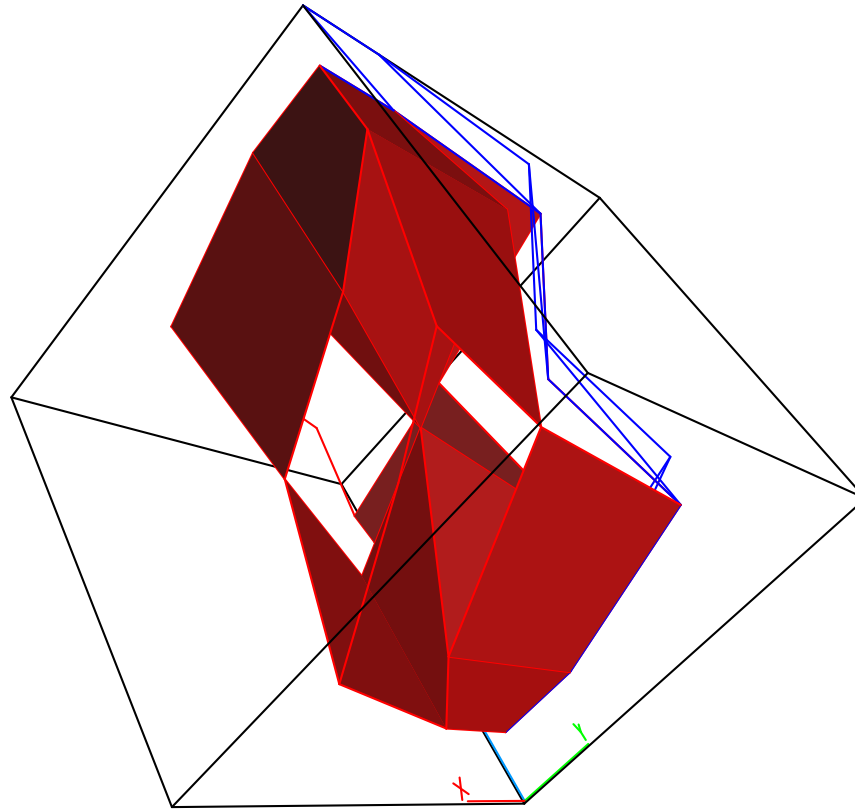
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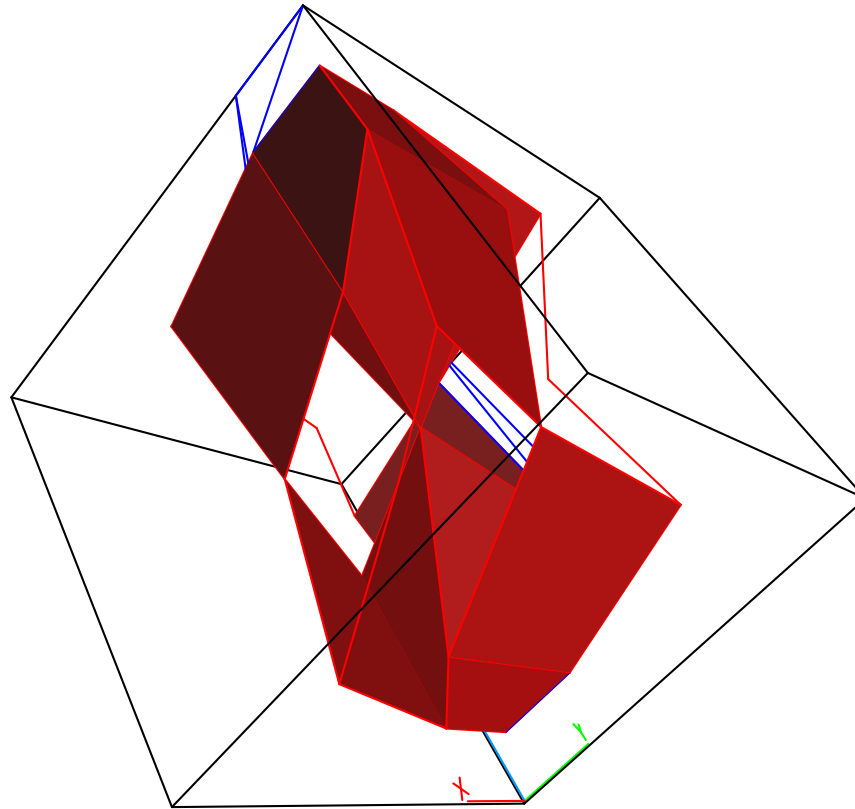
## Path 4

---



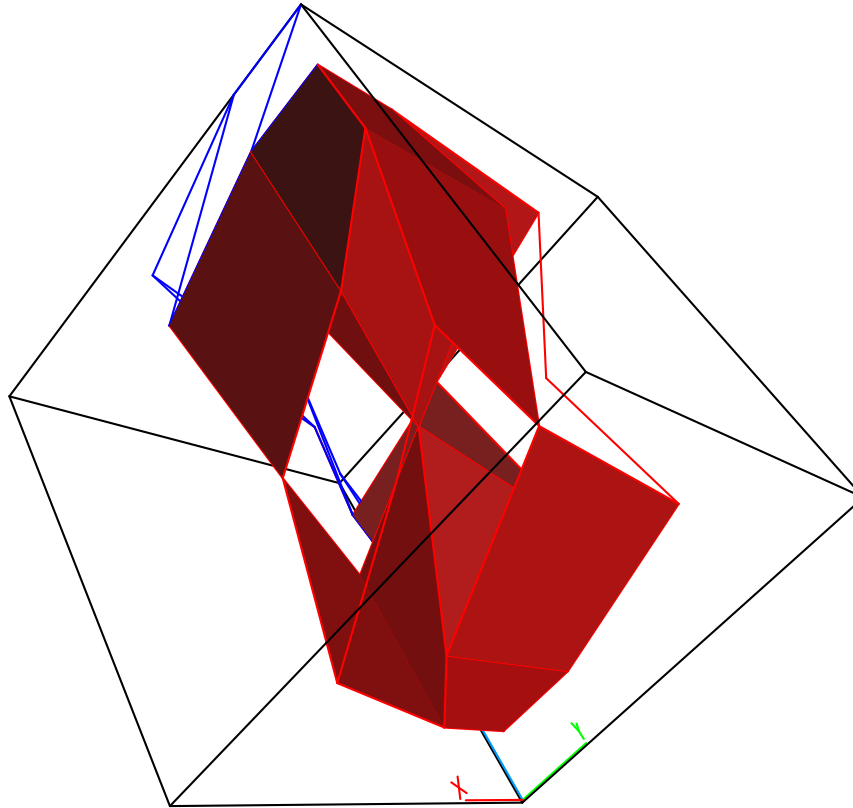
## Path 5

---



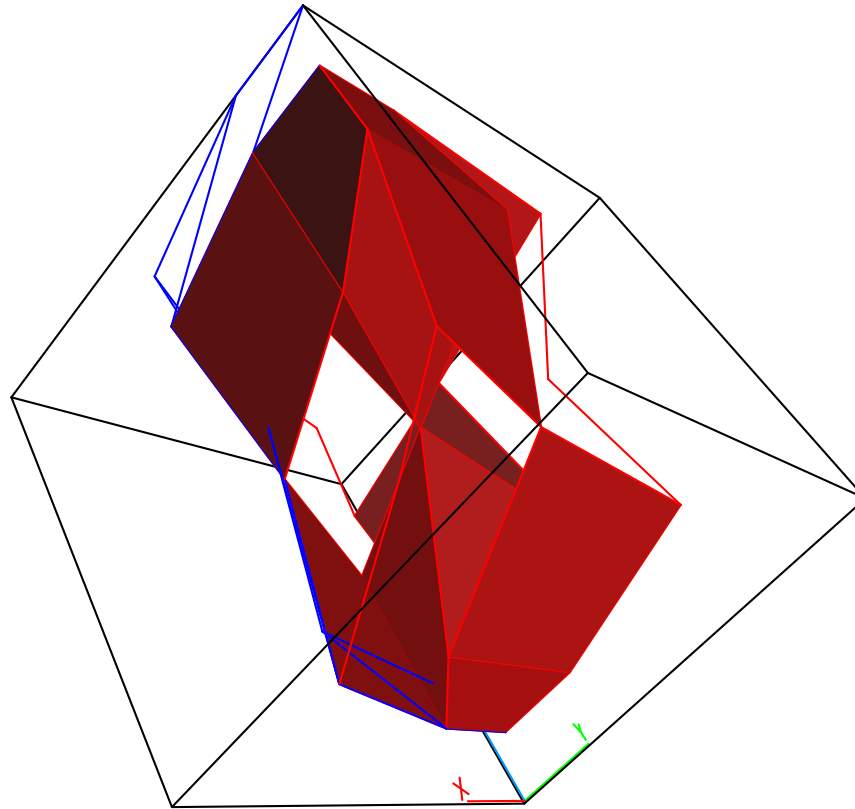
## Path 6

---



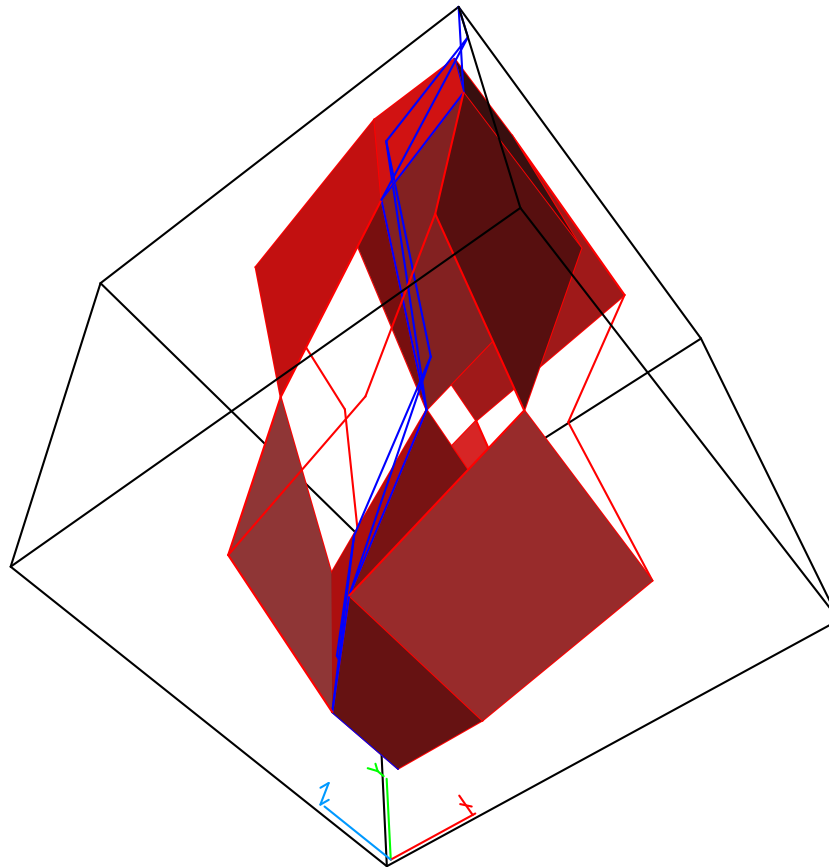
## Path 7

---



## Path 7: another view

---



## Remark

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- The spurious path is number 2,
- and is due to the fact that the intersection of the 3 components of type B/C of each of the 3 forbidden cubes contain a deadlock (containing the min point) and a unreachable region (containing the max point).
- This component should be split in two unlinked components.

## Algorithm for computing a representative of a dipath, corresponding to a given path in the component category

---

- Useful for static analysis of some “sequential” representatives of the concurrent program, that are enough for determining the properties of the program
- Based on:
  - initial point to max vertex of the same component path (using interleaving semantics - in a *disconnected component*!)
  - then vertex to max vertex in each component, until final component

## Example

---

```
> more simple.pv
A=Pa.Va.Pb.Vb
B=Pb.Vb.Pa.Va
PROG=A|B
> essentialpaths simple.pv
(...)
Number of paths=3 among 6
```



## Example

---

```
> more simple.pv_1.sp
{1}P(a);{1}V(a);{1}P(b);{1}V(b);{2}P(b);
  {2}V(b);{2}P(a);{2}V(a);
> more simple.pv_2.sp
{2}P(b);{2}V(b);{2}P(a);{2}V(a);{1}P(a);
  {1}V(a);{1}P(b);{1}V(b);
> more simple.pv_3.sp
{1}P(a);{2}P(b);{1}V(a);{2}V(b);{2}P(a);
  {2}V(a);{1}P(b);{1}V(b);
```

## Some figures...

---

- new3phil.pv: (0.05s) Objects: 27, Morphisms: 48, Relations: 18
- new4phil.pv: (0.07s) Objects: 85, Morphisms: 200, Relations: 132
- new7phil.pv: 147.36s; 81 Mo; (about one million transitions in a standard interleaving model) Objects: 2467, Morphisms: 10094, Relations: 15484
- new8phil.pv: 320.02s; 121Mo; (about 10 million transitions in a standard interleaving model)  
Objects: 3214, Morphisms: 14282, Relations: 24396
- After: not enough memory on my workstation (512Mo)  
[ameliorations...]