Flowering graphs

Interactive proximity test to codes on graphs by flowering

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Roadmap

1 Interactive Oracle Proofs of Proximity

2 Flowering protocol

3 Flowering graphs

- SNARKs
- Interactive Oracle Proof of Proximity
- Fast Reed-Solomon IOPP

Why verifiable computing: a sad story

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- $\mathcal{P}$ is very sad
- #emotion

SNARK means **S**uccinct **N**on-interactive **AR**gument of **K**nowledge.

It turns $\boldsymbol{A : x \mapsto y}$ (that runs in $\tau(|x|)$) into

$$\underbrace{A'}\ : \underbrace{x}_{\text{input}}, \underbrace{\alpha}_{\text{randomness}} \mapsto \underbrace{y}_{\text{output}}, \underbrace{\pi}_{\text{SNARK proof that "}A(x) = y\text{"}}$$

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It turns $A : x \mapsto y$ (that runs in $\tau(|x|)$) into

$$A' : x, \alpha \mapsto y, \pi$$

input, randomness, output, SNARK proof that "$A(x) = y$"

satisfying:

- $|\pi| \ll \tau$
- A' runs in $\tilde{O}(\tau)$
- there is a verifier $\mathcal{V}$ in $\mathrm{poly}(|\pi|)$ such that
  - **Completeness:** if $A(x) = y$ then $\underset{\alpha}{\mathbb{P}}(\mathcal{V}(\pi)$ accepts$) = 1$
  - **Soundness:** if $A(x) \neq y$ then $\underset{\alpha}{\mathbb{P}}(\mathcal{V}(\pi)$ accepts$) \leqslant s$.

- Proof of **Transaction** (Blockchains)
- Proof of **Authenticity** (Signature preservation)
- Proof of **Emulation** (Speedrun)
- Proof of **Training** (AI regulation)

Definition **Relative Hamming distance**

Let $u, v \in \mathbb{F}^N$.

$$\Delta(u, v) := \frac{1}{N} |\{i \in [N] \mid u_i \neq v_i\}|$$

A linear **error correcting code** is a linear subspace of $\mathbb{F}^N$.

Definition **Reed-Solomon codes**

Let $\mathcal{L}_N \subseteq \mathbb{F}$, $|\mathcal{L}_N| = N$ and $K < N$.

$$\mathsf{RS}[\mathcal{L}_N, K] := \{f : \mathcal{L}_N \to \mathbb{F} \mid f \in \mathbb{F}[X]_{\leqslant K-1}\}$$

$\mathsf{RS}[\mathcal{L}_N, K]$ has length N, dimension K and minimal distance $1 - \frac{K+1}{N}$.

Reduction from **checking computation** to **testing proximity** to $\mathsf{RS}[\mathcal{L}_N, K]$

$$\begin{aligned} y = A(x) \quad &\Longrightarrow \quad \mathsf{Arithmetization}(A, x, y) \in \mathsf{RS}[\mathcal{L}_N, K] \\ y \neq A(x) \quad &\Longrightarrow \quad \Delta(\mathsf{Arithmetization}(A, x, y), \mathsf{RS}[\mathcal{L}_N, K]) > \delta \end{aligned}$$

Idea:
Computation = arithmetic circuit = composed polynomials $\to$ Reed-Solomon code

Definition **Locally-testable code**

A code C is **(ℓ, δ, s)-locally-testable** if there is $\mathcal{V}$ **only ℓ accesses** to u such that

- **Completeness:** if $u \in C$ then $\mathbb{P}(\mathcal{V}^u \text{ accepts}) = 1$
- **Soundness:** if $\Delta(u, C) > \delta$ then $\mathbb{P}(\mathcal{V}^u \text{ accepts}) \leqslant s$.

C has **locality ℓ** if C is (ℓ, δ, s)-l.-t. for $s < 1$.

Example **Reed-Solomon codes are not locally-testable**

RS$[\mathcal{L}_N, K]$ doesn't have locality $\ell < K + 1$.

Proof.
Any K values correspond to a degree $\leqslant K - 1$ polynomial by interpolation.

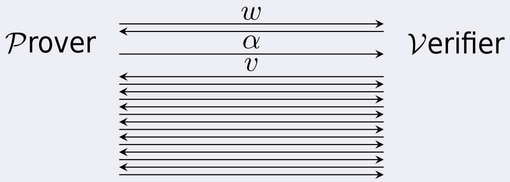
Definition **Non-interactive proof**

$$\mathcal{P}\text{rover} \xrightarrow{\quad w \quad} \mathcal{V}\text{erifier}$$

Definition **Sigma protocol**

$$\mathcal{P}\text{rover} \quad \begin{array}{c} \xrightarrow{\quad\quad w \quad\quad} \\ \xleftarrow{\quad\quad\quad\quad} \\ \alpha \\ \xrightarrow[\quad\quad v \quad\quad]{} \end{array} \quad \mathcal{V}\text{erifier}$$

Definition **Interactive Proof**



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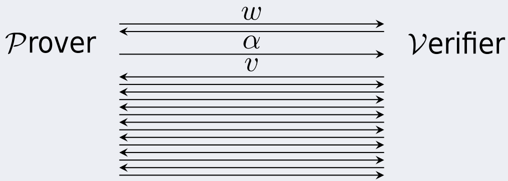


Euh ??? **We are doing SNARKs right?**

This is **not** Succinct Non-interactive.

Definition **Cryptographic hash function**

A cryptographic **hash function** $H : \{0,1\}^* \to \{0,1\}^{|H|}$

- looks **injective**: $\mathcal{P}$ cannot find collisions
- looks **random**: $\mathcal{P}$ cannot find input to get desired output

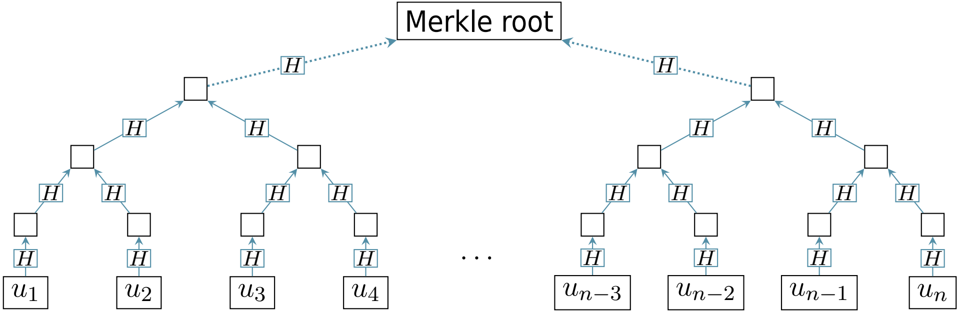
Fiat-Shamir replaces $\mathcal{V}$'s randomness by hash of previous messages:

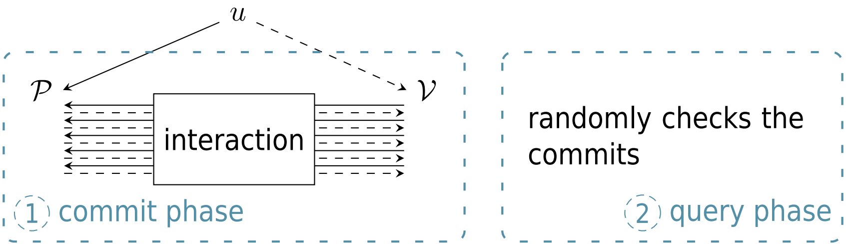
$$\mathcal{P} \begin{array}{c} \xrightarrow{\quad w \quad} \\ \xleftarrow{\quad \alpha \quad} \\ \xrightarrow[\quad v \quad]{} \end{array} \mathcal{V} \qquad \text{becomes} \qquad \alpha := H(w) \circlearrowleft \mathcal{P} \begin{array}{c} \xrightarrow{\quad w \quad} \\ \xrightarrow[\quad v \quad]{} \end{array} \mathcal{V}$$

Definition **Oracle access**

$\mathcal{P}$ provides to $\mathcal{V}$ an **oracle access** to $u \in \mathbb{F}^n$ by giving **black-box** access to u.

In practice, $\mathcal{P}$ provides the root of a **Merkle tree**.





- **Completeness:** if $u \in C$ then $\mathbb{P}(\mathcal{V}^{u,\leftrightarrow\mathcal{P}}$ accepts$) = 1$
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[BCS16] Eli Ben-Sasson, Alessandro Chiesa, and Nicholas Spooner. Interactive Oracle Proofs.
In *Theory of Cryptography: 14th International Conference, TCC 2016-B, Beijing, China, October 31-November 3, 2016, Proceedings, Part II 14*, pages 31–60. Springer, 2016

Idea: Test even and odd parts $f(X) =: f_{\text{even}}(X^2) + X f_{\text{odd}}(X^2)$.

Definition **Folding**

Let $f : \mathcal{L}_N \to \mathbb{F}$ and $\alpha \in \mathbb{F}$.

$$\text{Fold}[f, \alpha](X^2) := f_{\text{even}}(X^2) + \alpha f_{\text{odd}}(X^2) = \frac{f(X) + f(-X)}{2} + \alpha \frac{f(X) - f(-X)}{2X}$$

[BBHR18] Eli Ben-Sasson, Iddo Bentov, Yinon Horesh, and Michael Riabzev. Fast Reed-Solomon Interactive Oracle Proofs of Proximity. **In *45th international colloquium on automata, languages, and programming (ICALP 2018)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2018**

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- **Validity preservation:** $f \in \text{RS}[\mathcal{L}_N, K] \iff \mathbb{P}_{\alpha}(\text{Fold}[f,\alpha] \in \text{RS}[\mathcal{L}_{N/2}, K/2]) > \frac{1}{|\mathbb{F}|}$
- **Local check:** $\mathcal{V}$ computes $\text{Fold}[f,\alpha](x^2)$ with 2 queries to f

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FRI protocol

$$f_0 : \mathcal{L}_N \to \mathbb{F}$$

$\mathcal{P}$ $\mathcal{V}$

$$\alpha_0 \xleftarrow{\$} \mathbb{F}$$

$$f_1 : \mathcal{L}_{N/2} \to \mathbb{F}$$

$$f_1 = \mathsf{Fold}[f_0, \alpha_0]$$

1 commit phase

2 query phase

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$$f_1(x_1) \stackrel{?}{=} \mathsf{Fold}[f_0, \alpha_0](x_1)$$

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repeat m times

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Proposition **FRI complexities [BBHR18]**

FRI protocol for RS$[\mathcal{L}_N, K]$ with m repetitions has following complexity:

- Prover complexity: $< 8N$ (after encoding)
- Verifier complexity: $< 2m \log K$
- Number of queries: $2m \log K$
- Number of rounds: $\log K$

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Proposition **FRI completeness**

If $f_0 \in \mathsf{RS}[\mathcal{L}_N, K]$ then $\mathcal{V}$ accepts with probability 1.

Theorem **FRI soundness [BCI+23]**

If $\Delta(f_0, \mathsf{RS}[\mathcal{L}_N, K]) > \delta$ then for any $\tilde{\mathcal{P}}$ and $\varepsilon > 0$, $\mathcal{V}$ accepts with probability

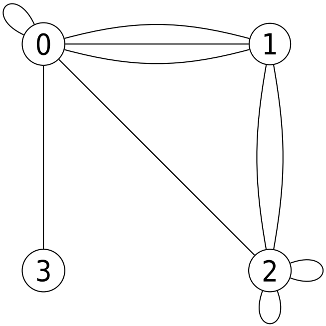
$$\leqslant \frac{K^2 \log K}{(2\varepsilon)^7 |\mathbb{F}|} + \left(1 - \min\left(\delta, 1 - \sqrt{K/N} - \varepsilon\right)\right)^m.$$

[BCI+23] Eli Ben-Sasson, Dan Carmon, Yuval Ishai, Swastik Kopparty, and Shubhangi Saraf. Proximity Gaps for Reed–Solomon Codes. *J. ACM*, 70(5), October 2023

- Graphs
- Folding graphs
- Flowering

$\Gamma = (V, E)$ is a n-RIM:

- **Multigraph:** multiple edges and loops



Regular Indexed Multigraphs (RIM)

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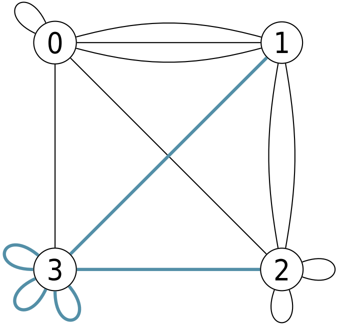
- ▶ **Regular:** same number n of edges

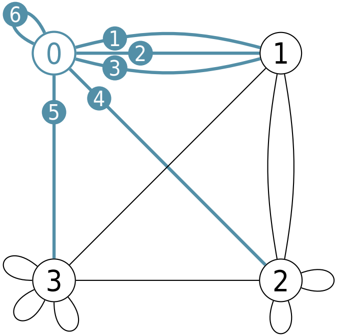


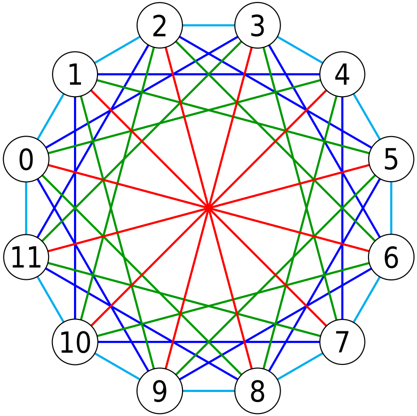
$\Gamma = (V, E)$ is a n-RIM:

▷ **Multigraph:** multiple edges and loops

▷ **Regular:** same number n of edges

▶ **Indexed:** edge is $(v, \ell) \in V \times [n]$
Write $\boldsymbol{E(v, \ell)} \in V$ the neighbor of v by ℓ

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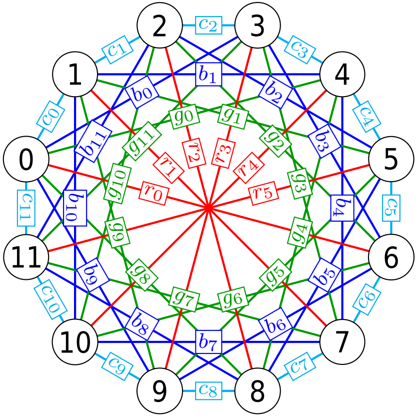
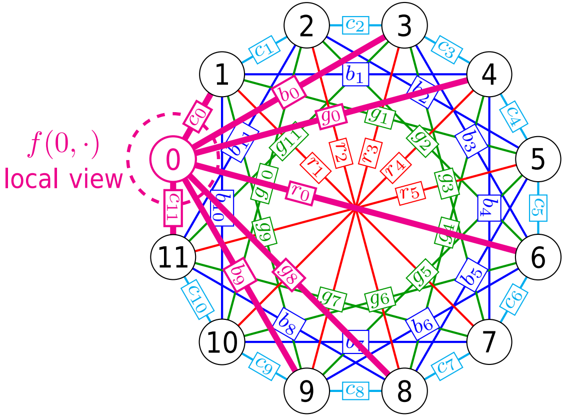


Word $f : V \times [n] \to \mathbb{F}$ on a graph Γ



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Definition **Code $\mathcal{C}[\Gamma, C_0]$**

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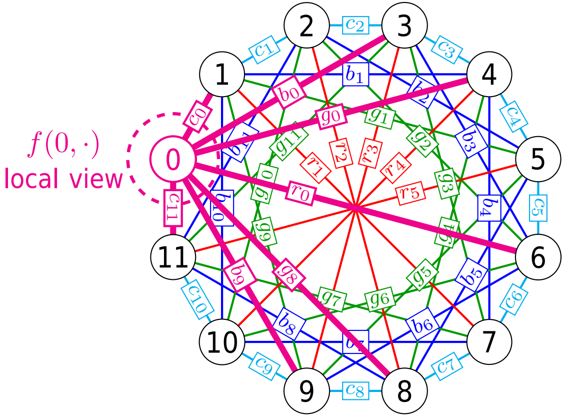
We'll only use $C_0 = \mathrm{RS}[n,k]$.



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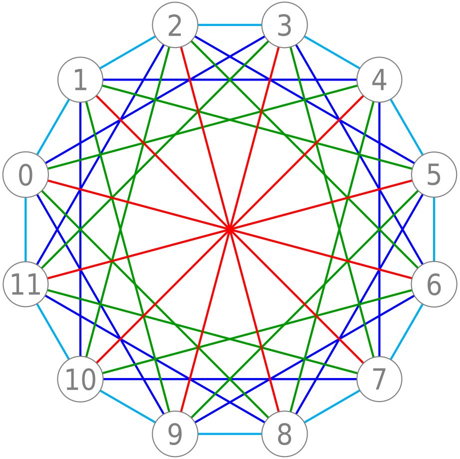
For 0, we must have
$(c_0, b_0, g_0, r_0, g_8, b_9, c_{11}) \in \mathsf{RS}[7,k]$.

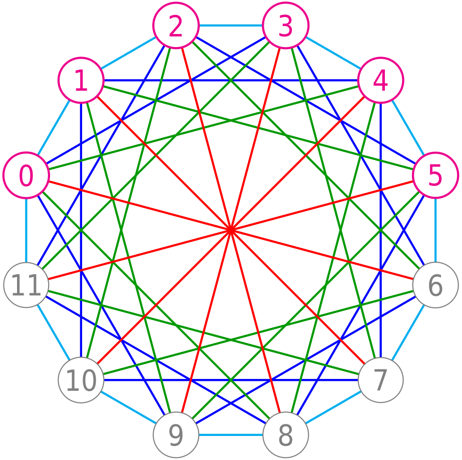
Cut-graph $\Gamma' = \mathsf{Cut}[\Gamma, V']$:



[DMR25] Hugo Delavenne, Tanguy Medevielle, and Élina Roussel. Interactive Oracle Proofs of Proximity to Codes on Graphs, 2025.
Submitted

Cut-graph $\Gamma' = \mathsf{Cut}[\Gamma, V']$:

- Choose vertices $V' \subseteq V$



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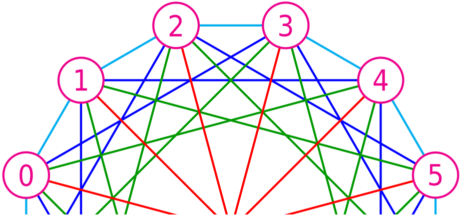

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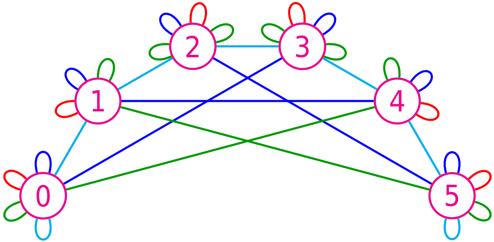
Cut-graph $\Gamma' = \text{Cut}[\Gamma, V']$:

- ▷ Choose vertices $V' \subseteq V$
- ▷ Cut the rest
- ▶ Enjoy your new graph

$$E_{V'}(v,\ell) = \begin{cases} E(v,\ell) & \text{if } E(v,\ell) \in V' \\ v & \text{otherwise} \end{cases}$$



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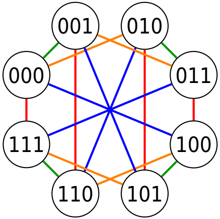
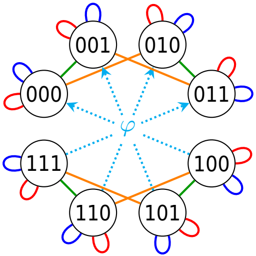
Definition **Graph isomorphism**

A bijection $\varphi : V' \to V''$ is an **isomorphism** $\Gamma' \to \Gamma''$ if

$$\forall (v', \ell) \in V' \times [n], \quad \varphi(E'(v', \ell)) = E''(\varphi(v'), \ell).$$





[DMR25] Hugo Delavenne, Tanguy Medevielle, and Élina Roussel. Interactive Oracle Proofs of Proximity to Codes on Graphs, 2025. **Submitted**

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A bijection $\varphi : V' \to V''$ is an **isomorphism** $\Gamma' \to \Gamma''$ if

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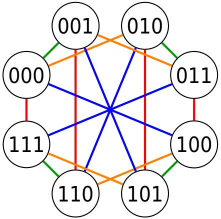
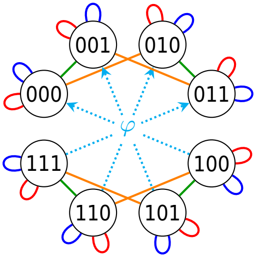
Definition **Flowering cut**

With $V'' = V \setminus V'$, if $\mathsf{Cut}[\Gamma, V'] \sim \mathsf{Cut}[\Gamma, V'']$, the cut is **flowering**.

The cut-word $\mathsf{Cut}[f, V']$ is the restriction $f_{|V' \times [n]}$.





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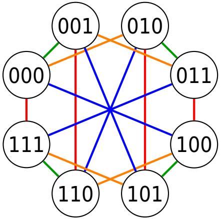

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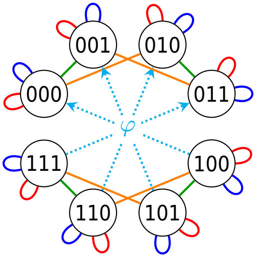

Definition **Folding**

For $\alpha \in \mathbb{F}$, $(v', \ell) \in V' \times [n]$,

$$\text{Fold}[f, \alpha](v', \ell) := \text{Cut}[f, V'](v, \ell) + \alpha \text{Cut}[f, V''](\varphi^{-1}(v'), \ell).$$

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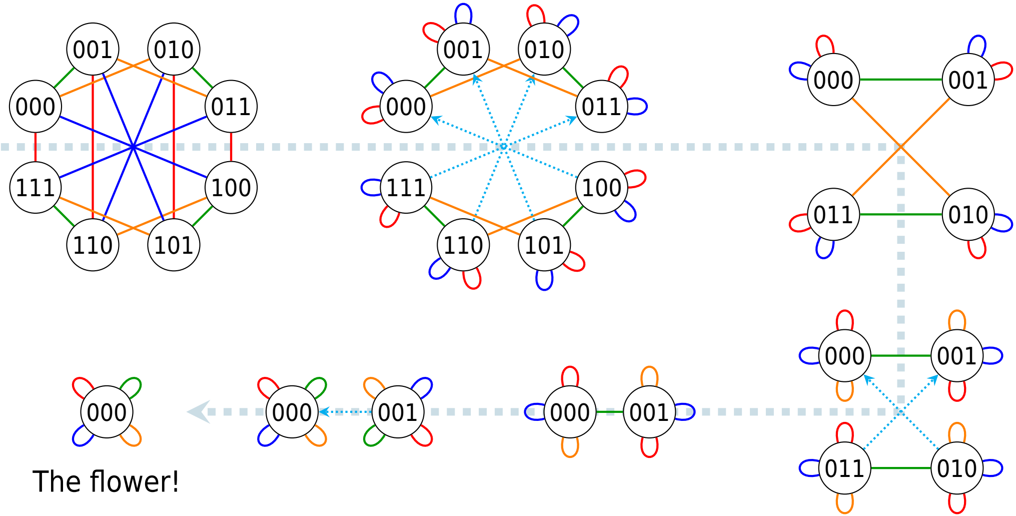
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The flower!

Flowering protocol

$$f_0 : V_0 \times [n] \to \mathbb{F}$$

$\mathcal{P}$ $\mathcal{V}$

$$\alpha_0 \xleftarrow{\$} \mathbb{F}$$

$$f_1 : V_1 \times [n] \to \mathbb{F}$$

$$f_1 = \mathsf{Fold}[f_0, \alpha_0]$$

1 commit phase

2 query phase

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repeat m times

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$$f_r \stackrel{?}{\in} \mathsf{RS}[n, k]$$

repeat m times

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[DMR25] Hugo Delavenne, Tanguy Medevielle, and Élina Roussel. Interactive Oracle Proofs of Proximity to Codes on Graphs, 2025. **Submitted**

Proposition **Flowering complexities**

Flowering with m repetitions has complexities: (recall FRI)

- Prover complexity: $< 3N$ — $< 8N$
- Verifier complexity: $4mnr$ — $< 2m \log K$
- Number of queries: $\sim 2mnr$ — $2m \log K$
- Number of rounds: r — $\log K$

[BBHR18] Eli Ben-Sasson, Iddo Bentov, Yinon Horesh, and Michael Riabzev. Fast Reed-Solomon Interactive Oracle Proofs of Proximity. **In *45th international colloquium on automata, languages, and programming (ICALP 2018)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2018**

[DMR25] Hugo Delavenne, Tanguy Medevielle, and Élina Roussel. Interactive Oracle Proofs of Proximity to Codes on Graphs, 2025. **Submitted**

Complexity comparison

Proposition **Flowering complexities**

Flowering with m repetitions has complexities: (with our first graphs) (recall FRI)

- Prover complexity: $< 3N$ — $< 8N$
- Verifier complexity: $4mnr$ ($< 4m\log^2 N$) — $< 2m\log K$
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Completeness and soundness

Proposition **Flowering completeness**

If $f \in \mathcal{C}[\Gamma, \mathsf{RS}[n,k]]$ then $\mathcal{V}$ accepts with probability 1.

Theorem **Flowering soundness**

If $\Delta(f, \mathcal{C}[\Gamma, \mathsf{RS}[n,k]]) > \delta$ then for any $\tilde{\mathcal{P}}$ and $\varepsilon > 0$, $\mathcal{V}$ accepts with probability

$$\leqslant \frac{r}{\varepsilon|\mathbb{F}|} + (1 - \delta + \varepsilon r)^m .$$

Recall FRI: $$\frac{K^2 \log K}{(2\varepsilon)^7 q|\mathbb{F}|} + \left(1 - \min\left(\delta, 1 - \sqrt{K/N} - \varepsilon\right)\right)^m$$

[BCI+23] Eli Ben-Sasson, Dan Carmon, Yuval Ishai, Swastik Kopparty, and Shubhangi Saraf. Proximity Gaps for Reed–Solomon Codes. ***J. ACM*, 70(5), October 2023**

[DMR25] Hugo Delavenne, Tanguy Medevielle, and Élina Roussel. Interactive Oracle Proofs of Proximity to Codes on Graphs, 2025. **Submitted**

- First graphs
- Expander graphs
- New cuts (*current work*)

Cayley graphs

Definition **Cayley graph Cay(G, S)**

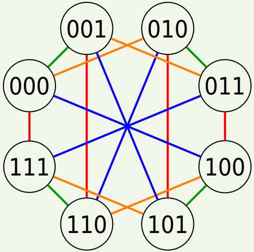
Fix $(G, \cdot)$. Let $S \subseteq G$ symmetric generating. Define $\Gamma = (G, E)$ where $E(g, s) = g \cdot s$.

We take

- $G = (\mathbb{F}_2^r, +)$
- $S \subseteq G$ of size n
- $\Gamma = \mathsf{Cay}[G, S]$
- $C = \mathcal{C}[\Gamma, \mathsf{RS}[n, k]]$

Example

With $G = (\mathbb{F}_2^3, +)$ and $S = \{100, 010, 001, 111\}$,



[Cay78] Arthur Cayley. Desiderata and Suggestions: No. 2. The Theory of Groups: Graphical Representation. ***American Journal of Mathematics*, 1(2):174–176, 1878**

Using S the columns of a parity check matrix of a $[n, n-r, d]_2$ binary code

Proposition **Parameters of the code**

- $N = n2^{r-1}$
- $\operatorname{rate} C \geqslant \frac{2k}{n} - 1$
- $\frac{1}{2}\delta \leqslant \Delta(C) \leqslant \delta$, with $\delta = \frac{1}{2^{r-d+1}}\left(1 - \frac{k-1}{n}\right) = \frac{n2^{d-2}}{N}\left(1 - \frac{k-1}{n}\right)$

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If $d \ll r$ (i.e. far from MDS), δ is **terrible** ($O(1/N)$).

[DMR25] Hugo Delavenne, Tanguy Medevielle, and Élina Roussel. Interactive Oracle Proofs of Proximity to Codes on Graphs, 2025. **Submitted**

Expander graphs

Definition **Graph expansion**

Let $\Gamma = (V, E)$ a n-regular graph and $A \in \{0,1\}^{|V|\times|V|}$ its adjacency matrix.
Let $\Lambda_1 \geqslant \Lambda_2 \geqslant \dots \geqslant \Lambda_n \in \mathbb{R}$ be the eigenvalues of A.

Then Γ is **λ-expander** if $|\Lambda_i| \leqslant n\lambda$ for $i \geqslant 2$.

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Lemma Minimal distance expansion lower bound [AC88]

If Γ is λ-expander, with $\delta = 1 - \frac{k+1}{n}$, $\mathcal{C}[\Gamma, \mathsf{RS}[n,k]]$ has minimal distance $\geqslant \delta(\delta - \lambda)$.

Thus if $(\Gamma_i)_{i\in\mathbb{N}}$ has **constant expansion**,
then $(\mathcal{C}[\Gamma_i, \mathsf{RS}[n_i, \gamma n_i]])_{i\in\mathbb{N}}$ has **constant minimal distance**.

[AC88] Noga Alon and Fan Chung. Explicit construction of linear sized tolerant networks. ***Discrete Mathematics*, 72(1-3):15–19, 1988**

Expander Cayley graph examples

Let

- $G_p = \mathrm{SL}_3(\mathbb{F}_p)$
- S_p symmetric generating
- $\Gamma_p = \mathsf{Cay}(G_p, S_p)$

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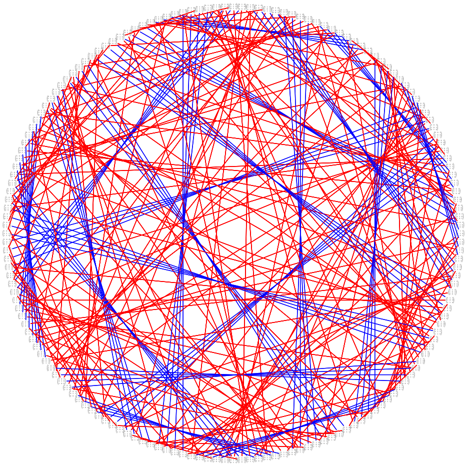
We obtain Γ_2 with

$$S_2 = \left\{ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^{\pm 1} \right\}:$$



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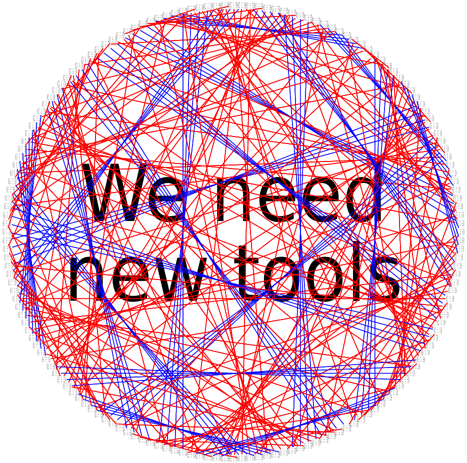
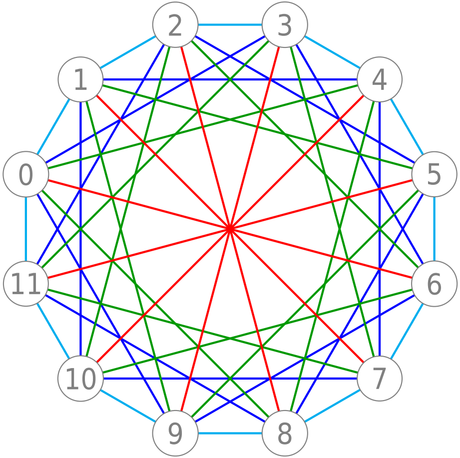
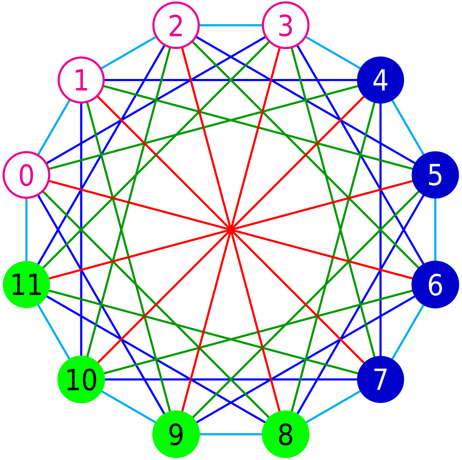


We can cut into $m > 2$ cuts



We can cut into $m > 2$ cuts

- Choose m sets of vertices $V_0, ..., V_{m-1} \subseteq V$



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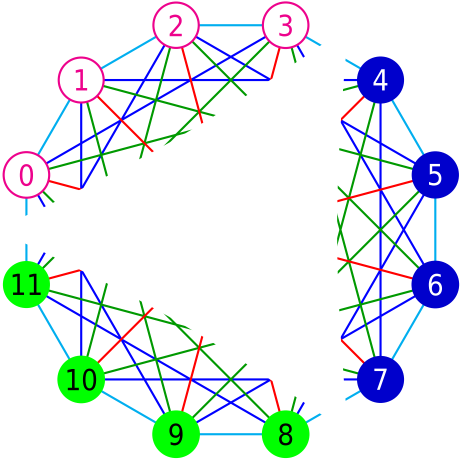
- Cut outgoing edges



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- ▷ Choose m sets of vertices $V_0, ..., V_{m-1} \subseteq V$
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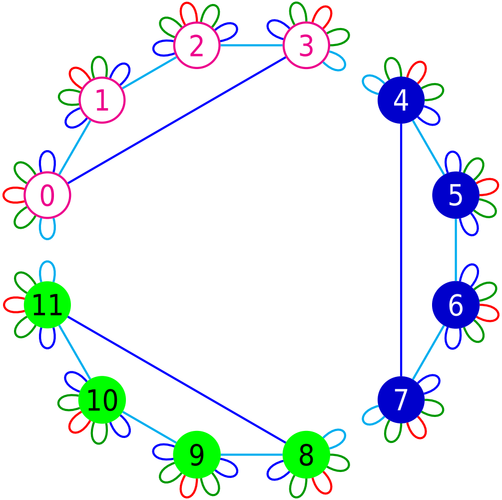
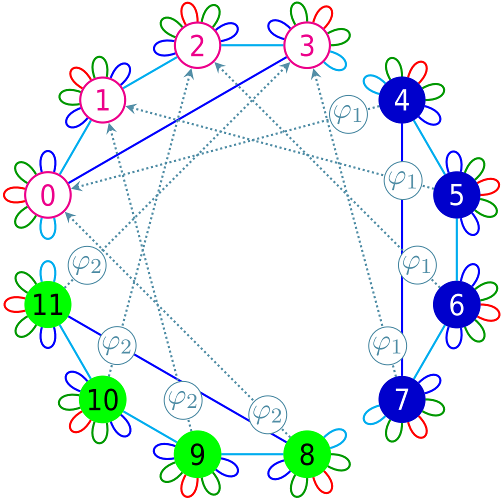
- ▶ Get new graphs with petals



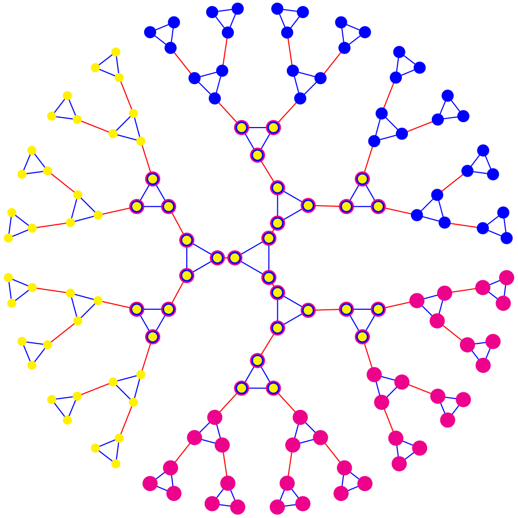
We can cut into $m > 2$ cuts

- ▷ Choose m sets of vertices $V_0, \ldots, V_{m-1} \subseteq V$
- ▷ Cut outgoing edges
- ▷ Get new graphs with petals
- ▶ Define the new Fold with the isomorphisms

$$\mathsf{Fold}[f, \alpha](v, \ell) := \sum_{i=0}^{m-1} \alpha^i f(\varphi_i^{-1}(v), \ell)$$



Cuts $V_1, ..., V_m$ may not be disjoint:



Let $S = \{s_0^{\pm 1}, s_2^{\pm 1}, \ldots, s_{\tilde{n}-1}^{\pm 1}\}$ and $G = \langle S \rangle$ be finite. Write $\tilde{s}_i := s_{i \bmod \tilde{n}}$. Let

$$r_g := \min\{k \in \mathbb{N} \mid g = \tilde{s}_0^{j_1} \cdots \tilde{s}_{r_g}^{j_{r_g}}\}$$
$$r := \max_{g \in G} r_g \leqslant \tilde{n} \cdot \operatorname{diam} \mathsf{Cay}(G, S) = O(\tilde{n} \log |G|)$$

The cuts are $\quad V_{i,j} := \left\{\tilde{s}_i^{j} \cdot \tilde{s}_{i+1}^{j_{i+1}} \cdots \tilde{s}_r^{j_r} \mid j_{i+1}, \ldots, j_k \in \mathbb{N}\right\}$.

- $\Gamma_{i,j} \sim \Gamma_{i,0}$ with $\varphi_{i,j}(g) = \tilde{s}_i^{-j} g \quad \rightarrow \mathbf{order}(\tilde{s}_i)$ **cuts**
- $\Gamma_{r,0}$ is a flower $\quad \rightarrow O(\tilde{n} \log |G|)$ **rounds**

Competing parameters with FRI

- We have a better soundness
- Our complexity could be improved

New cuts, new graphs

- 2 disjoint cuts V', V'' $\rightarrow$ m covering cuts $V_1, V_2, ..., V_m$
- Compute complexity for general Cayley graphs

Make this actually useful

- Arithmetize circuits to graphs: colored De Bruijn
- Encode words on graphs into bigger flowering graphs