Flowering graphs

Interactive Oracle Proofs of Proximity to codes on graphs by flowering

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Definition **Oracle access**

An algorithm $\mathcal{V}$ has **oracle access** to $u \in \mathbb{F}^n$ if it has **black-box** access to u.

- Denoted $\mathcal{V}^u$
- Can limit or count number of queries to oracle

Definition **Locally-testable code**

A code C is (ℓ, δ, s)**-locally-testable** if there is $\mathcal{V}$ with ℓ**-oracle access** such that

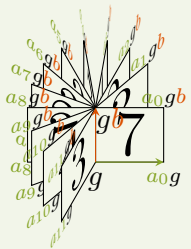
- **Completeness:** if $u \in C$ then $\mathbb{P}(\mathcal{V}^u \text{ accepts}) = 1$
- **Soundness:** if $\Delta(u, C) > \delta$ then $\mathbb{P}(\mathcal{V}^u \text{ accepts}) \leqslant s$.

Example **Reed-Solomon codes are not locally-testable**

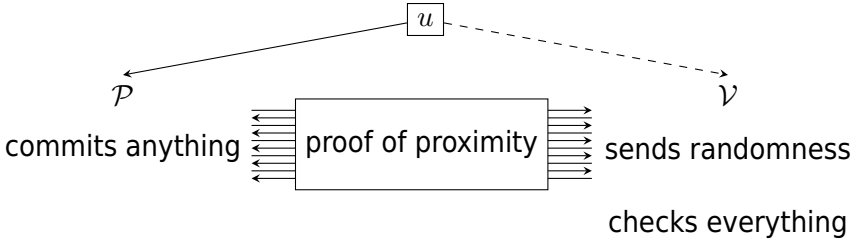
$\mathrm{RS}[n,k] := \{(f(x_1),...,f(x_n)) \mid f \in \mathbb{F}_q[X]_{<k}\}$ has locality $k+1$.

Example **Codes with constant rate, minimal distance and locality [DEL$^+$22]**

There exists codes with constant rate, minimal distance and locality:



[DEL$^+$22] Irit Dinur, Shai Evra, Ron Livne, Alexander Lubotzky, and Shahar Mozes. Locally testable codes with constant rate, distance, and locality.
In *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing*, pages 357–374, 2022



- **Completeness:** if $u \in C$ then $\mathbb{P}(\mathcal{V}^{u,\leftrightarrow\mathcal{P}} \text{ accepts}) = 1$
- **Soundness:** if $\Delta(u, C) > \delta$ then for any $\mathcal{P}$, $\mathbb{P}(\mathcal{V}^{u,\leftrightarrow\mathcal{P}} \text{ accepts}) \leqslant s$
- Can be turned **non-interactive** [FS86, BCS16]

[FS86] Amos Fiat and Adi Shamir. How To Prove Yourself: Practical Solutions to Identification and Signature Problems.
In Andrew M. Odlyzko, editor, *Advances in Cryptology — CRYPTO' 86*, pages 186–194, Berlin, Heidelberg, 1986. Springer Berlin Heidelberg

[BCS16] Eli Ben-Sasson, Alessandro Chiesa, and Nicholas Spooner. Interactive Oracle Proofs.
In *Theory of Cryptography: 14th International Conference, TCC 2016-B, Beijing, China, October 31-November 3, 2016, Proceedings, Part II 14*, pages 31–60. Springer, 2016

Idea. Test even and odd parts $f(X) =: f_{\mathsf{even}}(X^2) + X f_{\mathsf{odd}}(X^2)$.

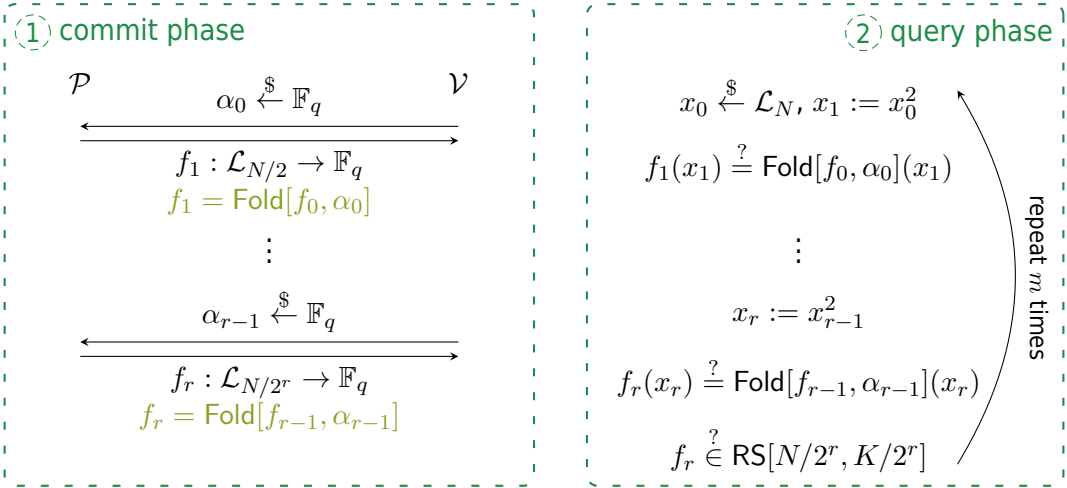
Definition **Folding**

$$\mathsf{Fold}[f, \alpha](Y) := f_{\mathsf{even}}(Y) + \alpha f_{\mathsf{odd}}(Y) = \frac{f(X) + f(-X)}{2} + \alpha \frac{f(X) - f(-X)}{2X},$$

with "$Y = X^2$", $\alpha \in \mathbb{F}_q$

- **Validity preservation:** $f \in \mathsf{RS}[N, K] \iff \mathbb{P}_{\alpha}(\mathsf{Fold}[f, \alpha] \in \mathsf{RS}[N/2, K/2]) > \frac{1}{q}$
- **Local check:** $\mathcal{V}$ computes $\mathsf{Fold}[f, \alpha](x^2)$ with 2 queries to f
- **Field restriction:** $\mathcal{L}_{N/2} := \{x^2 \mid x, -x \in \mathcal{L}_N\}$
  - $\triangleright$ $\mathbb{F}_q$ must have 2^N roots of unity

[BBHR18] Eli Ben-Sasson, Iddo Bentov, Yinon Horesh, and Michael Riabzev. Fast Reed-Solomon Interactive Oracle Proofs of Proximity. In ***45th international colloquium on automata, languages, and programming (ICALP 2018)***. **Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2018**



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Proposition **FRI complexities**

FRI protocol for RS$[N, K]$ with m repetitions has following complexity:

- Prover complexity: $< 6N$
- Verifier complexity: $< 2m \log N$
- Query complexity: $< 2m \log K$
- Round complexity: $\log K$

[BBHR18] Eli Ben-Sasson, Iddo Bentov, Yinon Horesh, and Michael Riabzev. Fast Reed-Solomon Interactive Oracle Proofs of Proximity. **In *45th international colloquium on automata, languages, and programming (ICALP 2018)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2018**

Proposition **FRI completeness**

If $f \in \mathsf{RS}[N, K]$ then $\mathcal{V}$ accepts with probability 1.

Theorem **FRI soundness [BCI+23]**

If $\Delta(f, \mathsf{RS}[N, K]) > \delta$ then for any $\mathcal{P}$, $\mathcal{V}$ accepts with probability

$$\leqslant \frac{K^2 \log K}{(2\varepsilon)^7 q} + \left(1 - \min\left(\delta, 1 - \sqrt{K/N}\right)\right)^m,$$

with m a complexity parameter, and $\varepsilon = \sqrt{K/N}/20$.

[BCI+23] Eli Ben-Sasson, Dan Carmon, Yuval Ishai, Swastik Kopparty, and Shubhangi Saraf. Proximity Gaps for Reed–Solomon Codes.
***J. ACM*, 70(5), October 2023**

1 Proximity tests

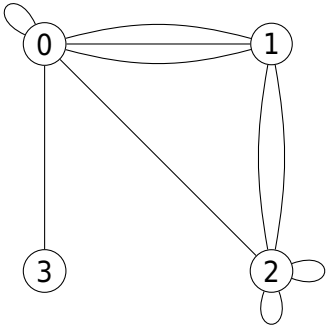
2 Graphs

3 Flowering protocol

- Regular Indexed Multigraphs
- Words and codes on graphs
- Hamming distance vs vertex distance

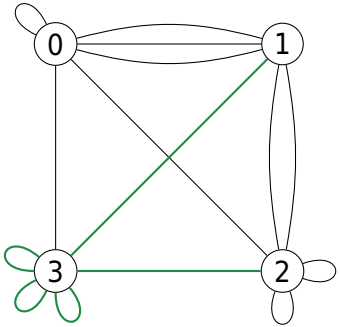
$\Gamma = (V, E)$ is a n-RIM:

- **Multigraph:** multiple edges and loops



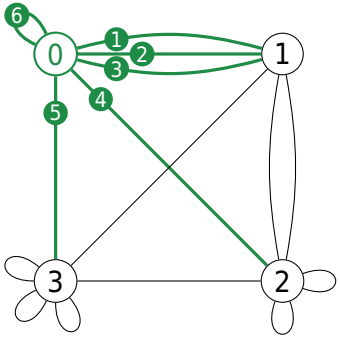
$\Gamma = (V, E)$ is a n-RIM:

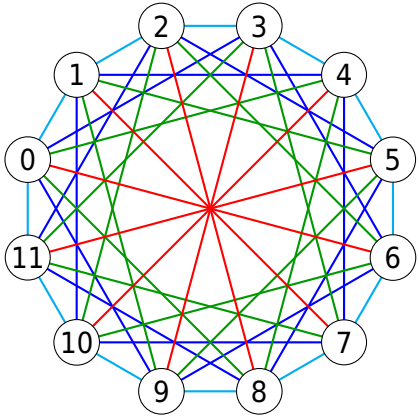
- ▷ **Multigraph:** multiple edges and loops
- ▶ **Regular:** same number n of edges



$\Gamma = (V, E)$ is a n-RIM:

- ▷ **Multigraph:** multiple edges and loops
- ▷ **Regular:** same number n of edges
- ▶ **Indexed:** edge is $(v, \ell) \in V \times [n]$
  Write $E(v, \ell) \in V$ the neighbor of v by ℓ

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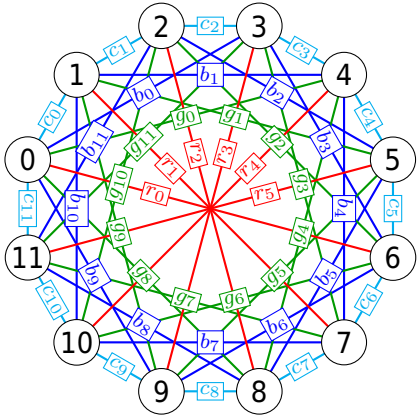
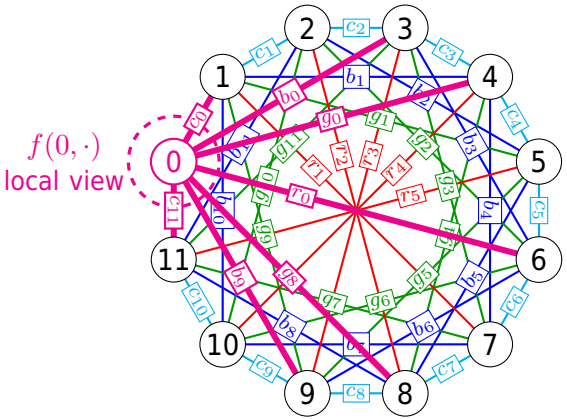


Word $f(v, \ell)$ on a graph Γ



Word $f(v, \ell)$ on a graph Γ

Definition **Code $\mathcal{C}[\Gamma, C_0]$**

Given Γ a n-RIM and $C_0 \subseteq \mathbb{F}_q^n$,

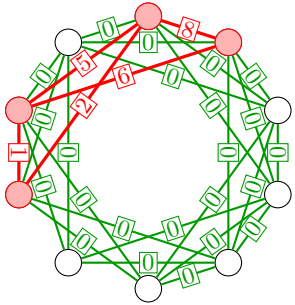
$$f \in \mathcal{C}[\Gamma, C_0] \iff \forall v,\ f(v, \cdot) \in C_0.$$

We'll only use $C_0 = \mathsf{RS}[n, k]$.

Hamming distance vs vertex distance

Hamming: $\Delta(f, f') := \frac{\#\text{diff edges}}{\#\text{edges}} = \frac{5}{30} \approx 0.167$

Vertex: $\Delta_V(f, f') := \frac{\#\text{diff vertices}}{\#\text{vertices}} = \frac{4}{10} = 0.4$



$f - f'$

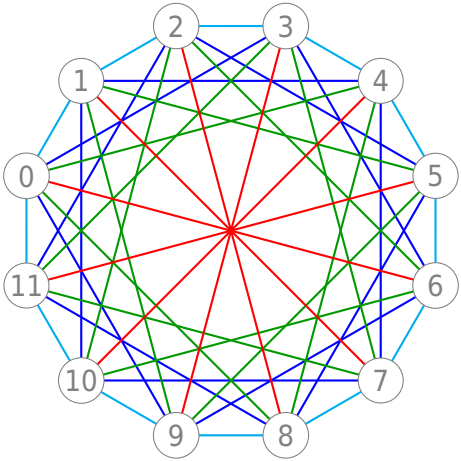
Proposition **Hamming distance is more fine grain than vertex distance**

For any f, f', $\Delta(f, f') \leqslant \Delta_V(f, f')$. (if nodes of Γ have same number of loops)

[DMR25] Hugo Delavenne, Tanguy Medevielle, and Élina Roussel. Interactive Oracle Proofs of Proximity to Codes on Graphs, 2025. **Submitted**

- Folding graphs
  - Cutting graphs
  - Graph isomorphism & folding
  - A flowering example
- Flowering
  - Flowering protocol
  - Complexity comparison
  - Completeness and soundness
- Our codes
  - Cayley graphs
  - Our choice of G and S

Cut-graph $\Gamma' = \mathsf{Cut}[\Gamma, V']$:



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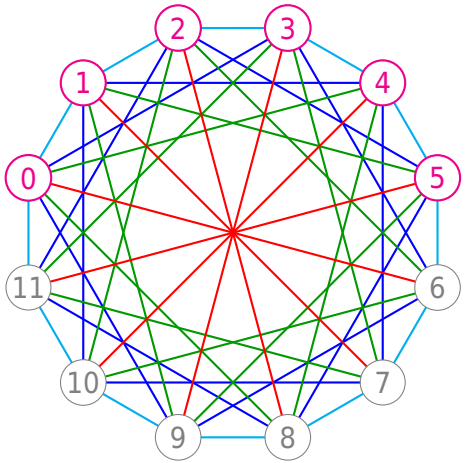

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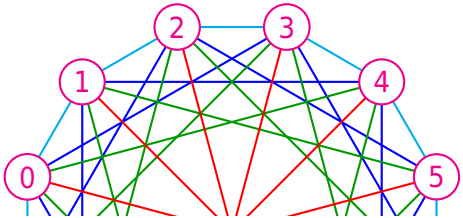

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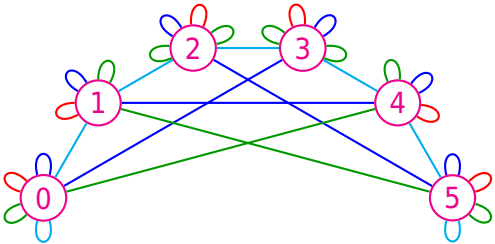
Cut-graph $\Gamma' = \mathsf{Cut}[\Gamma, V']$:

- ▷ Choose vertices $V' \subseteq V$
- ▷ Cut the rest
- ▶ Enjoy your new graph

$$E_{V'}(v, \ell) = \begin{cases} E(v, \ell) & \text{if } E(v, \ell) \in V' \\ v & \text{otherwise} \end{cases}$$



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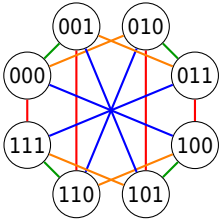
Definition **Graph isomorphism**

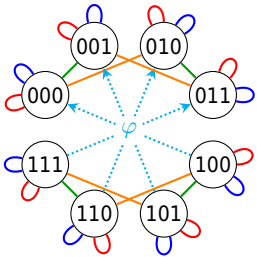
A bijection $\varphi : V' \to V''$ is an **isomorphism** $\Gamma' \to \Gamma''$ if

$$\forall v, \forall \ell, \varphi(E'(v, \ell)) = E''(\varphi(v), \ell).$$



If $V'' = V \setminus V'$ and $\mathsf{Cut}[\Gamma, V'] \sim \mathsf{Cut}[\Gamma, V'']$, the cut is **flowering**.

$\mathsf{Cut}[f, V']$ is the restriction $f_{|V' \times [n]}$.



Definition **Folding**

$$\mathsf{Fold}[f, \alpha](v, \ell) := \mathsf{Cut}[f, V'](v, \ell) + \alpha \mathsf{Cut}[f, V''](\varphi(v), \ell)$$

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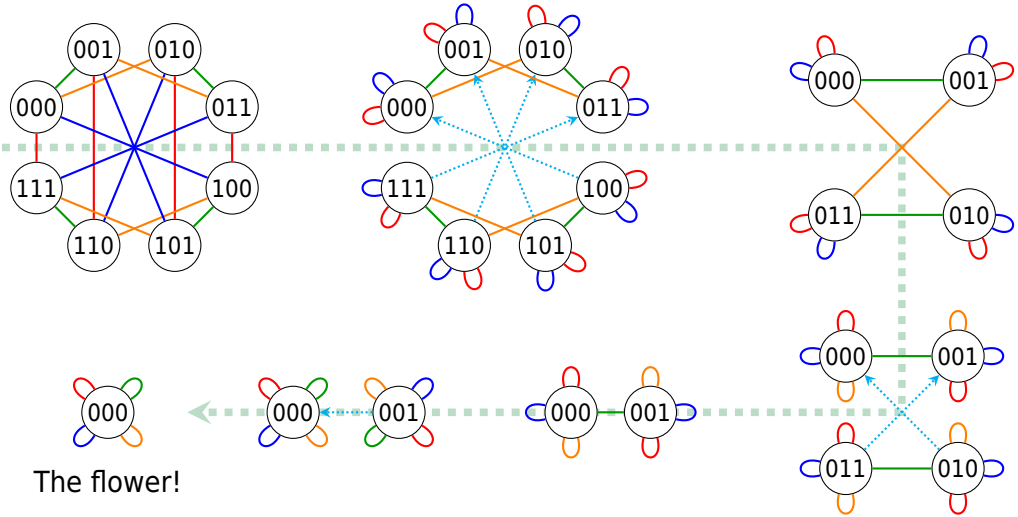
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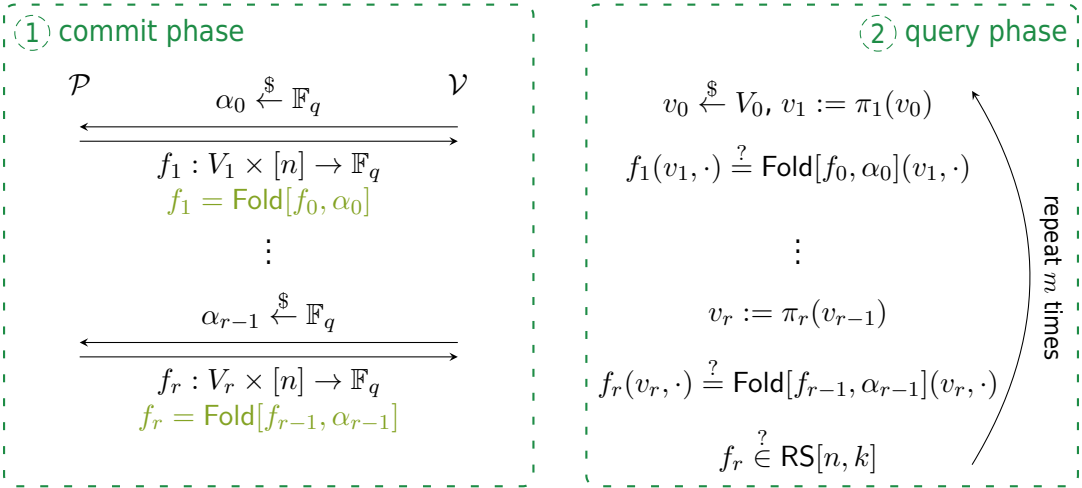
The flower!



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Proposition **Flowering complexities**

Flowering with m repetitions has complexities: (recall FRI)

- Prover complexity: $< 3N$ $< 6N$
- Verifier complexity: $< 4m \log^2 N$ $< 2m \log N$
- Query complexity: $< 2m \log^2 N$ $< 2m \log K$
- Round complexity: $< \log N$ $\log K$

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Proposition **Flowering completeness**

There exists $\mathcal{P}$ such that if $f \in \mathcal{C}[\Gamma, \mathsf{RS}[n,k]]$ then $\mathcal{V}$ accepts with probability 1.

Theorem **Flowering soundness**

If $\Delta_V(f, \mathcal{C}[\Gamma, \mathsf{RS}[n,k]]) > \delta$ then for any $\mathcal{P}$, $\mathcal{V}$ accepts with probability

$$\leqslant \min_{\varepsilon} \left(\frac{\log N}{\varepsilon q} + (1 - \delta + \varepsilon \log N)^m \right).$$

Recall the FRI soundness: $\dfrac{K^2 \log K}{(2\varepsilon)^7 q} + \left(1 - \min\left(\delta, 1 - \sqrt{K/N}\right)\right)^m$

[BCI+23] Eli Ben-Sasson, Dan Carmon, Yuval Ishai, Swastik Kopparty, and Shubhangi Saraf. Proximity Gaps for Reed–Solomon Codes. ***J. ACM*, 70(5), October 2023**

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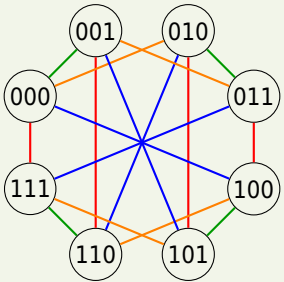
Definition **Cayley graph**

Fix $(G, \cdot)$. Let $S \subseteq G$ be symmetric. Define $\Gamma = (G, E)$ where $E(g, s) = g \cdot s$.

Example $G = (\mathbb{F}_2^3, +)$, $S = \{100, 010, 001, 111\}$



[Cay78] Arthur Cayley. Desiderata and Suggestions: No. 2. The Theory of Groups: Graphical Representation. ***American Journal of Mathematics*, 1(2):174–176, 1878**

We take

- $G = (\mathbb{F}_2^r, +)$
- S the columns of a parity check matrix of a $[n, n-r, d]_2$ binary code
- $C = \mathcal{C}[\Gamma, \mathsf{RS}[n, k]]$

Proposition **Parameters of the code**

- $N = n2^{r-1}$
- $\operatorname{rate} C \geqslant \frac{2k}{n} - 1$
- $\frac{1}{2}\delta \leqslant \Delta(C) \leqslant \delta$, with $\delta = \frac{1}{2^{r-d+1}}\left(1 - \frac{k-1}{n}\right) = \frac{n2^{d-2}}{N}\left(1 - \frac{k-1}{n}\right)$

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Our choice of G and S

We take

- $G = (\mathbb{F}_2^r, +)$
- S the columns of a parity check matrix of a $[n, n-r, d]_2$ binary code
- $C = \mathcal{C}[\Gamma, \mathsf{RS}[n, k]]$

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- $N = n2^{r-1}$
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- $\frac{1}{2}\delta \leqslant \Delta(C) \leqslant \delta$, with $\delta = \frac{1}{2^{r-d+1}}\left(1 - \frac{k-1}{n}\right) = \frac{n2^{d-2}}{N}\left(1 - \frac{k-1}{n}\right)$

If $d \ll r$ (i.e. far from MDS), δ is **terrible** ($O(1/N)$).

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▷ Future work

Future work

We want flowering-friendly graphs with good **minimal distance** code

- working with **operations research** people
- go deeper into graph **expansion**

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Thank you for your attention!