Flowering graphs

Proximity test to codes on graphs by flowering

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SNARK means **S**uccinct **N**on-interactive **AR**gument of **K**nowledge.

It turns $A : x \mapsto y$ (that runs in $\tau(|x|)$) into

$$A' : \underbrace{x}_{\text{input}}, \underbrace{\alpha}_{\text{randomness}} \mapsto \underbrace{y}_{\text{output}}, \underbrace{\pi}_{\text{SNARK proof that } ``A(x)=y"}$$

satisfying:

- $|\pi| \ll \tau$
- A' runs in $\tilde{O}(\tau)$
- there is a verifier $\mathcal{V}$ in $\mathrm{poly}(|\pi|)$ such that
  - **Completeness:** if $A(x) = y$ then $\mathbb{P}_{\alpha}(\mathcal{V}(\pi) \text{ accepts}) = 1$
  - **Soundness:** if $A(x) \neq y$ then $\mathbb{P}_{\alpha}(\mathcal{V}(\pi) \text{ accepts}) \leqslant s$.

Contents

1 Interactive proximity tests

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- Interactive Oracle Proof of Proximity
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  - Oracle access
  - Interactive Oracle Proofs of Proximity
- Fast Reed-Solomon IOPP
  - Folding Reed-Solomon codes
  - FRI protocol
  - Complexities
  - Completeness and soundness

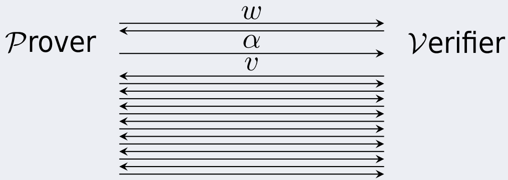
Definition **Non-interactive proof**

$$\mathcal{P}\text{rover} \xrightarrow{\quad w \quad} \mathcal{V}\text{erifier}$$

Definition **Sigma protocol**

$$\mathcal{P}\text{rover} \quad \begin{array}{c} \xrightarrow{\quad\quad w \quad\quad} \\ \xleftarrow[\quad\quad \alpha \quad\quad]{} \\ \xrightarrow[\quad\quad v \quad\quad]{} \end{array} \quad \mathcal{V}\text{erifier}$$

Definition **Interactive Proof**



Arithmetization reduces checking a computation to testing proximity to a code C:

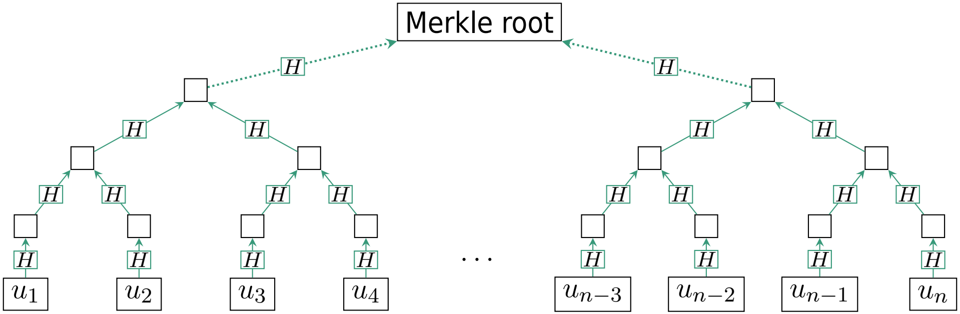
$$\begin{aligned} A(x) = y \quad &\Longrightarrow \quad \text{Arithmetization}(A, x, y) \in C \\ A(x) \neq y \quad &\Longrightarrow \quad \Delta(\text{Arithmetization}(A, x, y), C) > \delta \end{aligned}$$

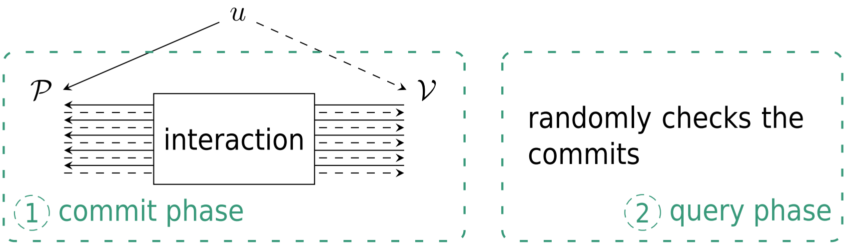
With arithmetization + proximity test + Fiat-Shamir we get a **NARK**!

Definition **Oracle access**

$\mathcal{P}$ can provide to $\mathcal{V}$ an **oracle access** to $u \in \mathbb{F}^n$ by giving **black-box** access to u.

In practice, $\mathcal{P}$ provides the root of a **Merkle tree**.





- **Completeness:** if $u \in C$ then $\mathbb{P}(\mathcal{V}^{u,\leftrightarrow \mathcal{P}}$ accepts$) = 1$
- **Soundness:** if $\Delta(u, C) > \delta$ then for any $\mathcal{P}$, $\mathbb{P}(\mathcal{V}^{u,\leftrightarrow \mathcal{P}}$ accepts$) \leqslant s$

[BCS16] Eli Ben-Sasson, Alessandro Chiesa, and Nicholas Spooner. Interactive Oracle Proofs.
In *Theory of Cryptography: 14th International Conference, TCC 2016-B, Beijing, China, October 31-November 3, 2016, Proceedings, Part II 14*, pages 31–60. Springer, 2016

We test proximity to **RS$[\mathcal{L}_N, K]$** $:= \{f : \mathcal{L}_N \to \mathbb{F} \mid f \in \mathbb{F}[X]_{<K}\}$, with $|\mathcal{L}_N| = N$.

Idea. Test even and odd parts $f(X) =: f_{\text{even}}(X^2) + X f_{\text{odd}}(X^2)$.

Definition Fold

$$\text{Fold}[f, \alpha](Y) := f_{\text{even}}(Y) + \alpha f_{\text{odd}}(Y) = \frac{f(X) + f(-X)}{2} + \alpha \frac{f(X) - f(-X)}{2X},$$

with "$Y = X^2$", $\alpha \in \mathbb{F}$

- **Field restriction:** $\mathcal{L}_{N/2} := \{x^2 \mid x, -x \in \mathcal{L}_N\}$ so $\mathbb{F}$ must have 2^N roots of unity
- **Validity preservation:** $f \in \text{RS}[\mathcal{L}_N, K] \iff \underset{\alpha}{\mathbb{P}}(\text{Fold}[f, \alpha] \in \text{RS}[\mathcal{L}_{N/2}, K/2]) > \frac{1}{|\mathbb{F}|}$
- **Local check:** $\mathcal{V}$ computes $\text{Fold}[f, \alpha](x^2)$ with 2 queries to f

[BBHR18] Eli Ben-Sasson, Iddo Bentov, Yinon Horesh, and Michael Riabzev. Fast Reed-Solomon Interactive Oracle Proofs of Proximity. In ***45th international colloquium on automata, languages, and programming (ICALP 2018)***. **Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2018**

FRI protocol

$$f_0 : \mathcal{L}_N \to \mathbb{F}$$

$\mathcal{P}$ $\mathcal{V}$

$$\alpha_0 \xleftarrow{\$} \mathbb{F}$$

$$f_1 : \mathcal{L}_{N/2} \to \mathbb{F}$$

$$f_1 = \mathsf{Fold}[f_0, \alpha_0]$$

$$\vdots$$

$$\alpha_{r-1} \xleftarrow{\$} \mathbb{F}$$

$$f_r : \mathcal{L}_{N/2^r} \to \mathbb{F}$$

$$f_r = \mathsf{Fold}[f_{r-1}, \alpha_{r-1}]$$

1 commit phase

$$x_0 \xleftarrow{\$} \mathcal{L}_N,\ x_1 := x_0^2$$

$$f_1(x_1) \stackrel{?}{=} \mathsf{Fold}[f_0, \alpha_0](x_1)$$

$$\vdots$$

$$x_r := x_{r-1}^2$$

$$f_r(x_r) \stackrel{?}{=} \mathsf{Fold}[f_{r-1}, \alpha_{r-1}](x_r)$$

$$f_r \stackrel{?}{\in} \mathsf{RS}[\mathcal{L}_{N/2^r}, K/2^r]$$

repeat m times

2 query phase

[BBHR18] Eli Ben-Sasson, Iddo Bentov, Yinon Horesh, and Michael Riabzev. Fast Reed-Solomon Interactive Oracle Proofs of Proximity. **In *45th international colloquium on automata, languages, and programming (ICALP 2018)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2018**

Proposition **FRI complexities [BBHR18]**

FRI protocol for RS$[\mathcal{L}_N, K]$ with m repetitions has following complexity:

- Prover complexity: $< 6N$ (without encoding)
- Verifier complexity: $< 2m \log K$
- Number of queries: $< 2m \log K$

[BBHR18] Eli Ben-Sasson, Iddo Bentov, Yinon Horesh, and Michael Riabzev. Fast Reed-Solomon Interactive Oracle Proofs of Proximity. **In *45th international colloquium on automata, languages, and programming (ICALP 2018)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2018**

Proposition **FRI completeness**

If $f_0 \in \mathsf{RS}[\mathcal{L}_N, K]$ then $\mathcal{V}$ accepts with probability 1.

Theorem **FRI soundness**

If $\Delta(f_0, \mathsf{RS}[\mathcal{L}_N, K]) > \delta$ then for any $\tilde{\mathcal{P}}$ and $\varepsilon > 0$, $\mathcal{V}$ accepts with probability

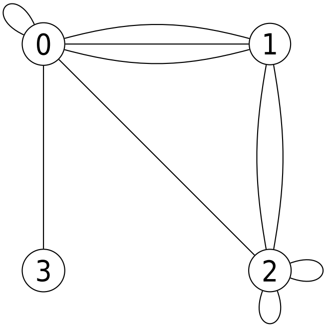
$$\leqslant \frac{K^2 \log K}{(2\varepsilon)^7 q} + \left(1 - \min\left(\delta, 1 - \sqrt{K/N} - \varepsilon\right)\right)^m.$$

[BCI+23] Eli Ben-Sasson, Dan Carmon, Yuval Ishai, Swastik Kopparty, and Shubhangi Saraf. Proximity Gaps for Reed–Solomon Codes. *J. ACM*, 70(5), October 2023

- Graphs
  - Regular Indexed Multigraphs (RIM)
  - Words and codes on graphs
- Folding graphs
  - Tutorial: Cutting graphs
  - Graph isomorphism & folding
  - « *Vous voulez des exemples ?* »
- Flowering
  - Flowering protocol
  - Complexity comparison
  - Completeness and soundness

$\Gamma = (V, E)$ is a n-RIM:

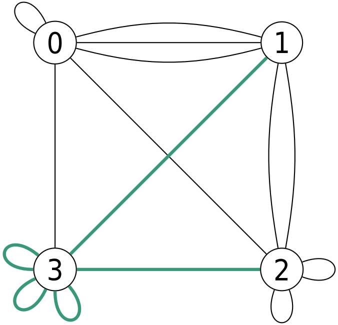
- **Multigraph:** multiple edges and loops



$\Gamma = (V, E)$ is a n-RIM:

▷ **Multigraph:** multiple edges and loops

▶ **Regular:** same number n of edges



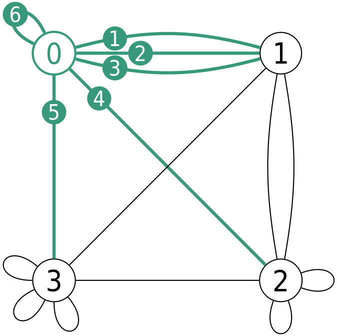
Regular Indexed Multigraphs (RIM)

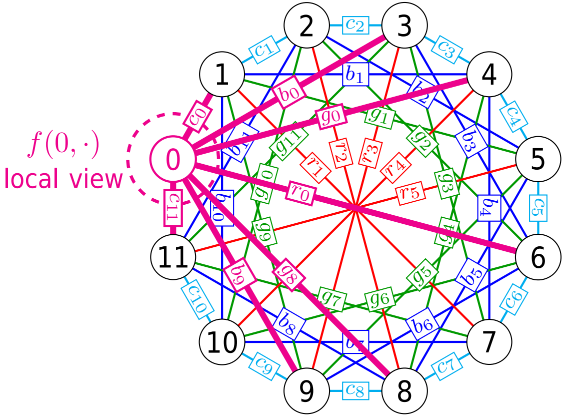
$\Gamma = (V, E)$ is a n-RIM:

▷ **Multigraph:** multiple edges and loops

▷ **Regular:** same number n of edges

▶ **Indexed:** edge is $(v, \ell) \in V \times [n]$
Write $\boldsymbol{E(v, \ell)} \in V$ the neighbor of v by ℓ





Word $f : V \times [n] \to \mathbb{F}$ on a graph Γ

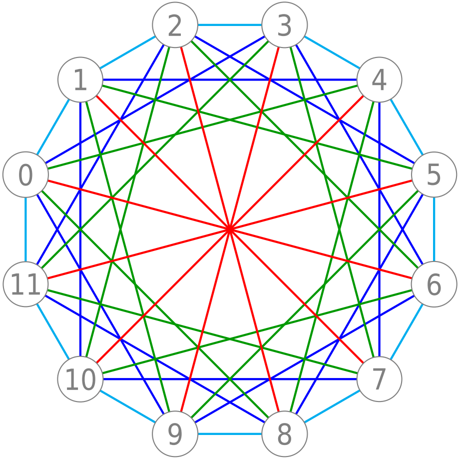
Definition **Code $\mathcal{C}[\Gamma, C_0]$**

Given Γ a n-RIM and $C_0 \subseteq \mathbb{F}^n$,

$$f \in \mathcal{C}[\Gamma, C_0] \iff \forall v,\ f(v,\cdot) \in C_0.$$

We'll only use $C_0 = \mathrm{RS}[n,k]$.

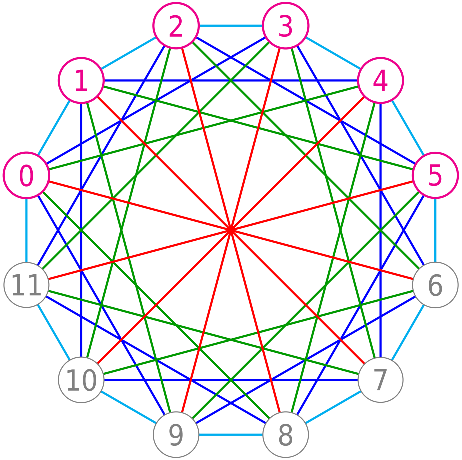
Cut-graph $\Gamma' = \mathrm{Cut}[\Gamma, V']$:



[DMR25] Hugo Delavenne, Tanguy Medevielle, and Élina Roussel. Interactive Oracle Proofs of Proximity to Codes on Graphs, 2025. **Submitted**

Cut-graph $\Gamma' = \mathrm{Cut}[\Gamma, V']$:

- Choose vertices $V' \subseteq V$



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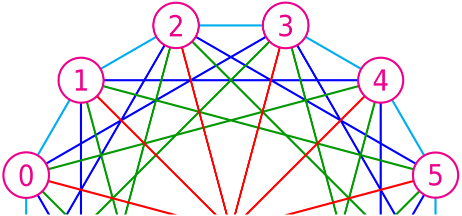
Cut-graph $\Gamma' = \mathrm{Cut}[\Gamma, V']$:

- Choose vertices $V' \subseteq V$
- Cut the rest



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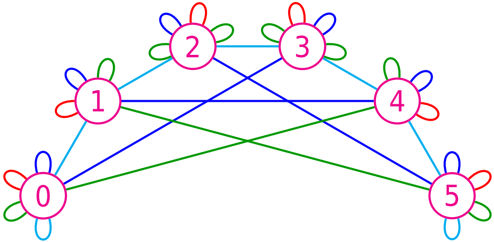
Cut-graph $\Gamma' = \text{Cut}[\Gamma, V']$:

- ▷ Choose vertices $V' \subseteq V$
- ▷ Cut the rest
- ▶ Enjoy your new graph

$$E_{V'}(v,\ell) = \begin{cases} E(v,\ell) & \text{if } E(v,\ell) \in V' \\ v & \text{otherwise} \end{cases}$$



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Definition **Graph isomorphism**

A bijection $\varphi : V' \to V''$ is an **isomorphism** $\Gamma' \to \Gamma''$ if

$$\forall (v', \ell) \in V' \times [n], \quad \varphi(E'(v', \ell)) = E''(\varphi(v'), \ell).$$

Definition **Flowering cut**

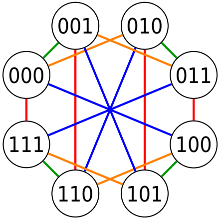
With $V'' = V \setminus V'$, if $\mathsf{Cut}[\Gamma, V'] \sim \mathsf{Cut}[\Gamma, V'']$, the cut is **flowering**.

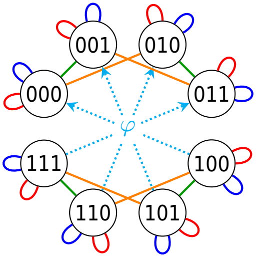
The cut-word $\mathsf{Cut}[f, V']$ is the restriction $f_{|V' \times [n]}$.

Definition **Folding**

For $\alpha \in \mathbb{F}$, $(v', \ell) \in V' \times [n]$,

$$\mathsf{Fold}[f, \alpha](v', \ell) := \mathsf{Cut}[f, V'](v, \ell) + \alpha \mathsf{Cut}[f, V''](\varphi(v'), \ell).$$





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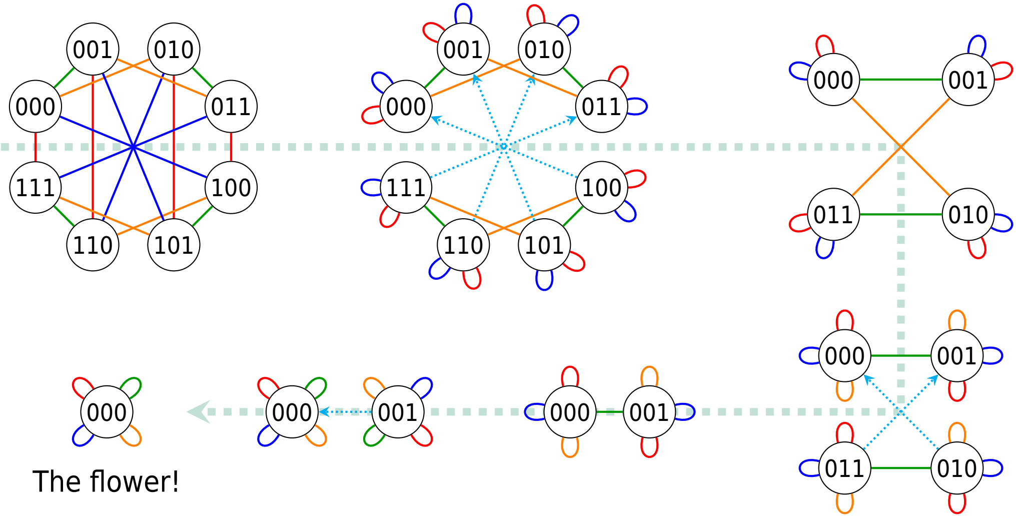
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The flower!

Flowering protocol

$$f_0 : V_0 \times [n] \to \mathbb{F}$$

$\mathcal{P}$ $\mathcal{V}$

$$\alpha_0 \xleftarrow{\$} \mathbb{F}$$

$$f_1 : V_1 \times [n] \to \mathbb{F}$$

$$f_1 = \mathsf{Fold}[f_0, \alpha_0]$$

$$\vdots$$

$$\alpha_{r-1} \xleftarrow{\$} \mathbb{F}$$

$$f_r : V_r \times [n] \to \mathbb{F}$$

$$f_r = \mathsf{Fold}[f_{r-1}, \alpha_{r-1}]$$

1 commit phase

$$v_0 \xleftarrow{\$} V_0,\ v_1 := \pi_1(v_0)$$

$$f_1(v_1, \cdot) \stackrel{?}{=} \mathsf{Fold}[f_0, \alpha_0](v_1, \cdot)$$

$$\vdots$$

$$v_r := \pi_r(v_{r-1})$$

$$f_r(v_r, \cdot) \stackrel{?}{=} \mathsf{Fold}[f_{r-1}, \alpha_{r-1}](v_r, \cdot)$$

$$f_r \stackrel{?}{\in} \mathsf{RS}[n, k]$$

repeat m times

2 query phase

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Proposition **Flowering complexities**

Flowering with m repetitions has complexities: (recall FRI)

- Prover complexity: $< 3N$ — $< 6N$
- Verifier complexity: $< 4mn \log N$ — $< 2m \log N$
- Number of queries: $< 2mn \log N$ — $< 2m \log K$

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Proposition **Flowering completeness**

If $f \in \mathcal{C}[\Gamma, \mathsf{RS}[n,k]]$ then $\mathcal{V}$ accepts with probability 1.

Theorem **Flowering soundness**

If $\Delta(f, \mathcal{C}[\Gamma, \mathsf{RS}[n,k]]) > \delta$ then for any $\tilde{\mathcal{P}}$ and $\varepsilon > 0$, $\mathcal{V}$ accepts with probability

$$\leqslant \frac{\log N}{\varepsilon q} + (1 - \delta + \varepsilon \log N)^m .$$

Recall for FRI: $$\frac{K^2 \log K}{(2\varepsilon)^7 q} + \left(1 - \min\left(\delta, 1 - \sqrt{K/N} - \varepsilon\right)\right)^m$$

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1 Interactive proximity tests

2 Flowering protocol

3 Flowering graphs

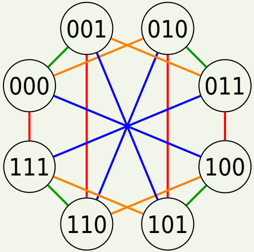
- Cayley graphs
- Our parameters
- Expander graphs
- Conclusion & future work

Definition **Cayley graph Cay(G, S)**

Fix $(G, \cdot)$. Let $S \subseteq G$ symmetric generating. Define $\Gamma = (G, E)$ where $E(g, s) = g{\cdot}s$.

Example

With $G = (\mathbb{F}_2^3, +)$ and $S = \{100, 010, 001, 111\}$,



We consider

- $G = (\mathbb{F}_2^r, +)$
- $S \subseteq G$ of size n

[Cay78] Arthur Cayley. Desiderata and Suggestions: No. 2. The Theory of Groups: Graphical Representation. ***American Journal of Mathematics*, 1(2):174–176, 1878**

Our parameters

We take

- $\Gamma = \mathsf{Cay}((\mathbb{F}_2^r, +), S)$
- S the columns of a parity check matrix of a $[n, n-r, d]_2$ binary code
- $C = \mathcal{C}[\Gamma, \mathsf{RS}[n,k]]$

Proposition **Parameters of the code**

- $N = n2^{r-1}$
- $\mathrm{rate}\, C \geqslant \frac{2k}{n} - 1$
- $\frac{1}{2}\delta \leqslant \Delta(C) \leqslant \delta$, with $\delta = \frac{1}{2^{r-d+1}}\left(1 - \frac{k-1}{n}\right) = \frac{n2^{d-2}}{N}\left(1 - \frac{k-1}{n}\right)$

If $d \ll r$ (i.e. far from MDS), δ is **terrible** ($O(1/N)$).

[DMR25] Hugo Delavenne, Tanguy Medevielle, and Élina Roussel. Interactive Oracle Proofs of Proximity to Codes on Graphs, 2025. **Submitted**

Lemma **Expansion bound [AC88]**

If Γ is λ-expander, with $\delta = 1 - \frac{k+1}{n}$,

$$\Delta(\mathcal{C}[\Gamma, \mathsf{RS}[n,k]]) \geqslant \delta(\delta - \lambda)$$

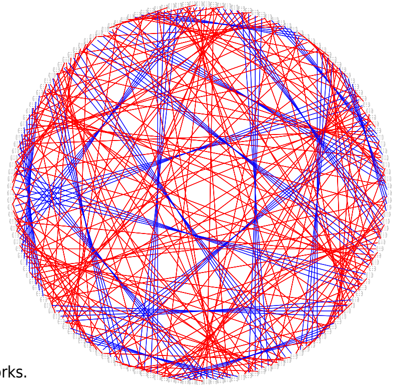
Thus **constant expansion** implies **constant minimal distance**.

Example **Expander graph family**

$\mathsf{Cay}(\mathrm{SL}_3(\mathbb{F}_p), S_p)$ has **constant expansion**.

We obtain $\mathsf{Cay}(\mathrm{SL}_3(\mathbb{F}_2), S_2)$ with

$$S_2 = \left\{ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^{\pm 1} \right\}$$



[AC88] Noga Alon and Fan Chung. Explicit construction of linear sized tolerant networks. ***Discrete Mathematics*, 72(1-3):15–19, 1988**

Competing parameters with FRI

- Better soundness
- Complexity could be improved

More possible cuts

- 2 cuts V', V'' $\quad\rightarrow\quad$ m cuts $V_1, V_2, ..., V_m$
- $V = V' \sqcup V''$ $\quad\rightarrow\quad$ $V = V_1 \cup V_2 \cup \cdots \cup V_m$