Flowering graphs

Interactive proximity test to codes on graphs by flowering

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1. Interactive Oracle Proofs of Proximity
2. Flowering protocol
3. Flowering graphs

1. Interactive Oracle Proofs of Proximity
2. Flowering protocol
3. Flowering graphs

- SNARKs
- Interactive Oracle Proof of Proximity
- Fast Reed-Solomon IOPP

Why verifiable computing: a sad story

- $\mathcal{P}$ has executed a very long algorithm $A : x \mapsto y$
- very proud, it wants to share y to $\mathcal{V}$
- $\mathcal{V}$ only trusts what it sees
- it can't accept $y = A(x)$ and doesn't want to compute it itself
- $\mathcal{P}$ is very sad
- #emotion

SNARK means **S**uccinct **N**on-interactive **AR**gument of **K**nowledge.

It turns $A : x \mapsto y$ (that runs in $\tau(|x|)$) into

$$A' : x, \alpha \mapsto y, \pi$$

input randomness output SNARK that "$A(x) = y$"

satisfying:

- $|\pi| \ll \tau$
- A' runs in $\tilde{O}(\tau)$
- there is a verifier $\mathcal{V}$ in $\mathrm{poly}(|\pi|)$ such that
  - **Completeness:** if $A(x) = y$ then $\underset{\alpha}{\mathbb{P}}(\mathcal{V}(\pi)$ accepts$) = 1$
  - **Soundness:** if $A(x) \neq y$ then $\underset{\alpha}{\mathbb{P}}(\mathcal{V}(\pi)$ accepts$) \leqslant s$.

Code-based SNARK workflow

A, x

$\mathcal{P}$rover

$\mathcal{V}$erifier

Arithmetize A

Arithmetize A

Executes $A(x)$ $\xrightarrow[\text{trace}]{\text{execution}}$

Interactive **proximity test**

Notation **Relative Hamming distance**

Let $u, v \in \mathbb{F}^N$.

$$\Delta(u,v) := \frac{1}{N} |\{i \in [N] \mid u_i \neq v_i\}|$$

Notation **Reed-Solomon code**

Let $\mathcal{L} \subseteq \mathbb{F}$ and $k < |\mathcal{L}| = N$.

$$\mathsf{RS}[\mathcal{L}, k] := \{f : \mathcal{L} \to \mathbb{F} \mid f \in \mathbb{F}[X]_{\leqslant k-1}\}$$

Assume $\langle g \rangle$ has order K. **We want to compute u_{K-1}** where $\begin{cases} u_0 = u_1 = 1 \\ u_{i+2} = u_{i+1}^2 + u_i^2. \end{cases}$

$T(1) = 1$	$T(g) = 1$	$\cdots$	$T(g^i)$	$T(g^{i+1})$	$T(g^{i+2})$	$\cdots$	$T(g^{K-1})$

$$T(g^{i+2}) \leftarrow T(g^{i+1})^2 + T(g^i)^2$$

With $Q(X, Y, Z) := Z - Y^2 - X^2$, the trace is valid iff

$$\forall i \in \{0, \ldots, K-3\}, \qquad \underbrace{Q(T(g^i), T(g^{i+1}), T(g^{i+2}))}_{=: Q \circ T(g^i)} = 0.$$

With $Z(X) := \prod_{i=0}^{K-3} (X - g^i)$, the trace is valid iff

$$\exists f(X) \in \mathbb{F}[X]_{\leqslant K}, \qquad Q \circ T(X) = f(X) Z(X).$$

Arithmetization: Why testing proximity suffices?

Lemma **Reduction to proximity testing**

Let $\boldsymbol{T}, \boldsymbol{f} : \mathcal{L} \to \mathbb{F}$.

Assume that:

- $\Delta(\boldsymbol{T}, \mathrm{RS}[\mathcal{L}, K]) < \delta$ and $\Delta(\boldsymbol{f}, \mathrm{RS}[\mathcal{L}, K+1]) < \delta$ (proximity tests!)
- For any $T' \in \mathbb{F}[X]_{\leqslant K-1}, f' \in \mathbb{F}[X]_{\leqslant K}$,

$$Q \circ T'(X) \neq f'(X)Z(X). \qquad \text{(no valid trace!)}$$

Then

$$\mathbb{P}_{x \in \mathcal{L}}(Q \circ \boldsymbol{T}(x) = \boldsymbol{f}(x)Z(x)) \leqslant \frac{2K}{|\mathcal{L}|} + 2\delta.$$

[Bor22] Sarah Bordage. *Efficient protocols for testing proximity to algebraic codes*.
Theses, Institut Polytechnique de Paris, June 2022

Definition **Locally-testable code**

C is **(ℓ, δ, s)-locally-testable** if there is $\mathcal{V}$ with **only ℓ queries** to u such that

- **Completeness:** if $u \in C$ then $\mathbb{P}(\mathcal{V}^u \text{ accepts}) = 1$
- **Soundness:** if $\Delta(u, C) > \delta$ then $\mathbb{P}(\mathcal{V}^u \text{ accepts}) \leqslant s$.

C has **locality ℓ** if C is (ℓ, δ, s)-locally-testable for $\delta, s < 1$.

Example **Reed-Solomon codes are not locally-testable**

RS$[\mathcal{L}, K]$ does **not** have locality $\ell \leqslant K$.

Proof.
Any K values correspond to a degree $\leqslant K - 1$ polynomial by interpolation.

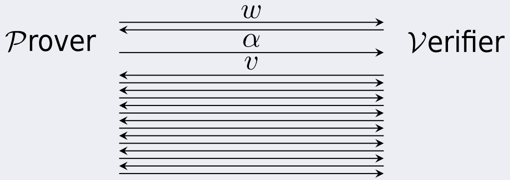
Definition **Non-interactive proof**

$$\mathcal{P}\text{rover} \xrightarrow{\quad w \quad} \mathcal{V}\text{erifier}$$

Definition **Sigma protocol**

$$\mathcal{P}\text{rover} \quad \begin{array}{c} \xrightarrow{\quad w \quad} \\ \xleftarrow{\quad \alpha \quad} \\ \xrightarrow[\quad v \quad]{} \end{array} \quad \mathcal{V}\text{erifier}$$

Definition **Interactive Proof**



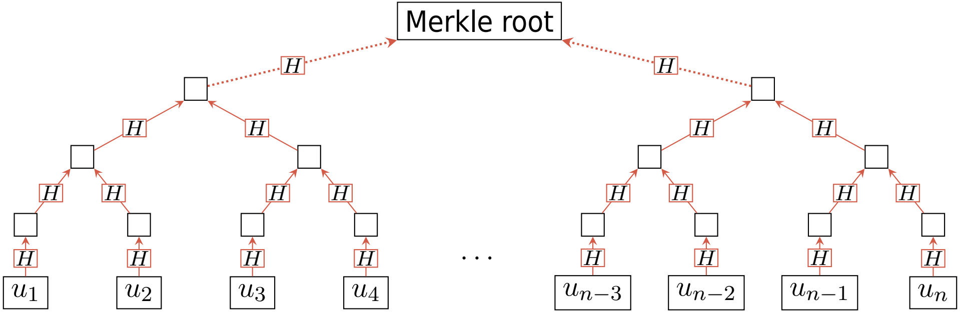
Remember: we want **S**uccinct **N**on-interactive **AR**guments of **K**nowledge.

- made non-interactive with **Fiat-Shamir**
- made succinct with **oracle accesses**

Definition **Oracle access**

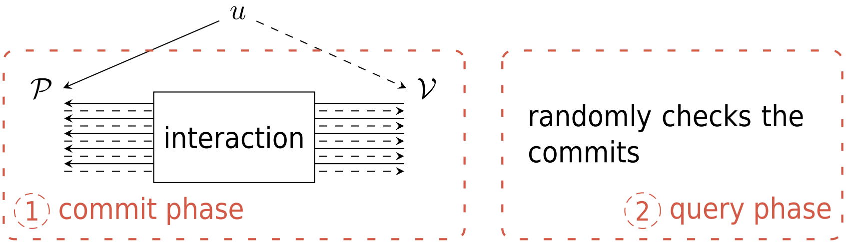
$\mathcal{P}$ provides to $\mathcal{V}$ an **oracle access** to $u \in \mathbb{F}^n$ by giving **black-box** access to u.

In practice, $\mathcal{P}$ provides the root of a **Merkle tree**.



Merkle root created in $O(n)$, Merkle branch opened in $O(\log n)$

Summary: IOPPs



- **Completeness:** if $u \in C$ then $\mathbb{P}(\mathcal{V}^{u,\leftrightarrow\mathcal{P}}$ accepts$) = 1$
- **Soundness:** if $\Delta(u, C) > \delta$ then for any $\mathcal{P}$, $\mathbb{P}(\mathcal{V}^{u,\leftrightarrow\mathcal{P}}$ accepts$) \leqslant s$

[BCS16] Eli Ben-Sasson, Alessandro Chiesa, and Nicholas Spooner. Interactive Oracle Proofs.
In *Theory of Cryptography: 14th International Conference, TCC 2016-B, Beijing, China, October 31-November 3, 2016, Proceedings, Part II 14*, pages 31–60. Springer, 2016

Idea: Test even and odd parts $f(X) =: f_{\text{even}}(X^2) + X f_{\text{odd}}(X^2)$.

Definition **Folding**

Let $f : \mathcal{L}_0 \to \mathbb{F}$ and $\alpha \in \mathbb{F}$.

$$\text{Fold}[f, \alpha](X^2) := f_{\text{even}}(X^2) + \alpha f_{\text{odd}}(X^2) = \frac{f(X) + f(-X)}{2} + \alpha \frac{f(X) - f(-X)}{2X}$$

- **Field restriction:** $\mathcal{L}_1 := \{x^2 \mid x, -x \in \mathcal{L}_0\}$ so $\mathbb{F}$ must have lots of roots of unity
- **Validity preservation:** $f \in \text{RS}[\mathcal{L}_0, K] \iff \mathbb{P}_{\alpha}(\text{Fold}[f, \alpha] \in \text{RS}[\mathcal{L}_1, K/2]) > \frac{1}{|\mathbb{F}|}$
- **Local check:** $\mathcal{V}$ computes $\text{Fold}[f, \alpha](x^2)$ with 2 queries to f

[BBHR18] Eli Ben-Sasson, Iddo Bentov, Yinon Horesh, and Michael Riabzev. Fast Reed-Solomon Interactive Oracle Proofs of Proximity. **In *45th international colloquium on automata, languages, and programming (ICALP 2018)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2018**

$$f_0 : \mathcal{L}_0 \to \mathbb{F}$$

① commit phase

$\mathcal{P}$ $\quad$ $\mathcal{V}$

$$\alpha_0 \xleftarrow{\$} \mathbb{F}$$

$$f_1 : \mathcal{L}_1 \to \mathbb{F}$$

$$f_1 = \mathsf{Fold}[f_0, \alpha_0]$$

$$\vdots$$

$$\alpha_{r-1} \xleftarrow{\$} \mathbb{F}$$

$$f_r : \mathcal{L}_r \to \mathbb{F}$$

$$f_r = \mathsf{Fold}[f_{r-1}, \alpha_{r-1}]$$

② query phase

$$x_0 \xleftarrow{\$} \mathcal{L}_0,\ x_1 := x_0^2$$

$$f_1(x_1) \stackrel{?}{=} \mathsf{Fold}[f_0, \alpha_0](x_1)$$

$$\vdots$$

$$x_r := x_{r-1}^2$$

$$f_r(x_r) \stackrel{?}{=} \mathsf{Fold}[f_{r-1}, \alpha_{r-1}](x_r)$$

repeat m times

$$f_r \stackrel{?}{\in} \mathsf{RS}[\mathcal{L}_r, K/2^r]$$

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Let $N := |\mathcal{L}_0|$.

Proposition **FRI complexities [BBHR18]**

FRI protocol for RS$[\mathcal{L}_0, K]$ with m repetitions has following complexity:

- Prover complexity: $< 8N$ (after encoding)
- Verifier complexity: $< 2m \log K$
- Number of queries: $2m \log K$
- Number of rounds: $\log K$

[BBHR18] Eli Ben-Sasson, Iddo Bentov, Yinon Horesh, and Michael Riabzev. Fast Reed-Solomon Interactive Oracle Proofs of Proximity. **In *45th international colloquium on automata, languages, and programming (ICALP 2018)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2018**

$$f_0 : \mathcal{L}_0 \to \mathbb{F}$$

$\mathcal{P}$ $\mathcal{V}$

$$\alpha_0 \xleftarrow{\$} \mathbb{F}$$

$$f_1 : \mathcal{L}_1 \to \mathbb{F}$$

$$f_1 = \mathsf{Fold}[f_0, \alpha_0]$$

$$\vdots$$

$$\alpha_{r-1} \xleftarrow{\$} \mathbb{F}$$

$$f_r : \mathcal{L}_r \to \mathbb{F}$$

$$f_r = \mathsf{Fold}[f_{r-1}, \alpha_{r-1}]$$

1 commit phase

A **commit error** is when

$$\Delta(\mathsf{Fold}[f_i, \alpha_i], C_{i+1}) < \Delta(f_i, C_i).$$

Proposition [BCI+23]

It happens w.p. over α_i

$$\leqslant \frac{10^7 N^{3.5}}{K^{1.5}|\mathbb{F}|}.$$

[BCI+23] Eli Ben-Sasson, Dan Carmon, Yuval Ishai, Swastik Kopparty, and Shubhangi Saraf. Proximity Gaps for Reed–Solomon Codes. ***J. ACM*, 70(5), October 2023**

A **query error** is when for some i

$$f_{i+1} \neq \mathsf{Fold}[f_i, \alpha_i]$$

but

$$f_{i+1}(x_{i+1}) = \mathsf{Fold}[f_i, \alpha_i](x_{i+1}).$$

Proposition [BCI+23]

If no commit error, it happens w.p.

$$\leqslant \left(1 - \min\left(\delta, 1 - \frac{21}{20}\sqrt{K/N}\right)\right)^m.$$

$$x_0 \xleftarrow{\$} \mathcal{L}_0,\ x_1 := x_0^2$$

$$f_1(x_1) \stackrel{?}{=} \mathsf{Fold}[f_0, \alpha_0](x_1)$$

$$\vdots$$

$$x_r := x_{r-1}^2$$

$$f_r(x_r) \stackrel{?}{=} \mathsf{Fold}[f_{r-1}, \alpha_{r-1}](x_r)$$

$$f_r \stackrel{?}{\in} \mathsf{RS}[\mathcal{L}_r, K/2^r]$$

repeat m times

2 query phase

[BCI+23] Eli Ben-Sasson, Dan Carmon, Yuval Ishai, Swastik Kopparty, and Shubhangi Saraf. Proximity Gaps for Reed–Solomon Codes. ***J. ACM*, 70(5), October 2023**

Proposition FRI completeness

If $f_0 \in \mathsf{RS}[\mathcal{L}_0, K]$ then $\mathcal{V}$ accepts with probability 1.

Theorem FRI soundness

If $\Delta(f_0, \mathsf{RS}[\mathcal{L}_0, K]) > \delta$ then for any $\tilde{\mathcal{P}}$ and $\varepsilon > 0$, $\mathcal{V}$ accepts with probability

$$\leqslant \frac{10^7 N^{3.5} \log K}{K^{1.5}|\mathbb{F}|} + \left(1 - \min\left(\delta, 1 - \frac{21}{20}\sqrt{K/N}\right)\right)^m \tag{1}$$

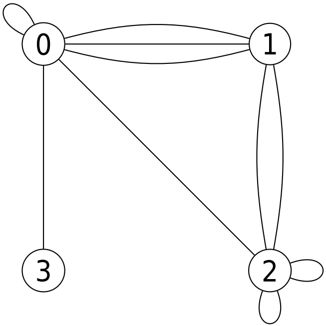
- $|\mathbb{F}| > 2^\lambda \cdot \frac{10^7 N^{3.5} \log K}{K^{1.5}}$, for security λ. Ex: extension of $\mathbb{F}_{2^{64}-2^{32}+1}$
- (1) does not take advantage of $\delta \in [1 - \sqrt{K/N}, 1 - K/N]$

[BCI+23] Eli Ben-Sasson, Dan Carmon, Yuval Ishai, Swastik Kopparty, and Shubhangi Saraf. Proximity Gaps for Reed–Solomon Codes. *J. ACM*, 70(5), October 2023

- Codes on graphs
- Folding graphs
- Flowering

$\Gamma = (V, E)$ is a n-RIM:

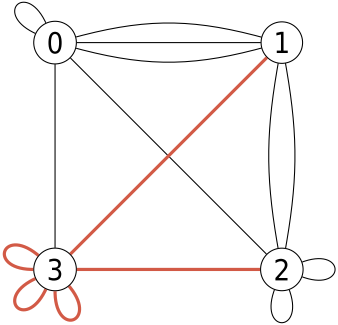
- **Multigraph:** multiple edges and loops



$\Gamma = (V, E)$ is a n-RIM:

▷ **Multigraph:** multiple edges and loops

▶ **Regular:** same number n of edges

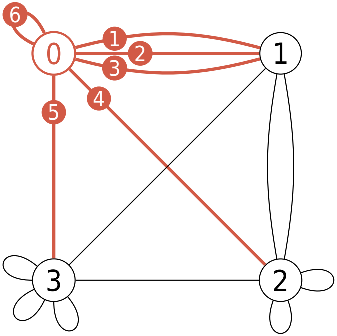


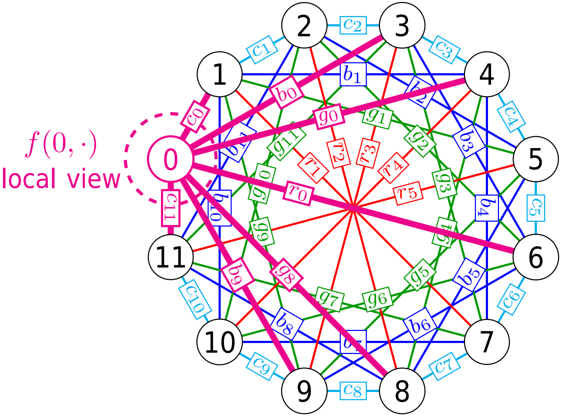
$\Gamma = (V, E)$ is a n-RIM:

▷ **Multigraph:** multiple edges and loops

▷ **Regular:** same number n of edges

▶ **Indexed:** edge is $(v, \ell) \in V \times [n]$
Write $\boldsymbol{E(v, \ell)} \in V$ the neighbor of v by ℓ





Word $f : V \times [n] \to \mathbb{F}$ on a graph Γ

Definition **Code $\mathcal{C}[\Gamma, C_0]$**

Given Γ a n-RIM and $C_0 \subseteq \mathbb{F}^n$,

$$f \in \mathcal{C}[\Gamma, C_0] \iff \forall v,\ f(v, \cdot) \in C_0.$$

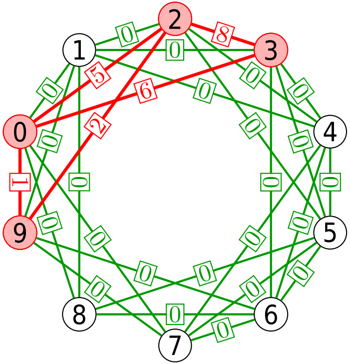
We'll only use $C_0 = \mathrm{RS}[n, k]$.

Example

For $v = 0$, being a codeword requires $(c_0, b_0, g_0, r_0, g_8, b_9, c_{11}) \in \mathrm{RS}[7, k]$.

Hamming: $\Delta(f, f') := \dfrac{\text{\#diff edges}}{\text{\#edges}} = \dfrac{5}{30} \approx 0.167$

Vertex: $\Delta_V(f, f') := \dfrac{\text{\#diff vertices}}{\text{\#vertices}} = \dfrac{4}{10} = 0.4$



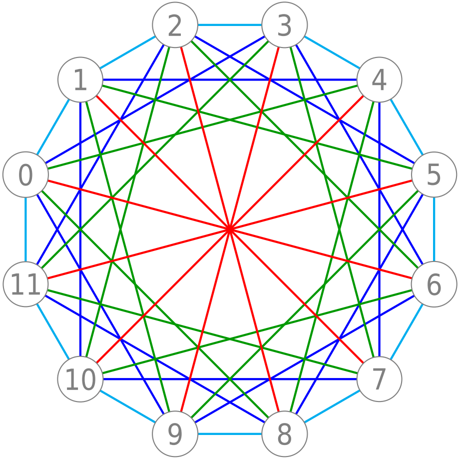
$f - f'$

Proposition **Hamming is more fine grain**

For any f, f', $\Delta(f, f') \leqslant \Delta_V(f, f')$.

[DMR25] Hugo Delavenne, Tanguy Medevielle, and Élina Roussel. Interactive Oracle Proofs of Proximity to Codes on Graphs, 2025. **Accepted to ISIT 2025**

Cut-graph $\mathbf{\Gamma' = Cut[\Gamma, \mathit{V'}]}$:



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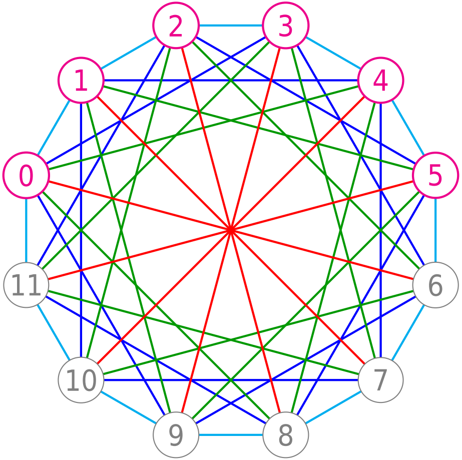
Cut-graph $\mathbf{\Gamma' = Cut[\Gamma, V']}$:

- Choose vertices $V' \subseteq V$



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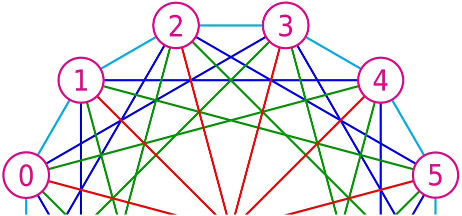
Cut-graph $\boldsymbol{\Gamma' = \mathbf{Cut}[\Gamma, V']}$:

- Choose vertices $V' \subseteq V$
- Cut the rest



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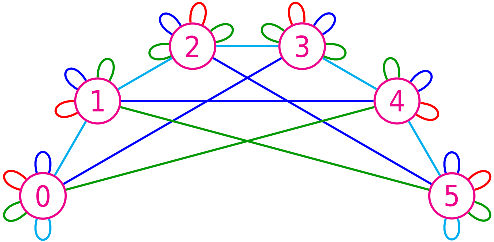
Cut-graph $\boldsymbol{\Gamma' = \mathbf{Cut}[\Gamma, V']}$:

- Choose vertices $V' \subseteq V$
- Cut the rest
- Enjoy your new graph

$$E_{\Gamma'}(v,\ell) = \begin{cases} E_\Gamma(v,\ell) & \text{if } E_\Gamma(v,\ell) \in V' \\ v & \text{otherwise} \end{cases}$$



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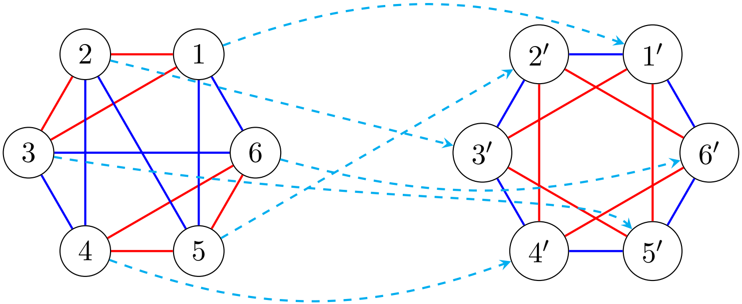
Definition **Graph isomorphism**

A bijection $\varphi : V \to V'$ is an **isomorphism** $\Gamma \to \Gamma'$ if

$$\forall (v, \ell) \in V \times [n], \quad \varphi(E(v, \ell)) = E'(\varphi(v), \ell).$$



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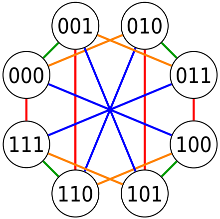
Definition **Flowering cut collection**

Let $V_0, \dots, V_{m-1} \subseteq V$, $\boldsymbol{V = \bigcup_{i=0}^{m} V_i}$.

The cuts are **flowering** if $(\mathrm{Cut}[\Gamma, V_i])_{i=0,\dots,m-1}$ are **isomorphic**.



The **cut-word** $\mathrm{Cut}[f, V_i]$ is the restriction $f_{|V_i \times [n]}$.

Definition **Folding**

For $\alpha \in \mathbb{F}$, $(v, \ell) \in V_0 \times [n]$, and $\varphi_i : \mathrm{Cut}[\Gamma, V_i] \xrightarrow{\sim} \mathrm{Cut}[\Gamma, V_0]$

$$\mathrm{Fold}[f, \alpha](v, \ell) := \sum_{i=0}^{m-1} \alpha^i \mathrm{Cut}[f, V_i](\varphi_i(v), \ell)$$

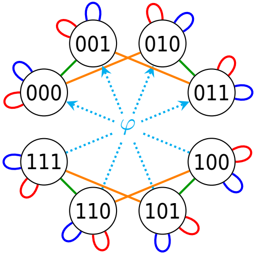


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Illustration: flowering cut collection

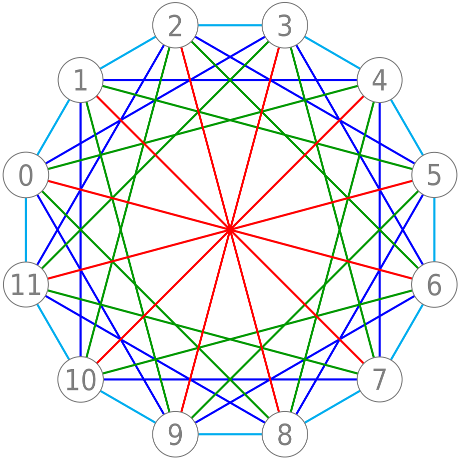
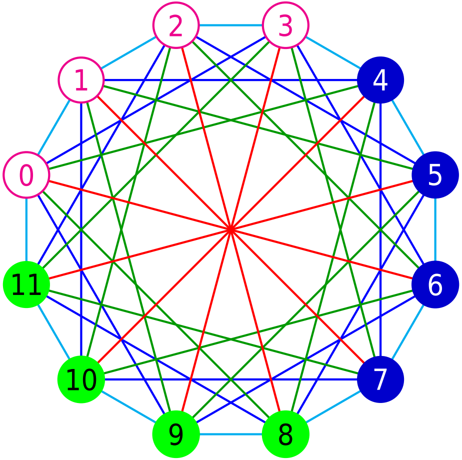
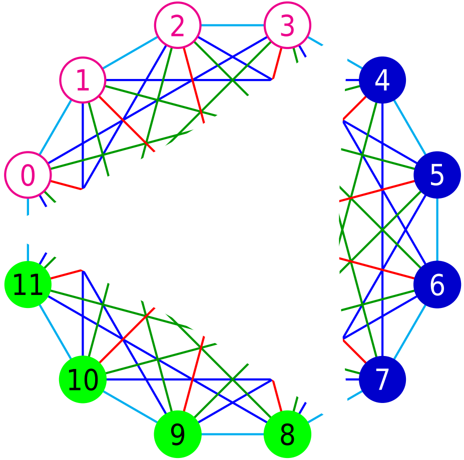


Illustration: flowering cut collection

- Choose 3 sets of vertices $V_0, V_1, V_2 \subseteq V$



- Choose 3 sets of vertices $V_0, V_1, V_2 \subseteq V$
- Cut outgoing edges



- Choose 3 sets of vertices $V_0, V_1, V_2 \subseteq V$
- Cut outgoing edges
- Get new graphs with petals

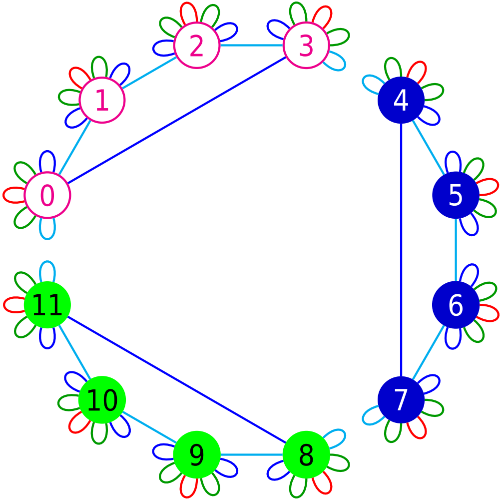
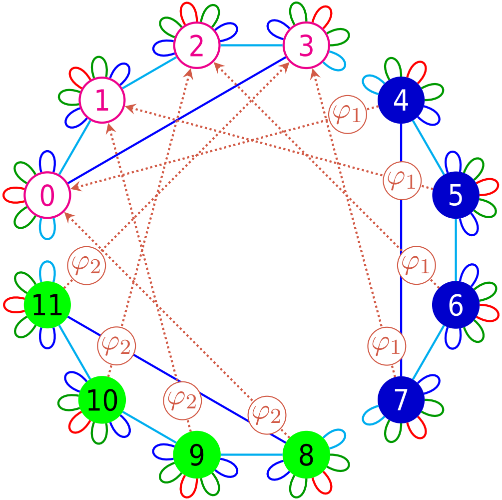
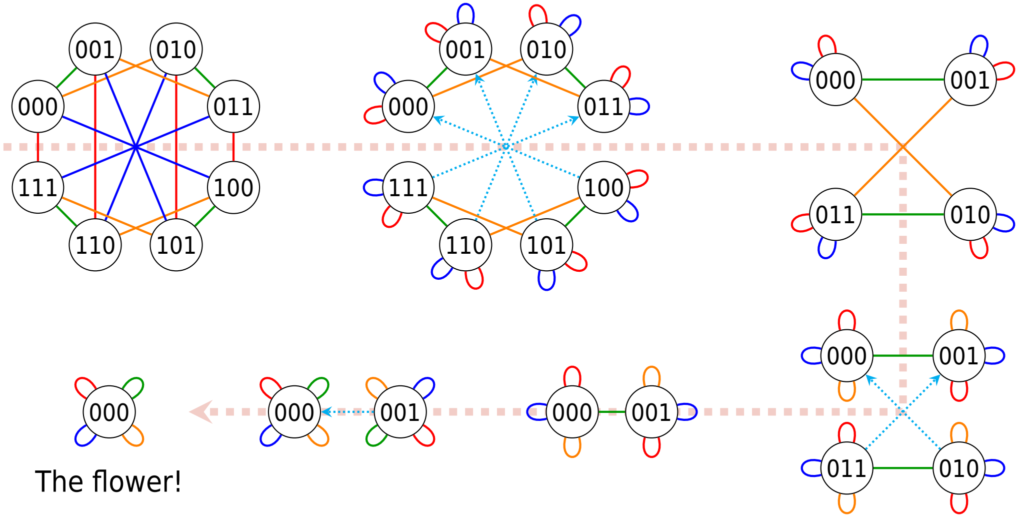


Illustration: flowering cut collection

- ▷ Choose 3 sets of vertices $V_0, V_1, V_2 \subseteq V$
- ▷ Cut outgoing edges
- ▷ Get new graphs with petals
- ▶ Define the the isomorphisms

$$\varphi_1 : i \mapsto i - 4 \qquad \varphi_2 : i \mapsto i - 8$$

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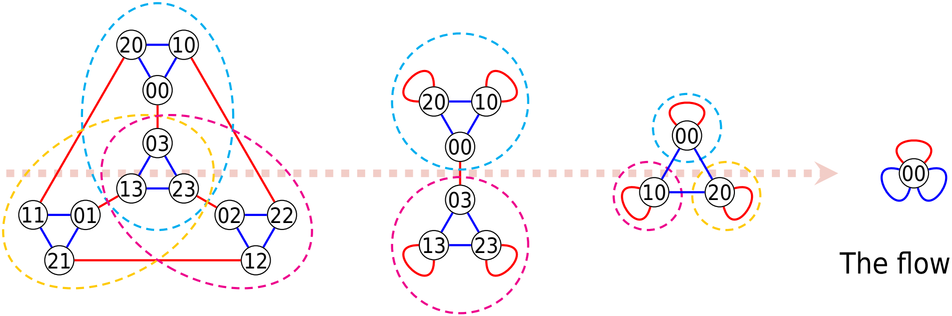
The flower!



The flower!

$$f_0 : V_0 \times [n] \to \mathbb{F}$$

$\mathcal{P}$ $\mathcal{V}$

$$\alpha_0 \xleftarrow{\$} \mathbb{F}$$

$$f_1 : V_1 \times [n] \to \mathbb{F}$$

$$f_1 = \mathsf{Fold}[f_0, \alpha_0]$$

$$\vdots$$

$$\alpha_{r-1} \xleftarrow{\$} \mathbb{F}$$

$$f_r : V_r \times [n] \to \mathbb{F}$$

$$f_r = \mathsf{Fold}[f_{r-1}, \alpha_{r-1}]$$

1 commit phase

$$v_0 \xleftarrow{\$} V_0,\ v_1 := \varphi_{1,i}(v_0)\ (v_0 \in V_{1,i})$$

$$f_1(v_1, \cdot) \stackrel{?}{=} \mathsf{Fold}[f_0, \alpha_0](v_1, \cdot)$$

$$\vdots$$

$$v_r := \varphi_{r,i}(v_{r-1})\ (v_{r-1} \in V_{r,i})$$

$$f_r(v_r, \cdot) \stackrel{?}{=} \mathsf{Fold}[f_{r-1}, \alpha_{r-1}](v_r, \cdot)$$

repeat m times

$$f_r \stackrel{?}{\in} \mathsf{RS}[n, k]$$

2 query phase

[DMR25] Hugo Delavenne, Tanguy Medevielle, and Élina Roussel. Interactive Oracle Proofs of Proximity to Codes on Graphs, 2025. **Accepted to ISIT 2025**

Flowering commit soundness

$f_0 : V_0 \times [n] \to \mathbb{F}$

$\mathcal{P}$ $\mathcal{V}$

$\alpha_0 \xleftarrow{\$} \mathbb{F}$

$f_1 : V_1 \times [n] \to \mathbb{F}$

$f_1 = \mathsf{Fold}[f_0, \alpha_0]$

$\vdots$

$\alpha_{r-1} \xleftarrow{\$} \mathbb{F}$

$f_r : V_r \times [n] \to \mathbb{F}$

$f_r = \mathsf{Fold}[f_{r-1}, \alpha_{r-1}]$

1 commit phase

A **commit error** is when

$$\Delta_V(\mathsf{Fold}[f_i, \alpha_i], C_{i+1}) < \Delta_V(f_i, C_i).$$

Proposition [DMR25]

It happens w.p. over α_i

$$\leqslant \frac{N}{|\mathbb{F}|}$$

[DMR25] Hugo Delavenne, Tanguy Medevielle, and Élina Roussel. Interactive Oracle Proofs of Proximity to Codes on Graphs, 2025. **Accepted to ISIT 2025**

A **query error** is when for some i

$$f_{i+1} \neq \mathsf{Fold}[f_i, \alpha_i]$$

but

$$f_{i+1}(v_{i+1}, \cdot) = \mathsf{Fold}[f_i, \alpha_i](v_{i+1}, \cdot)$$

Proposition [DMR25]

If no commit error, it happens w.p.

$$\leqslant (1-\delta)^m$$

$$v_0 \xleftarrow{\$} V_0,\ v_1 := \varphi_{1,i}(v_0)\ (v_0 \in V_{1,i})$$

$$f_1(v_1, \cdot) \stackrel{?}{=} \mathsf{Fold}[f_0, \alpha_0](v_1, \cdot)$$

$$\vdots$$

$$v_r := \varphi_{r,i}(v_{r-1})\ (v_{r-1} \in V_{r,i})$$

$$f_r(v_r, \cdot) \stackrel{?}{=} \mathsf{Fold}[f_{r-1}, \alpha_{r-1}](v_r, \cdot)$$

$$f_r \stackrel{?}{\in} \mathsf{RS}[n, k]$$

repeat m times

2 query phase

[DMR25] Hugo Delavenne, Tanguy Medevielle, and Élina Roussel. Interactive Oracle Proofs of Proximity to Codes on Graphs, 2025. **Accepted to ISIT 2025**

Proposition **Flowering completeness**

If $f \in \mathcal{C}[\Gamma, \mathsf{RS}[n,k]]$ then $\mathcal{V}$ accepts with probability 1.

Theorem **Flowering soundness**

If $\Delta(f, \mathcal{C}[\Gamma, \mathsf{RS}[n,k]]) > \delta$ then for any $\tilde{\mathcal{P}}$ and $\varepsilon > 0$, $\mathcal{V}$ accepts with probability

$$\leqslant \frac{Nr}{|\mathbb{F}|} + (1-\delta)^m$$

Recall FRI:

$$\frac{10^7 N^{3.5} \log K}{K^{1.5}|\mathbb{F}|} + \left(1 - \min\left(\delta, 1 - \frac{21}{20}\sqrt{K/N}\right)\right)^m$$

[DMR25] Hugo Delavenne, Tanguy Medevielle, and Élina Roussel. Interactive Oracle Proofs of Proximity to Codes on Graphs, 2025.
Accepted to ISIT 2025

Complexity comparison

For codes $[N, K]$: RS$[\mathcal{L}, K]$ or $\mathcal{C}[\Gamma, \text{RS}[n, k]]$

Protocol	FRI	[DMR25]	[DL25]
Prover	$< 8N$	$< 3N$	$< 5\kappa N \log N$
Verifier	$< 2m \log K$	$< 4mn \log N$	$< 8\kappa mn \log N$
Query	$< 2m \log K$	$2mn \log N$	$3\kappa mn \log N$
Rounds	$\log K$	$< \log N$	$< \kappa \log N$

Complexity depends on the **graphs used** and the **cutting strategy**.

[DMR25] Hugo Delavenne, Tanguy Medevielle, and Élina Roussel. Interactive Oracle Proofs of Proximity to Codes on Graphs, 2025.
Accepted to ISIT 2025

[DL25] Hugo Delavenne and Louise Lallemand. Codes on any Cayley graph have an Interactive Oracle Proof of Proximity, 2025.
Submitted

1 Interactive Oracle Proofs of Proximity

2 Flowering protocol

3 Flowering graphs

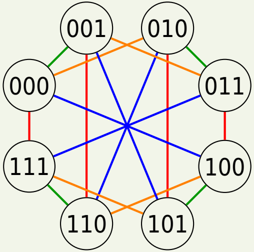
- First graphs [DMR25]
- Expander graphs [DL25]
- Current work

Definition **Cayley graph Cay(G, S)**

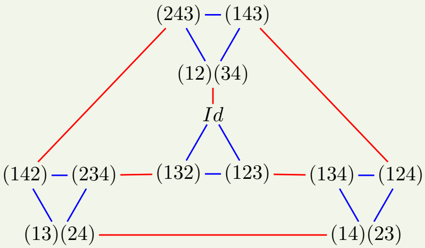
Let $(G, \cdot)$ and $S \subseteq G$ such that $\langle S \rangle = G$ and $S^{-1} = S$.
Define **Cay(G, S)** $= (G, E)$ by $E(g, s) = g \cdot s$.

Example

$G = (\mathbb{F}_2^3, +)$, $S = \{100, 010, 001, 111\}$

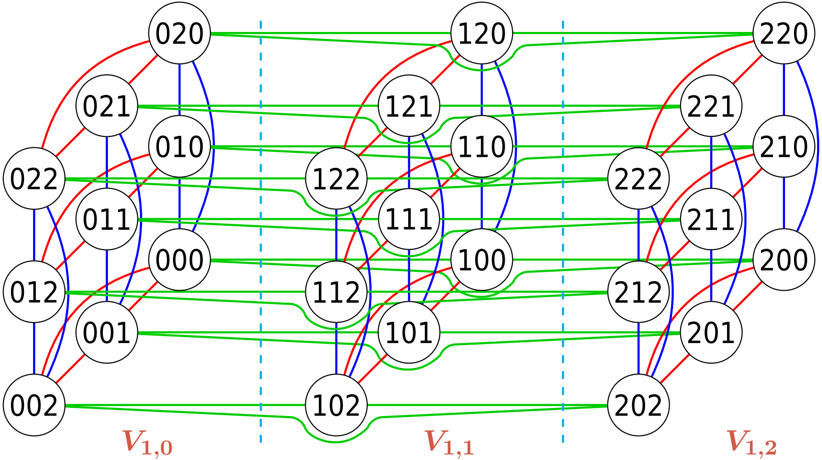


$G = \mathcal{A}_4$, $S = \{(123), (132), (12)(34)\}$



[Cay78] Arthur Cayley. Desiderata and Suggestions: No. 2. The Theory of Groups: Graphical Representation. ***American Journal of Mathematics*, 1(2):174–176, 1878**

We take $G = (\mathbb{F}_m^r, +)$ and $S \subseteq G$ of size n. The cuts are $V_{i,j} = \{j\}^i \times \mathbb{F}_m^{r-i}$



Let $G = (\mathbb{F}_2^r, +)$.

Using S the columns of a parity check matrix of a $[n, n-r, d]_2$ binary code

Proposition **Parameters of the code**

- $N = n2^{r-1}$
- $\text{rate}\, C \geqslant \frac{2k}{n} - 1$
- $\frac{1}{2}\delta \leqslant \Delta(C) \leqslant \delta$, with $\delta = \frac{1}{2^{r-d+1}}\left(1 - \frac{k-1}{n}\right) = \frac{n2^{d-2}}{N}\left(1 - \frac{k-1}{n}\right)$

If $d \ll r$ (i.e. far from MDS), δ is **terrible** ($O(1/N)$).

[DMR25] Hugo Delavenne, Tanguy Medevielle, and Élina Roussel. Interactive Oracle Proofs of Proximity to Codes on Graphs, 2025. **Accepted to ISIT 2025**

Definition **Graph expansion**

Let $\Gamma = (V, E)$ a n-regular graph and $A \in \{0,1\}^{|V|\times|V|}$ its adjacency matrix.
Let $\Lambda_1 \geqslant \Lambda_2 \geqslant \ldots \geqslant \Lambda_n \in \mathbb{R}$ be the eigenvalues of A.

Then Γ is **λ-expander** if $|\Lambda_i| \leqslant n\lambda$ for $i \geqslant 2$.

Lemma **Minimal distance expansion lower bound [AC88]**

If Γ is λ-expander, with $\delta = 1 - \frac{k+1}{n}$, $\mathcal{C}[\Gamma, \mathsf{RS}[n,k]]$ has minimal distance $\geqslant \delta(\delta - \lambda)$.

Thus if $(\Gamma_i)_{i\in\mathbb{N}}$ has **constant expansion**,

then $(\mathcal{C}[\Gamma_i, \mathsf{RS}[n_i, \rho n_i]])_{i\in\mathbb{N}}$ has **constant minimal distance**.

[AC88] Noga Alon and Fan Chung. Explicit construction of linear sized tolerant networks.
***Discrete Mathematics*, 72(1-3):15–19, 1988**

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Then Γ is **λ-expander** if $|\Lambda_i| \leqslant n\lambda$ for $i \geqslant 2$.

$\lambda \in [0, 1]$ characterizes random walk propagation in Γ.

Small λ means good expansion.

Lemma **Minimal distance expansion lower bound [AC88]**

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[AC88] Noga Alon and Fan Chung. Explicit construction of linear sized tolerant networks.
***Discrete Mathematics*, 72(1-3):15–19, 1988**

Example of expander graphs

Let $p \neq q$ be primes such that $p \equiv 1 \mod 4$ and $q \equiv 1 \mod 4$. Let

$$G_{p,q} := \begin{cases} \mathrm{PSL}_2(\mathbb{F}_q) & \text{if } p \text{ is a quadratic residue mod } q \\ \mathrm{PGL}_2(\mathbb{F}_q) & \text{otherwise} \end{cases}$$

There is a generating set $S_{p,q}$ of size $p+1$ such that $(\text{Cay}(G_{p,q}, S_{p,q}))_q$ is Ramanujan (optimal expander).

Let $S = \{s_0, \ldots, s_{n-1}\}$ and $G = \langle S \rangle$. Write $\tilde{s}_i := s_{i \bmod \tilde{n}}$.

Proposition

Let $\Gamma = \text{Cay}[G, S]$. There exists $r \leqslant n \operatorname{diam}(\Gamma)$ such that

$$G = \{\tilde{s}_0^{\alpha_0} \cdot \cdots \cdot \tilde{s}_{r-1}^{\alpha_{r-1}} \mid \alpha_0, \ldots, \alpha_{r-1} \in \{0, 1\}\}$$

At round i there are two cuts for $j \in \{0, 1\}$

$$V_{i,j} := \left\{\tilde{s}_i^j \cdot \tilde{s}_{i+1}^{\alpha_{i+1}} \cdots \tilde{s}_{r-1}^{\alpha_{r-1}} \mid \alpha_{i+1}, \ldots, \alpha_k \in \{0, 1\}\right\}$$

- $\boldsymbol{\Gamma_{i,1} \cong \Gamma_{i,0}}$ with $\varphi_i(g) = \tilde{s}_i^{-1} \cdot g$
- $\Gamma_{r,0}$ is a flower so there are $\boldsymbol{r}$ **rounds**

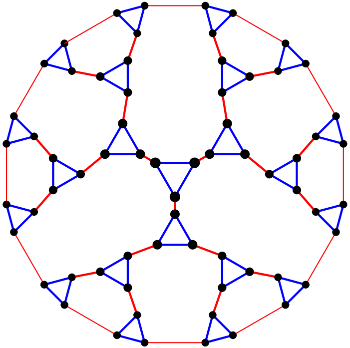
[DL25] Hugo Delavenne and Louise Lallemand. Codes on any Cayley graph have an Interactive Oracle Proof of Proximity, 2025. **Submitted**

Example

Let $S = \{b, b^{-1}, r\}$.

Assume that $G = \langle S \rangle$ is

$$G = \{b^{-1,0,1} r^{0,1} b^{-1,0,1} r^{0,1} b^{-1,0,1} r^{0,1} b^{-1,0,1}\}$$



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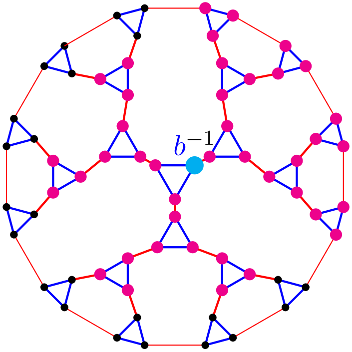
The first cuts are

$$\boldsymbol{V_{0,-1}} = \{b^{-1} \cdot r^{0,1} b^{-1,0,1} r^{0,1} b^{-1,0,1} r^{0,1} b^{-1,0,1}\}$$



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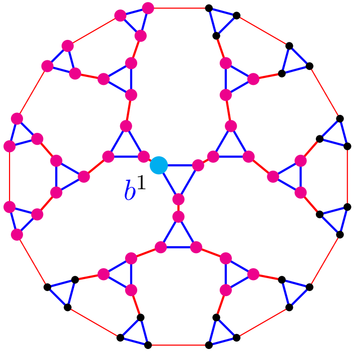
The first cuts are

$$\boldsymbol{V_{0,1}} = \{b^1 \cdot r^{0,1} b^{-1,0,1} r^{0,1} b^{-1,0,1} r^{0,1} b^{-1,0,1}\}$$



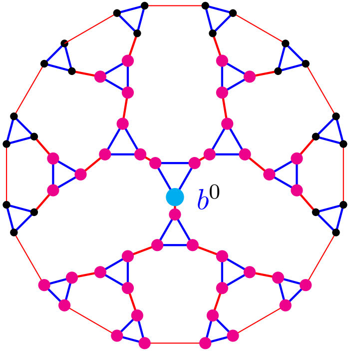
Let $S = \{b, b^{-1}, r\}$.

Assume that $G = \langle S \rangle$ is

$$G = \{b^{-1,0,1} r^{0,1} b^{-1,0,1} r^{0,1} b^{-1,0,1} r^{0,1} b^{-1,0,1}\}$$

The first cuts are

$$\boldsymbol{V_{0,0}} = \{b^0 \cdot r^{0,1} b^{-1,0,1} r^{0,1} b^{-1,0,1} r^{0,1} b^{-1,0,1}\}$$



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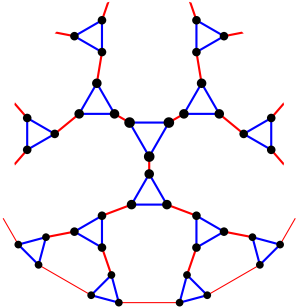

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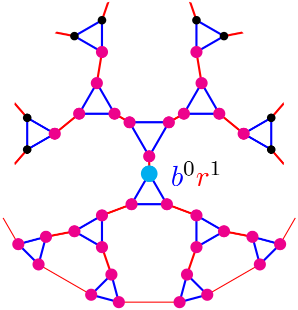
The second cuts are

$$\boldsymbol{V_{1,1}} = \{b^0 r^1 \cdot b^{-1,0,1} r^{0,1} b^{-1,0,1} r^{0,1} b^{-1,0,1}\}$$



Let $S = \{b, b^{-1}, r\}$.

Assume that $G = \langle S \rangle$ is

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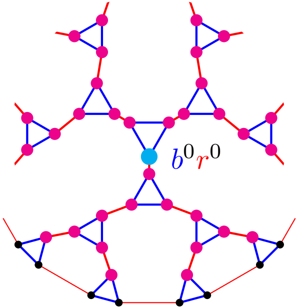
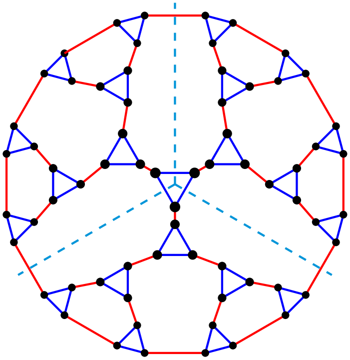
The second cuts are

$$V_{1,0} = \{b^0 r^0 \cdot b^{-1,0,1} r^{0,1} b^{-1,0,1} r^{0,1} b^{-1,0,1}\}$$

The cuts **overlap** a lot!

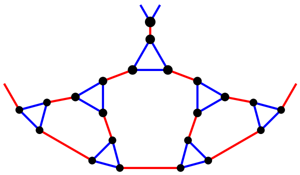


- This cut is defined with shortest paths.



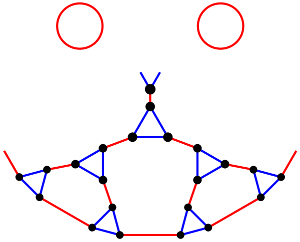
Why can’t we do the obvious cut?

- This cut is defined with shortest paths.
- Not suitable because
  - We lose **symmetries** and **algebraic properties**.
  - We still **can’t bound** the size of cuts.



Why can't we do the obvious cut?

- This cut is defined with shortest paths.
- Not suitable because
  - We lose **symmetries** and **algebraic properties**.
  - We still **can't bound** the size of cuts.
- Here we can still define new cuts :-)



Let $\Gamma = (V, E)$ and $v \in V$. Define the **ball** $B(v,d) := \{v' \in V \mid d_\Gamma(v',v) \leqslant d\}$.

Definition **Distance-d dominating set**

$M \subseteq V$ is **distance-d dominating** if $V = \bigcup_{v \in M} B(v,d)$.

Finding a small such set is $\mathbf{NP}$-hard... but expansion gives an **existence bound**!

Proposition

Assume Γ is Ramanujan of regularity n.
Then there is a a **distance-d dominating set** of size

$$\leqslant \frac{|V| \log |V|}{\left(\frac{3}{2} - \frac{1}{\sqrt{n}}\right)^d}$$

Let $\Gamma = \text{Cay}[G, S]$. Then

$$B(g,d) = \{g \cdot s_0 \cdots s_{d-1} \mid s_0, \ldots, s_{d-1} \in S \cup \{1_G\}\}$$

So $\boldsymbol{B(g_0, d) \cong B(g_1, d)}$ with $\varphi : g \mapsto g_1 g_0^{-1} \cdot g$.

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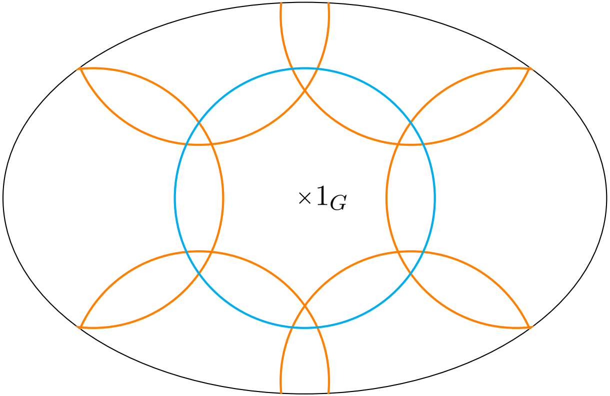
- Balls still lose properties



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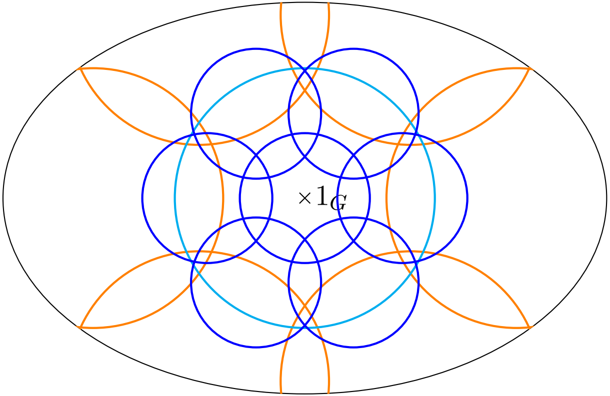
- Balls still lose properties
- Cover balls with smaller balls



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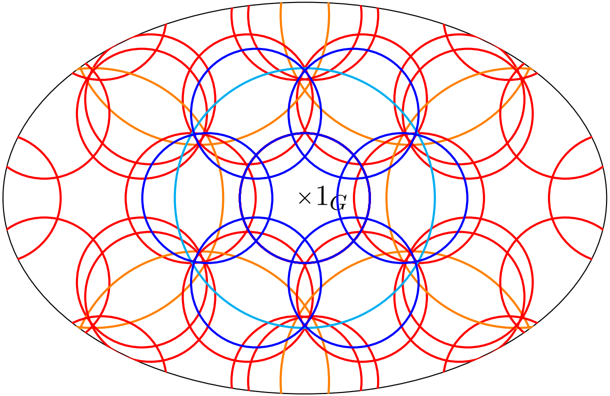
- Balls still lose properties
- Cover balls with smaller balls
- Looks like a mess



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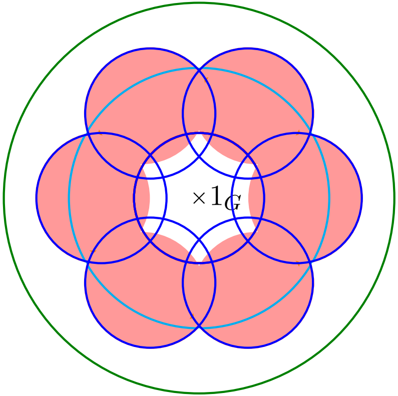
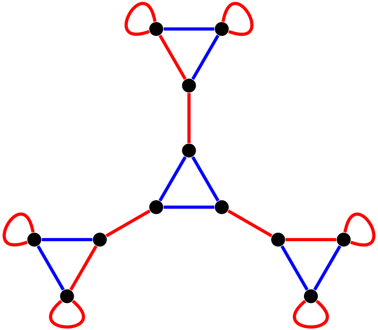
- Balls still lose properties
- Cover balls with smaller balls
- Looks like a mess
- ...but we can control overlap:

$$B(g, d_1) \subseteq \bigcup_i B(g_i, d_2) \subseteq B(g, d_1 + d_2)$$



Modify graphs to **get rid of good minimal distance** and obtain **linear encoding**.

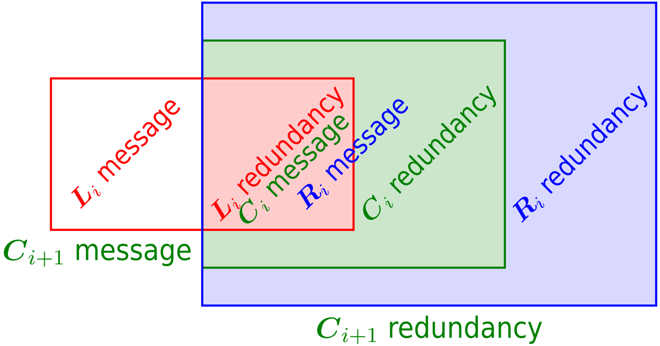
Here $\mathcal{C}[\Gamma, \mathbf{RS}[3, 2]]$ is linearly encodable



Message is in **blue**. Redundancy is in **red**.

[LM09] Jin Lu and José MF Moura. Linear time encoding of LDPC codes.
IEEE Transactions on Information Theory**, 56(1):233–249, 2009**

Let $(\boldsymbol{L}_i)_{i\in\mathbb{N}}$ and $(\boldsymbol{R}_i)_{i\in\mathbb{N}}$ be **systematic linear-time encodable** codes.
Build $\boldsymbol{C}_{i+1}$ to be **linear-time encodable** with good minimal distance:



Question: Can we test proximity on $\boldsymbol{L}_i$'s and $\boldsymbol{R}_i$'s to get proximity on $\boldsymbol{C}_i$'s?

[Spi95] Daniel A Spielman. Linear-time encodable and decodable error-correcting codes.
In *Proceedings of the twenty-seventh annual ACM symposium on Theory of computing*, pages 388–397, 1995

Competing parameters with FRI

- Better soundness
- Less constraint on the field
- Complexity to be improved

Make this actually useful

- Arithmetize circuits to graphs: colored De Bruijn graphs
- Adapt Spielman's construction to have a linear time encoding