Mixed Integer Non Linear Optimization: Methods and Applications

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Branch and Bound

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Branch and Bound

Based on **upper and lower bounds** on the optimal solution value and on **branching** which divide iteration after iteration the feasible region in **smaller subproblems**. A. H. Land & A. G. Doig (1960).

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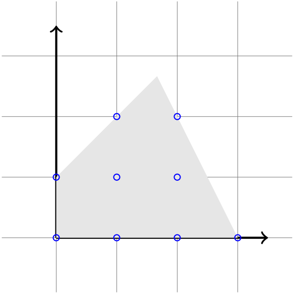
    - Otherwise continue

Branch and Bound: example

$$
\begin{aligned}
\min -x_1 - x_2 & & \\
2x_1 + x_2 & \leq & 6 \\
-x_1 + x_2 & \leq & 1 \\
x_1, x_2 & \geq & 0 \\
x_1, x_2 & & \text{integer}
\end{aligned}
$$



$L^0 = \lceil -\frac{13}{3} \rceil = -4$

P_0

Branch and Bound: example

$x^* = (\frac{5}{3}, \frac{8}{3})$, $c^\top x^* = -\frac{13}{3}$

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Branch on x_1:

- Subproblem P_1: $P_0 \cap \{x \mid x_1 \leq \lfloor \frac{5}{3} \rfloor = 1\}$

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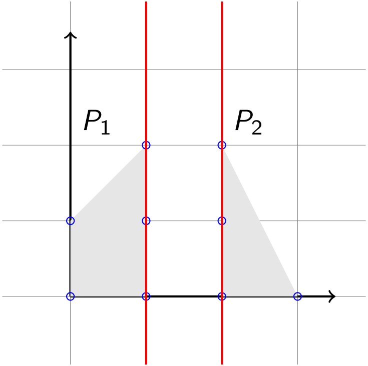
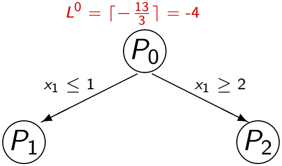
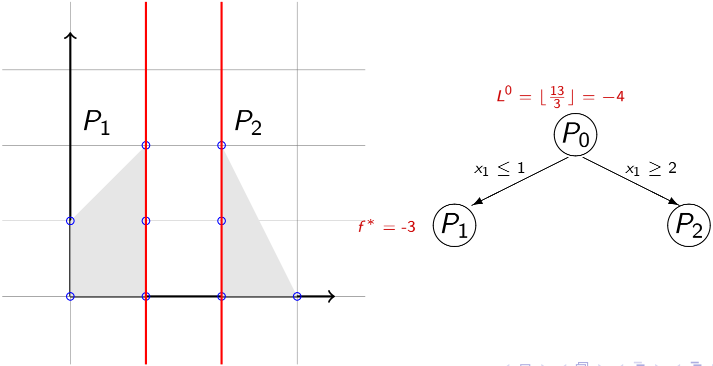




Branch and Bound: example

Explore P_1: optimal solution $(1, 2)$ of value -3. No further branching, upper bound $x^{\text{UB}} = (1, 2)$.



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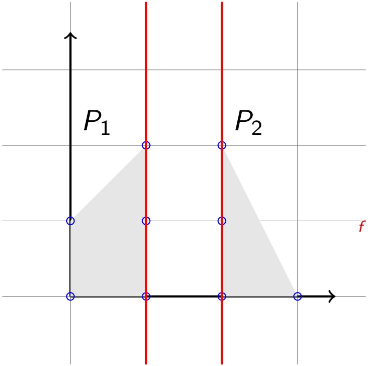
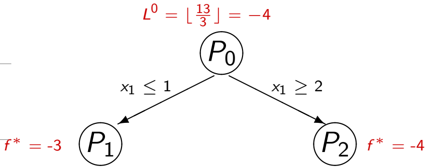
Explore P_1: optimal solution $(1, 2)$ of value -3. No further branching, upper bound $x^{\text{UB}} = (1, 2)$.

Explore P_2: optimal solution is $(2, 2)$ of value -4. No further branching, upper bound $x^{\text{UB}} = (2, 2)$.





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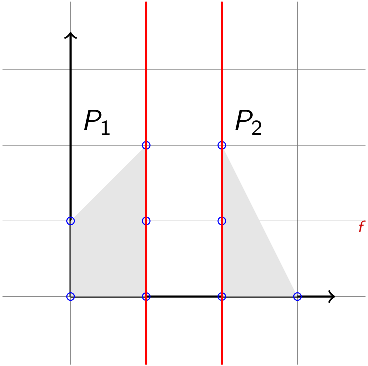
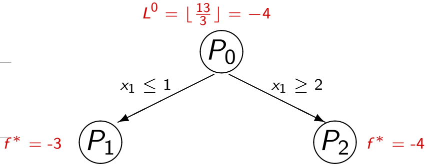
No subproblems left to explore $\rightarrow$ optimal solution $(2, 2)$ of value -4.





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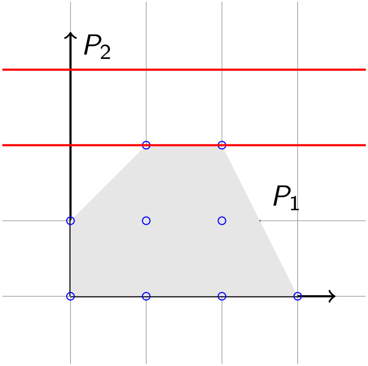

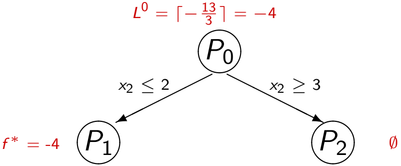

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end while
return x^*