Mixed Integer Non Linear Optimization: Methods and Applications

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Cutting Planes

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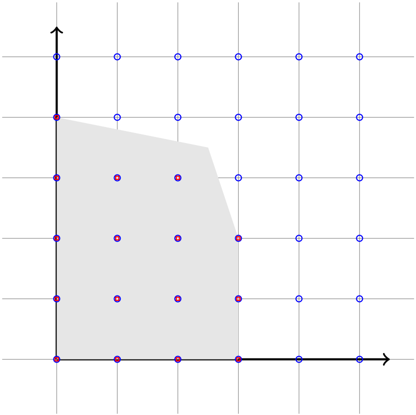
MILP Methods



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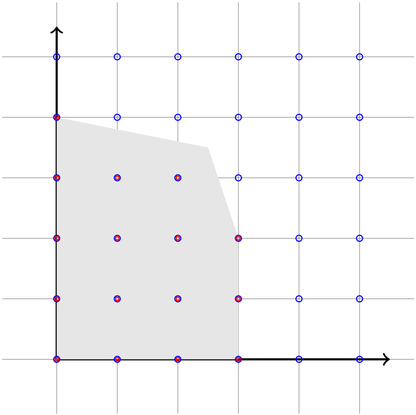

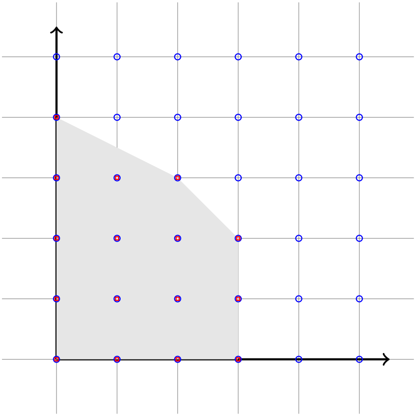

MILP Methods

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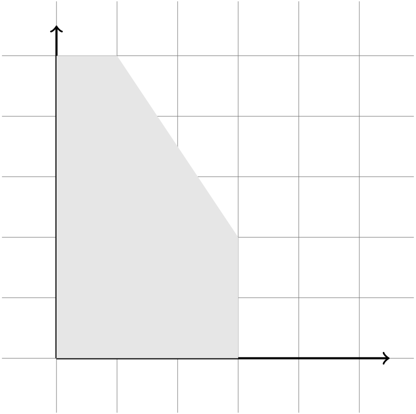
The ideal formulation is usually **very difficult** to find or can include an **exponential** number of constraints.

A few definitions



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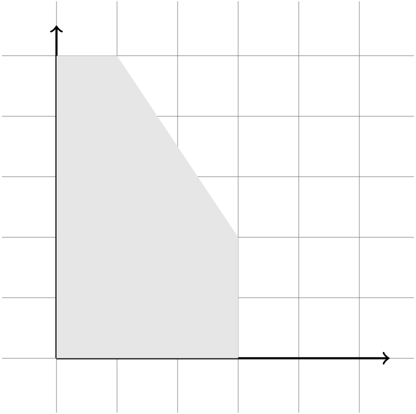

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- Face : subset of polyhedron s.t. $F = P$ or $F = P \cap H$ where H is some supporting hyperplane

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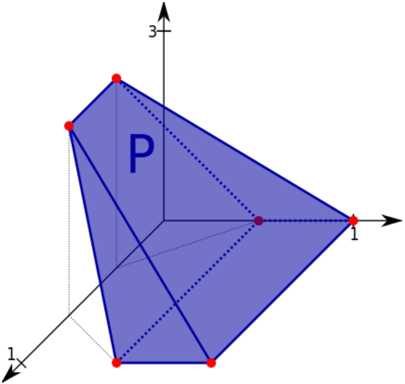

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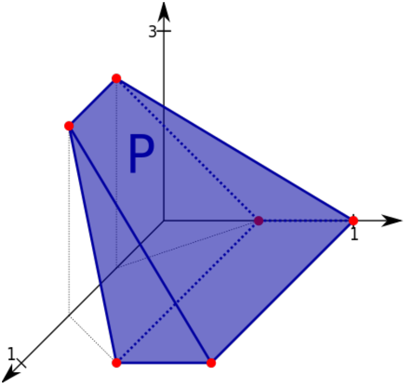

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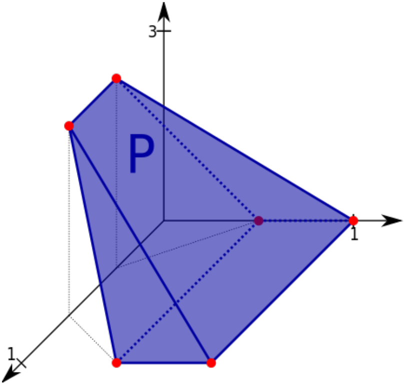

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Cutting plane

Based on **continuous relaxation strengthening** through valid and non trivial inequalities which **cut** iteration after iteration part of the feasible region of the relaxation (but no feasible point of the MILP problems). R. E. Gomory (1958).

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General-purpose solvers and the cuts added are of several types but all of them are generic.

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Example of generic separation problem and cuts: **Chvátal Inequalities**.

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$\rightarrow$ CP with Chvátal inequalities is an exact method for solving (M)ILPs.

Chvátal Inequalities: example

Let us consider the following IP:

$$
\begin{aligned}
\max x_1 + x_2 & & \\
2x_1 + x_2 &\leq & 6 \\
-x_1 + x_2 &\leq & 1 \\
x_1, x_2 &\geq & 0 \\
x_1, x_2 & & \text{integer}
\end{aligned}
$$

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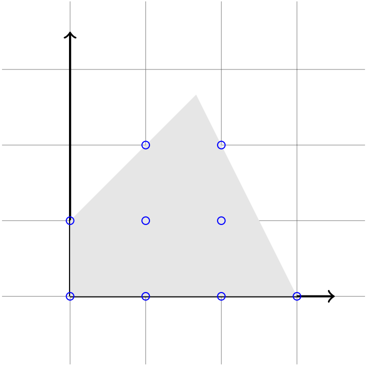

Chvátal Inequalities: example