Mixed Integer Non Linear Optimization: Methods and Applications
–
Mixed Integer Linear Programming: Reformulations, Relaxations, Restrictions

Claudia D'Ambrosio
dambrosio@lix.polytechnique.fr

Mixed Integer Linear Programming

$$
\begin{aligned}
\min_x \; & c^\top x \\
Ax &\leq b \\
\underline{x} \leq x &\leq \overline{x} \\
x_j &\in \mathbb{Z} \qquad \forall j \in Z
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where

- x is an n-dimensional vector of the decision variables,

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- c is the cost vector, A the constraints matrix, and b the right-hand-side vector,
- the set Z includes the indexes of the integer variables.

(Mixed) Integer Linear Programming

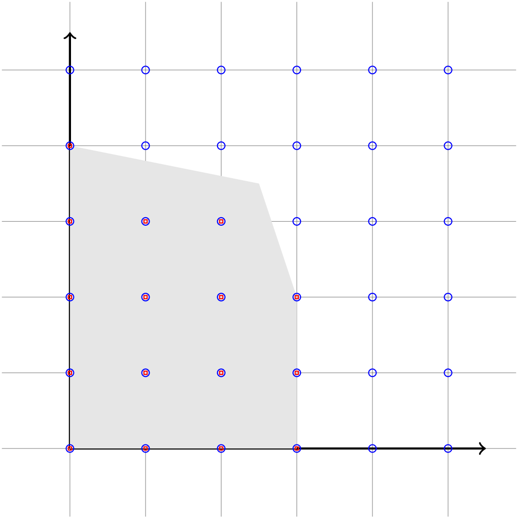

Figure: Lattice points (in blue), feasible region of the continuous relaxation (in gray), and their intersection (in red).

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- Reformulating a problem is interest when
  - The reformulation shows nicer **mathematical properties**
  - The reformulation is more **tractable**

Reformulation Examples: equivalent forms of LPs

General form

$$
\begin{aligned}
\min c^\top x & & & \\
a_i^\top x &= b_i & i &\in M \\
a_i^\top x &\geq b_i & i &\in \overline{M} \\
x_j &\geq 0 & j &\in N \\
x_j &\lesseqqgtr 0 & j &\in \overline{N}
\end{aligned}
$$

Canonical form

$$
\begin{aligned}
\min c^\top x & \\
Ax &\geq b \\
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Standard form

$$
\begin{aligned}
\min c^\top x & \\
Ax &= b \\
x &\geq 0
\end{aligned}
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$$x_j \overset{\leq}{\underset{>}{=}} 0 \quad \rightarrow \quad \begin{cases} x_j = x_j^+ - x_j^- \\ x_j^+ \geq 0 \\ x_j^- \geq 0 \end{cases}$$

Relaxation

$$
\left.\begin{array}{rrcll}
\min\limits_{y\in\mathbb{R}^q} & \bar{f}(y) & & & \\
 & \bar{g}_i(y) & \leq & 0 & \forall i = 1,\ldots,r \\
\underline{y} \leq & y & \leq & \overline{y} & \\
 & y_j & & \text{integer} & \forall j \in W
\end{array}\right\}
$$

where $W \subseteq Z$,
$q \geq n$,
$\bar{f}(y) \leq f(x)$ for all $x \subseteq y$,
and $\{x \mid g(x) \leq 0, \underline{x} \leq x \leq \overline{x}\} \subseteq \mathrm{Proj}_x\{y \mid \bar{g}(y) \leq 0, \underline{y} \leq y \leq \overline{y}\}$.

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- **continuous**: when $W = \emptyset$, $q = n$, $\overline{f}(y) = f(x)$, $\overline{g}(y) = g(x)$, $\underline{y} = \underline{x}$, $\overline{y} = \overline{x}$

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- **linear**: obtained by defining $\bar{f}(y) = c^\top y$ and $\bar{g}(y) = Ay - b$

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- **linear**: obtained by defining $\bar{f}(y) = c^\top y$ and $\bar{g}(y) = Ay - b$
- **convex**: obtained by defining $\bar{f}$ and $\bar{g}$ to be convex functions

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- The feasible region of the restriction is a **subset of the feasible region** of the original problem (when mapped in the same space).
- The restrictions are useful to obtain an **upper bound** on the optimal value (feasible solutions) of the original problem.

Restriction: example

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\begin{aligned}
\max x_1 + 2x_2 + 10x_3 \\
x_1 + x_2 \leq 4 \\
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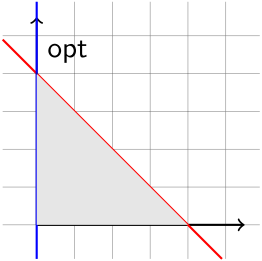

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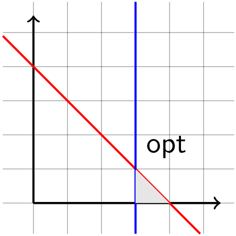

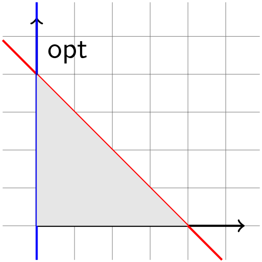

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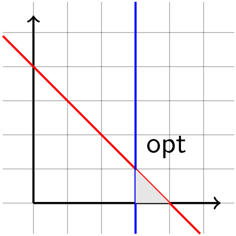

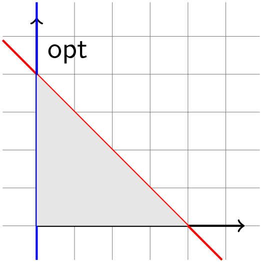

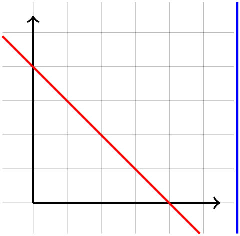

Complete enumeration?

Any purely binary program can be solved by considering all the 2^n potential solutions.

As n grows, the time needed to compute all the 2^n potential solutions grows exponentially in n.

n	2^n
10	1,024
100	1.26765060022823e+30
1,000	1.07150860718627e+301

Not an applicable approach in practice.

Which methods are used in practice?

Main ingredients

Ingredients for solving MILPs:

- Lower bound(s)
- Upper bound(s)

If LB $=$ UB, then we found an optimal solution of the (M)ILP.

Otherwise: improve LB and UB.

We focus on how to improve the LB.