Mixed Integer Non Linear Optimization: Methods and Applications

–

Mixed Integer Linear Programming

Claudia D'Ambrosio
dambrosio@lix.polytechnique.fr

Mixed Integer Linear Programming

$$
\begin{aligned}
\min_x c^\top x & \\
Ax \leq b & \\
\underline{x} \leq x \leq \overline{x} & \\
x_j \in \mathbb{Z} & \qquad \forall j \in Z
\end{aligned}
$$

where

- x is an n-dimensional vector of the decision variables,

Mixed Integer Linear Programming

$$\begin{aligned} \min_x \; & c^\top x \\ Ax &\leq b \\ \underline{x} \leq x &\leq \overline{x} \\ x_j &\in \mathbb{Z} \qquad\qquad \forall j \in Z \end{aligned}$$

where

- x is an *n*-dimensional vector of the decision variables,
- $\underline{x}$ and $\overline{x}$ are the given vectors of lower and upper bounds on the variables,

Mixed Integer Linear Programming

$$\begin{aligned} \min_x \; & c^\top x \\ Ax &\leq b \\ \underline{x} \leq x &\leq \overline{x} \\ x_j &\in \mathbb{Z} \qquad \forall j \in Z \end{aligned}$$

where

- x is an n-dimensional vector of the decision variables,
- $\underline{x}$ and $\overline{x}$ are the given vectors of lower and upper bounds on the variables,
- c is the cost vector, A the constraints matrix, and b the right-hand-side vector,

Mixed Integer Linear Programming

$$
\begin{aligned}
\min_{x} c^{\top} x & \\
Ax \leq b & \\
\underline{x} \leq x \leq \overline{x} & \\
x_j \in \mathbb{Z} & \qquad \forall j \in Z
\end{aligned}
$$

where

- x is an n-dimensional vector of the decision variables,
- $\underline{x}$ and $\overline{x}$ are the given vectors of lower and upper bounds on the variables,
- c is the cost vector, A the constraints matrix, and b the right-hand-side vector,
- the set Z includes the indexes of the integer variables.

(Mixed) Integer Linear Programming

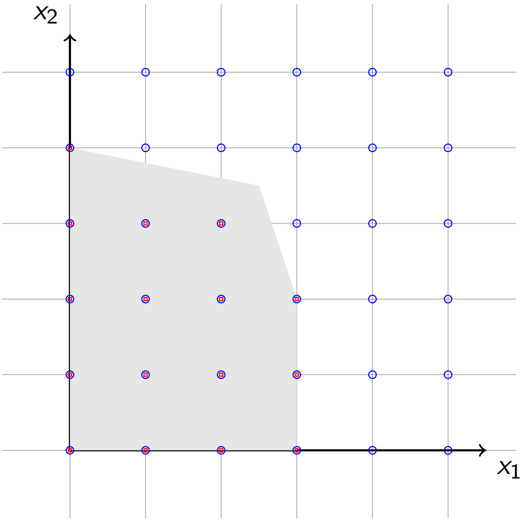


Figure: Lattice points (in blue), feasible region of the continuous relaxation (in gray), and their intersection (in red).

Applications

- **Finance**, e.g., robust portfolio selection
- **Power systems** , e.g., unit commitment, optimal power flow
- **Air traffic management** , e.g., aircraft conflicts detection and resolution
- **Transportation** , e.g., vehicle routing problem
- **etc.**

A motivating example: the (less simplified) Unit Commitment

$$\min_{p} \sum_{t \in T} \sum_{j \in J} c_j p_{jt} \tag{1}$$

$$\sum_{j \in J} p_{jt} = D_t \qquad \forall t \in T \tag{2}$$

$$0 \leq p_{jt} \leq \overline{p}_j \qquad \forall j \in J, t \in T \tag{3}$$

where (4) is the objective function minimizing the cost for producing electricity, (5) is the set of constraints satisfying the demand D_t at each time period $t \in T$, and (6) are the simple bounds on the electricity production by each unit j at each time period j.

A motivating example: the (less simplified) Unit Commitment

$$\min_{p} \sum_{t\in T} \sum_{j\in J} c_j p_{jt} \tag{1}$$

$$\sum_{j\in J} p_{jt} = D_t \qquad \forall t \in T \tag{2}$$

$$0 \leq p_{jt} \leq \overline{p}_j \qquad \forall j \in J, t \in T \tag{3}$$

where (4) is the objective function minimizing the cost for producing electricity, (5) is the set of constraints satisfying the demand D_t at each time period $t \in T$, and (6) are the simple bounds on the electricity production by each unit j at each time period j.

The production of unit $\tilde{\jmath}$ can only be either 0 or within $[p^l_{\tilde{\jmath}}, p^u_{\tilde{\jmath}}]$

A motivating example: the (less simplified) Unit Commitment

$$\min_p \sum_{t \in T} \sum_{j \in J} c_j p_{jt} \tag{1}$$

$$\sum_{j \in J} p_{jt} = D_t \qquad \forall t \in T \tag{2}$$

$$0 \leq p_{jt} \leq \overline{p}_j \qquad \forall j \in J, t \in T \tag{3}$$

where (4) is the objective function minimizing the cost for producing electricity, (5) is the set of constraints satisfying the demand D_t at each time period $t \in T$, and (6) are the simple bounds on the electricity production by each unit j at each time period j.

The production of unit $\tilde{\jmath}$ can only be either 0 or within $[p^l_{\tilde{\jmath}}, p^u_{\tilde{\jmath}}]$

The production of unit $\hat{\jmath}$ can be only in $\{0, p^1_{\hat{\jmath}}, p^2_{\hat{\jmath}}, \ldots, p^k_{\hat{\jmath}}\}$

A motivating example: the (less simplified) Unit Commitment

$$\min_p \sum_{t \in T} \sum_{j \in J} c_j p_{jt} \tag{4}$$

$$\sum_{j \in J} p_{jt} = D_t \qquad \forall t \in T \tag{5}$$

$$0 \leq p_{jt} \leq \overline{p}_j \qquad \forall j \in J, t \in T \tag{6}$$

where (4) is the objective function minimizing the cost for producing electricity, (5) is the set of constraints satisfying the demand D_t at each time period $t \in T$, and (6) are the simple bounds on the electricity production by each unit j at each time period j.

Motivation

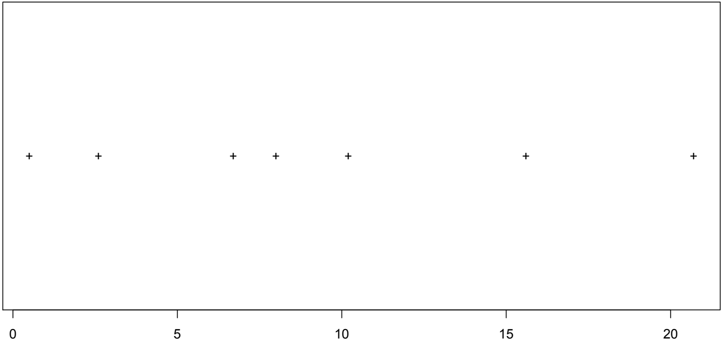
- **Discrete domain** of one (or more) variables
- Domain **discontinuity**
- **Conditional** constraints
- **Fixed cost**
- **Disjunctive** constraints
- **Absolute value** of a variable

Motivation

- **Discrete domain** of one (or more) variables
- Domain discontinuity
- Conditional constraints
- Fixed cost
- Disjunctive constraints
- Absolute value of a variable

Examples of modeling with integer/binary variables: Discrete domain of one (or more) variables

How to model the condition: $x \in \mathbb{R}$ and $x \in \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{\tilde{k}}\}$ where $\tilde{x}_k \in \mathbb{R}$ for $k = 1, \ldots, \tilde{k}$ within a MILP?



Examples of modeling with integer/binary variables: Discrete domain of one (or more) variables

How to model the condition: $x \in \mathbb{R}$ and $x \in \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{\tilde{k}}\}$ where $\tilde{x}_k \in \mathbb{R}$ for $k = 1, \ldots, \tilde{k}$ within a MILP?

Examples of modeling with integer/binary variables: Discrete domain of one (or more) variables

How to model the condition: $x \in \mathbb{R}$ and $x \in \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{\tilde{k}}\}$ where $\tilde{x}_k \in \mathbb{R}$ for $k = 1, \ldots, \tilde{k}$ within a MILP?

- Additional **binary variables**: $y \in \{0, 1\}^{\tilde{k}}$

Examples of modeling with integer/binary variables: Discrete domain of one (or more) variables

How to model the condition: $x \in \mathbb{R}$ and $x \in \{\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_{\tilde{k}}\}$ where $\tilde{x}_k \in \mathbb{R}$ for $k = 1, \dots, \tilde{k}$ within a MILP?

- Additional **binary variables**: $y \in \{0,1\}^{\tilde{k}}$

$$x = \sum_{k=1}^{\tilde{k}} \tilde{x}_k y_k$$

$$\sum_{k=1}^{\tilde{k}} y_k = 1.$$

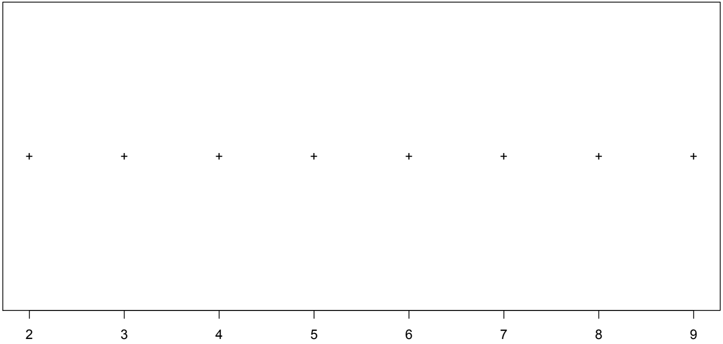
Examples of modeling with integer/binary variables: Discrete domain of one (or more) variables

How to model the condition: $x \in \mathbb{R}$ and $x \in \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{\tilde{k}}\}$ where $\tilde{x}_k \in \mathbb{R}$ for $k = 1, \ldots, \tilde{k}$ within a MILP?

Examples of modeling with integer/binary variables: Discrete domain of one (or more) variables

How to model the condition: $x \in \mathbb{R}$ and $x \in \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{\tilde{k}}\}$ where $\tilde{x}_k \in \mathbb{R}$ for $k = 1, \ldots, \tilde{k}$ within a MILP?

When $\tilde{x}_1 \in \mathbb{Z}$ and $\tilde{x}_k - \tilde{x}_{k-1} = 1$ for $k = 2, \ldots, \tilde{k}$



Examples of modeling with integer/binary variables: Discrete domain of one (or more) variables

How to model the condition: $x \in \mathbb{R}$ and $x \in \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{\tilde{k}}\}$ where $\tilde{x}_k \in \mathbb{R}$ for $k = 1, \ldots, \tilde{k}$ within a MILP?

Examples of modeling with integer/binary variables: Discrete domain of one (or more) variables

How to model the condition: $x \in \mathbb{R}$ and $x \in \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{\tilde{k}}\}$ where $\tilde{x}_k \in \mathbb{R}$ for $k = 1, \ldots, \tilde{k}$ within a MILP?

When $\tilde{x}_1 \in \mathbb{Z}$ and $\tilde{x}_k - \tilde{x}_{k-1} = 1$ for $k = 2, \ldots, \tilde{k}$

Examples of modeling with integer/binary variables: Discrete domain of one (or more) variables

How to model the condition: $x \in \mathbb{R}$ and $x \in \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{\tilde{k}}\}$ where $\tilde{x}_k \in \mathbb{R}$ for $k = 1, \ldots, \tilde{k}$ within a MILP?

When $\tilde{x}_1 \in \mathbb{Z}$ and $\tilde{x}_k - \tilde{x}_{k-1} = 1$ for $k = 2, \ldots, \tilde{k}$

- An **integer variable** x

Examples of modeling with integer/binary variables: Discrete domain of one (or more) variables

How to model the condition: $x \in \mathbb{R}$ and $x \in \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{\tilde{k}}\}$ where $\tilde{x}_k \in \mathbb{R}$ for $k = 1, \ldots, \tilde{k}$ within a MILP?

When $\tilde{x}_1 \in \mathbb{Z}$ and $\tilde{x}_k - \tilde{x}_{k-1} = 1$ for $k = 2, \ldots, \tilde{k}$

- An **integer variable** x

$$\begin{aligned} \tilde{x}_1 \leq x &\leq \tilde{x}_{\tilde{k}} \\ x &\in \text{integer}. \end{aligned}$$

Examples of modeling with integer/binary variables: Discrete domain of one (or more) variables

How to model the condition: $x \in \mathbb{R}$ and $x \in \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{\tilde{k}}\}$ where $\tilde{x}_k \in \mathbb{R}$ for $k = 1, \ldots, \tilde{k}$ within a MILP?

When $\tilde{x}_k - \tilde{x}_{k-1} = 1$ **for** $k = 2, \ldots, \tilde{k}$, another alternative

Examples of modeling with integer/binary variables: Discrete domain of one (or more) variables

How to model the condition: $x \in \mathbb{R}$ and $x \in \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{\tilde{k}}\}$ where $\tilde{x}_k \in \mathbb{R}$ for $k = 1, \ldots, \tilde{k}$ within a MILP?

When $\tilde{x}_k - \tilde{x}_{k-1} = 1$ for $k = 2, \ldots, \tilde{k}$, another alternative

- Additional **binary variables**: $y \in \{0, 1\}^{(\ell+1)}$ (ℓ is the smallest integer such that $\tilde{x}_{\tilde{k}} - \tilde{x}_1 < 2^{\ell+1}$)

Examples of modeling with integer/binary variables: Discrete domain of one (or more) variables

How to model the condition: $x \in \mathbb{R}$ and $x \in \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{\tilde{k}}\}$ where $\tilde{x}_k \in \mathbb{R}$ for $k = 1, \ldots, \tilde{k}$ within a MILP?

When $\tilde{x}_k - \tilde{x}_{k-1} = 1$ for $k = 2, \ldots, \tilde{k}$, another alternative

- Additional **binary variables**: $y \in \{0,1\}^{(\ell+1)}$ (ℓ is the smallest integer such that $\tilde{x}_{\tilde{k}} - \tilde{x}_1 < 2^{\ell+1}$)

$$
\begin{aligned}
x &= \tilde{x}_1 + \sum_{i=0}^{\ell} 2^i y_i \\
\tilde{x}_1 \leq x &\leq \tilde{x}_{\tilde{k}} \\
y_i &\in \{0,1\} \quad \forall i = 0, \ldots, \ell
\end{aligned}
$$

Motivation

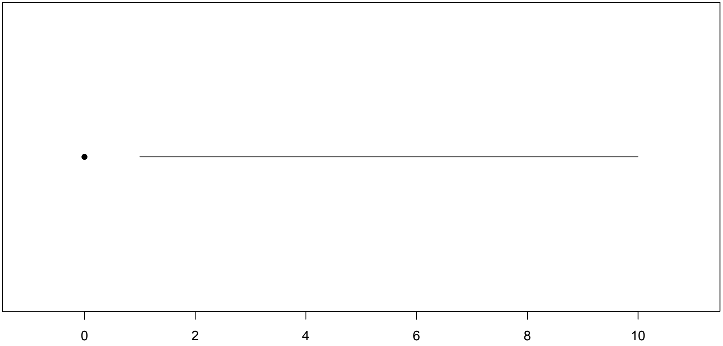
- **Discrete domain** of one (or more) variables
- Domain **discontinuity**
- **Conditional** constraints
- **Fixed cost**
- **Disjunctive** constraints
- **Absolute value** of a variable

Motivation

- Discrete domain of one (or more) variables
- Domain **discontinuity**
- Conditional constraints
- Fixed cost
- Disjunctive constraints
- Absolute value of a variable

Examples of modeling with integer/binary variables: Domain discontinuity

Discontinuous domain $x \in \{0\} \cup [\underline{x}, \overline{x}]$



Examples of modeling with integer/binary variables: Domain discontinuity

Discontinuous domain $x \in \{0\} \cup [\underline{x}, \overline{x}]$

Examples of modeling with integer/binary variables: Domain discontinuity

Discontinuous domain $x \in \{0\} \cup [\underline{x}, \overline{x}]$

MILP formulation:

$$\begin{aligned} \underline{x}y \leq x \leq \overline{x}y \\ y \in \{0, 1\}. \end{aligned}$$

Examples of modeling with integer/binary variables: Domain discontinuity

Discontinuous domain $x \in \{0\} \cup [\underline{x}, \overline{x}]$

MILP formulation:

$$\begin{aligned} \underline{x}y \leq x \leq \overline{x}y \\ y \in \{0, 1\}. \end{aligned}$$

For $y = 0$, $x = 0$

Examples of modeling with integer/binary variables: Domain discontinuity

Discontinuous domain $x \in \{0\} \cup [\underline{x}, \overline{x}]$

MILP formulation:

$$\underline{x}y \leq x \leq \overline{x}y$$
$$y \in \{0, 1\}.$$

For $y = 0$, $x = 0$

For $y = 1$, $x \in [\underline{x}, \overline{x}]$.

Motivation

- **Discrete domain** of one (or more) variables
- Domain **discontinuity**
- **Conditional** constraints
- **Fixed cost**
- **Disjunctive** constraints
- **Absolute value** of a variable

Motivation

- Discrete domain of one (or more) variables
- Domain discontinuity
- **Conditional** constraints
- Fixed cost
- Disjunctive constraints
- Absolute value of a variable

Examples of modeling with integer/binary variables: Conditional constraints

Impose a constraint $a_i^\top x \leq b_i$ only **under certain conditions** .

Examples of modeling with integer/binary variables: Conditional constraints

Impose a constraint $a_i^\top x \leq b_i$ only **under certain conditions** .

For example: If $x_1 \geq \tilde{x}_1$ then $a_i^\top x \leq b_i$.

Examples of modeling with integer/binary variables: Conditional constraints

Impose a constraint $a_i^\top x \leq b_i$ only **under certain conditions** .

For example: If $x_1 \geq \tilde{x}_1$ then $a_i^\top x \leq b_i$.

- Additional **binary variable**, say $y_i \in \{0, 1\}$, allows to activate or deactivate both the condition and the conditional constraint

Examples of modeling with integer/binary variables: Conditional constraints

Impose a constraint $a_i^\top x \leq b_i$ only **under certain conditions** .

For example: If $x_1 \geq \tilde{x}_1$ then $a_i^\top x \leq b_i$.

- Additional **binary variable**, say $y_i \in \{0, 1\}$, allows to activate or deactivate both the condition and the conditional constraint

$$\begin{aligned} \tilde{x}_1(1 - y_i) \leq x_1 &\leq \tilde{x}_1 + (1 - y_i)(\overline{x}_1 - \tilde{x}_1) \\ a_i^\top x &\leq b_i + My_i \end{aligned}$$

where M is the so-called *big-M* , i.e., a large enough parameter (hp. $x_1 \geq 0$).

Examples of modeling with integer/binary variables: Conditional constraints

Impose a constraint $a_i^\top x \leq b_i$ only **under certain conditions** .

For example: If $x_1 \geq \tilde{x}_1$ then $a_i^\top x \leq b_i$.

- Additional **binary variable**, say $y_i \in \{0, 1\}$, allows to activate or deactivate both the condition and the conditional constraint

$$
\begin{aligned}
\tilde{x}_1(1 - y_i) \leq x_1 \quad &\leq \quad \tilde{x}_1 + (1 - y_i)(\overline{x}_1 - \tilde{x}_1) \\
a_i^\top x \quad &\leq \quad b_i + My_i
\end{aligned}
$$

where M is the so-called *big-M* , i.e., a large enough parameter (hp. $x_1 \geq 0$).

$y_i = 0 \rightarrow x_1 \geq \tilde{x}_1 \rightarrow a_i^\top x \leq b$

Examples of modeling with integer/binary variables: Conditional constraints

Impose a constraint $a_i^\top x \leq b_i$ only **under certain conditions** .

For example: If $x_1 \geq \tilde{x}_1$ then $a_i^\top x \leq b_i$.

- Additional **binary variable**, say $y_i \in \{0, 1\}$, allows to activate or deactivate both the condition and the conditional constraint

$$\begin{aligned} \tilde{x}_1(1 - y_i) \leq x_1 &\leq \tilde{x}_1 + (1 - y_i)(\overline{x}_1 - \tilde{x}_1) \\ a_i^\top x &\leq b_i + My_i \end{aligned}$$

where M is the so-called *big-M* , i.e., a large enough parameter (hp. $x_1 \geq 0$).

$y_i = 0 \to x_1 \geq \tilde{x}_1 \to a_i^\top x \leq b$

$y_i = 1$ the constraint is deactivated and $x_1 \geq 0$, $x_1 \leq \tilde{x}_1$.

Examples of modeling with integer/binary variables: Conditional constraints

Impose a constraint $a_i^\top x \leq b_i$ only **under certain conditions** .

For example: If $x_1 \geq \tilde{x}_1$ then $a_i^\top x \leq b_i$.

- Additional **binary variable**, say $y_i \in \{0, 1\}$, allows to activate or deactivate both the condition and the conditional constraint

$$
\begin{aligned}
\tilde{x}_1(1 - y_i) \leq x_1 &\leq \tilde{x}_1 + (1 - y_i)(\overline{x}_1 - \tilde{x}_1) \\
a_i^\top x &\leq b_i + M y_i
\end{aligned}
$$

where M is the so-called *big-M* , i.e., a large enough parameter (hp. $x_1 \geq 0$).

$y_i = 0 \rightarrow x_1 \geq \tilde{x}_1 \rightarrow a_i^\top x \leq b$

$y_i = 1$ the constraint is deactivated and $x_1 \geq 0$, $x_1 \leq \tilde{x}_1$.

How to set the value of the big-M?

Examples of modeling with integer/binary variables: Conditional constraints

Example

If $x_1 \geq 5$, then $10x_2 + 5x_3 \leq 25$

Examples of modeling with integer/binary variables: Conditional constraints

Example

If $x_1 \geq 5$, then $10x_2 + 5x_3 \leq 25$

$$\begin{aligned} 10x_2 + 5x_3 &\leq 25 + My_1 \\ y_1 &\in \{0, 1\}. \end{aligned}$$

Examples of modeling with integer/binary variables: Conditional constraints

Example

If $x_1 \geq 5$, then $10x_2 + 5x_3 \leq 25$

$$\begin{aligned} 10x_2 + 5x_3 &\leq 25 + My_1 \\ y_1 &\in \{0, 1\}. \end{aligned}$$

Hp: $x_2 \leq 10$ and $x_3 \leq 10$

Examples of modeling with integer/binary variables: Conditional constraints

Example

If $x_1 \geq 5$, then $10x_2 + 5x_3 \leq 25$

$$\begin{aligned} 10x_2 + 5x_3 &\leq 25 + My_1 \\ y_1 &\in \{0, 1\}. \end{aligned}$$

Hp: $x_2 \leq 10$ and $x_3 \leq 10$

Value of M:

Examples of modeling with integer/binary variables: Conditional constraints

Example

If $x_1 \geq 5$, then $10x_2 + 5x_3 \leq 25$

$$\begin{aligned} 10x_2 + 5x_3 &\leq 25 + My_1 \\ y_1 &\in \{0, 1\}. \end{aligned}$$

Hp: $x_2 \leq 10$ and $x_3 \leq 10$

Value of M:
Max LHS : $10 \cdot 10 + 5 \cdot 10 = 150$

Examples of modeling with integer/binary variables: Conditional constraints

Example

If $x_1 \geq 5$, then $10x_2 + 5x_3 \leq 25$

$$\begin{aligned} 10x_2 + 5x_3 &\leq 25 + My_1 \\ y_1 &\in \{0, 1\}. \end{aligned}$$

Hp: $x_2 \leq 10$ and $x_3 \leq 10$

Value of M:
Max LHS : $10 \cdot 10 + 5 \cdot 10 = 150$
$M \geq 150 - 25 = 125.$

Examples of modeling with integer/binary variables: Conditional constraints

Example

If $x_1 \geq 5$, then $10x_2 + 5x_3 \leq 25$

$$\begin{aligned} 10x_2 + 5x_3 &\leq 25 + My_1 \\ y_1 &\in \{0, 1\}. \end{aligned}$$

Hp: $x_2 \leq 10$ and $x_3 \leq 10$

Value of M:
Max LHS : $10 \cdot 10 + 5 \cdot 10 = 150$
$M \geq 150 - 25 = 125.$

Tricky to set big-M value. Overestimate (valid model)

Motivation

- **Discrete domain** of one (or more) variables
- Domain **discontinuity**
- **Conditional** constraints
- **Fixed cost**
- **Disjunctive** constraints
- **Absolute value** of a variable

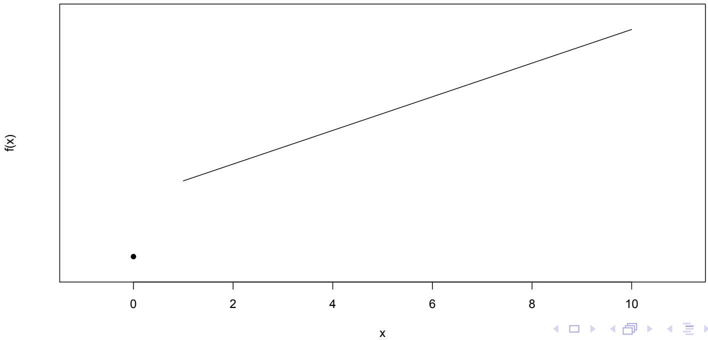
Motivation

- Discrete domain of one (or more) variables
- Domain discontinuity
- Conditional constraints
- **Fixed cost**
- Disjunctive constraints
- Absolute value of a variable

Examples of modeling with integer/binary variables: Fixed cost

A cost: composed of a **fixed part** and a variables part (**discontinuous**):

$$f(x) = \begin{cases} cx + d & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$



Examples of modeling with integer/binary variables: Fixed cost

A cost: composed of a **fixed part** and a variables part (**discontinuous**):

$$f(x) = \begin{cases} cx + d & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Examples of modeling with integer/binary variables: Fixed cost

A cost: composed of a **fixed part** and a variables part (**discontinuous**):

$$f(x) = \begin{cases} cx + d & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

MILP modeling:

$$\begin{aligned} \min f(x) &= cx + dy \\ x &\leq \bar{x}y \\ y &\in \{0, 1\} \end{aligned}$$

Examples of modeling with integer/binary variables: Fixed cost

A cost: composed of a **fixed part** and a variables part (**discontinuous**):

$$f(x) = \begin{cases} cx + d & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

MILP modeling:

$$\begin{aligned} \min f(x) &= cx + dy \\ x &\leq \bar{x}y \\ y &\in \{0, 1\} \end{aligned}$$

If $x > 0$, $y = 1$, thus $f(x) = cx + d$.

Examples of modeling with integer/binary variables: Fixed cost

A cost: composed of a **fixed part** and a variables part (**discontinuous**):

$$f(x) = \begin{cases} cx + d & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

MILP modeling:

$$\begin{aligned} \min f(x) &= cx + dy \\ x &\leq \bar{x}y \\ y &\in \{0,1\} \end{aligned}$$

If $x > 0$, $y = 1$, thus $f(x) = cx + d$.
If $x = 0$, $y = 0$ (because of min of the obj function)

Motivation

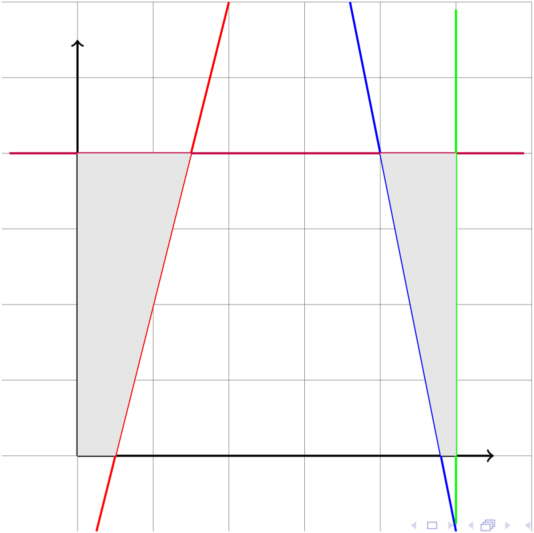
- **Discrete domain** of one (or more) variables
- Domain **discontinuity**
- **Conditional** constraints
- **Fixed cost**
- **Disjunctive** constraints
- **Absolute value** of a variable

Motivation

- Discrete domain of one (or more) variables
- Domain discontinuity
- Conditional constraints
- Fixed cost
- **Disjunctive** constraints
- Absolute value of a variable

Examples of modeling with integer/binary variables: Disjunctive constraints

Disjunction : satisfy $a_i^\top x \leq b_i$ or $a_k^\top x \leq b_k$.



Examples of modeling with integer/binary variables: Disjunctive constraints

Disjunction : satisfy $a_i^\top x \leq b_i$ or $a_k^\top x \leq b_k$.

Examples of modeling with integer/binary variables: Disjunctive constraints

Disjunction : satisfy $a_i^\top x \leq b_i$ or $a_k^\top x \leq b_k$.

$$
\begin{aligned}
a_i^\top x &\leq b_i + M_i y_i \\
a_k^\top x &\leq b_k + M_k y_k \\
y_i + y_k &\leq 1 \\
y_i &\in \{0, 1\} \\
y_k &\in \{0, 1\}
\end{aligned}
$$

Motivation

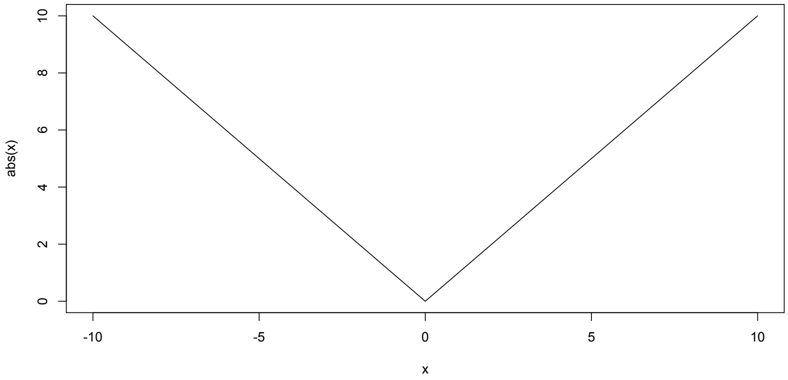
- **Discrete domain** of one (or more) variables
- Domain **discontinuity**
- **Conditional** constraints
- **Fixed cost**
- **Disjunctive** constraints
- **Absolute value** of a variable

Motivation

- Discrete domain of one (or more) variables
- Domain discontinuity
- Conditional constraints
- Fixed cost
- Disjunctive constraints
- **Absolute value** of a variable

Examples of modeling with integer/binary variables: Absolute value of a variable

MILP modeling of $|x|$?



Examples of modeling with integer/binary variables: Absolute value of a variable

MILP modeling of $|x|$?

Examples of modeling with integer/binary variables: Absolute value of a variable

MILP modeling of $|x|$?

$$
\begin{aligned}
|x| &= x^+ + x^- \\
x &= x^+ - x^- \\
0 \leq x^+ &\leq \overline{x} y \\
0 \leq x^- &\leq -\underline{x}(1-y) \\
y &\in \{0,1\}.
\end{aligned}
$$

Examples of modeling with integer/binary variables: Absolute value of a variable

MILP modeling of $|x|$?

$$\begin{aligned} |x| &= x^+ + x^- \\ x &= x^+ - x^- \\ 0 \leq x^+ &\leq \overline{x}y \\ 0 \leq x^- &\leq -\underline{x}(1-y) \\ y &\in \{0,1\}. \end{aligned}$$

If $x \leq 0$, $y = 0$, $x^+ = 0$, and $x^- \in [0, -\underline{x}]$.
If $x \geq 0$, $y = 1$, $x^- = 0$, and $x^+ \in [0, \overline{x}]$.

Examples of modeling with integer/binary variables: Absolute value of a variable

MILP modeling of $|x|$?

$$
\begin{aligned}
|x| &= x^+ + x^- \\
x &= x^+ - x^- \\
0 \leq x^+ &\leq \overline{x} y \\
0 \leq x^- &\leq -\underline{x}(1-y) \\
y &\in \{0,1\}.
\end{aligned}
$$

If $x \leq 0$, $y = 0$, $x^+ = 0$, and $x^- \in [0, -\underline{x}]$.
If $x \geq 0$, $y = 1$, $x^- = 0$, and $x^+ \in [0, \overline{x}]$.

Where $\underline{x} \leq x \leq \overline{x}$, hp. $\underline{x} < 0$ w.l.o.g.

Exercises

Formulate the following as mixed integer linear programs:

1. $u = \min\{x_1, x_2\}$, assuming that $0 \leq x_j \leq C$ for $j = 1, 2$.
2. $v = \|x_1 - x_2\|_\infty$ with $0 \leq x_j \leq C$ for $j = 1, 2$.
3. the set $X \setminus \{x^*\}$ where $X = \{x \in \mathbb{Z}^n \mid Ax \leq b\}$ and $x^* \in X$.