# Mixed Integer Non Linear Optimization: Methods and Applications
# –
# Solving LPs

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ÉCOLE POLYTECHNIQUE
IP PARIS

# LP problems and methods

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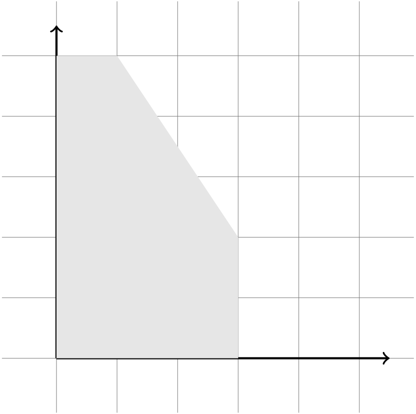
In this course: **graphical solution** of LPs and intuition on the **primal simplex method**

# LP problems and methods

Possible outcomes:

- **optimal**: when $X = \{x \mid Ax \leq b, \underline{x} \leq x \leq \overline{x}\} \neq \emptyset$, bounded. In this case, an optimal solution is found, i.e., a feasible point $x^*$ s.t. $c^\top x^* \leq c^\top x$ for all feasible $x \in X$
- **infeasible**: when $X = \{x \mid Ax \leq b, \underline{x} \leq x \leq \overline{x}\} = \emptyset$
- **unbounded**: when the $\min\{c^\top x \mid Ax \leq b, \underline{x} \leq x \leq \overline{x}\} = -\infty$

# A few definitions



- Polytope : a bounded polyhedron ($\exists M > 0$ s.t. $\|x\| \leq M$ for all $x \in P$) (see Minkowski, 1896).

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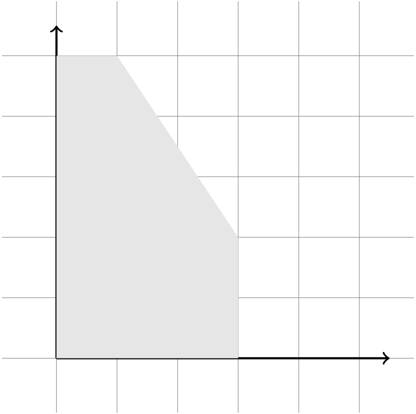

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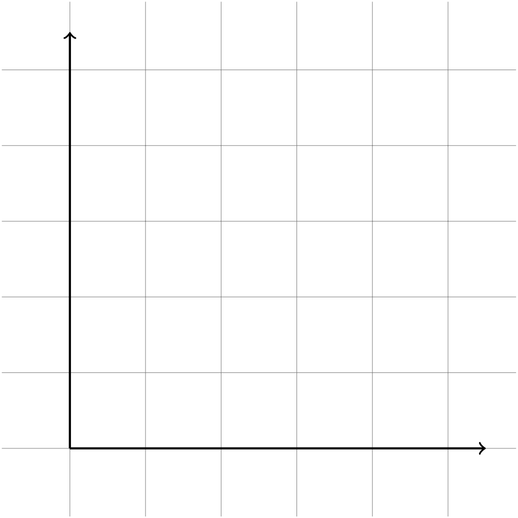
- Polytope dimension : dimension of the smallest subspace of $\mathbb{R}^n$ which contains all the polytope points.

# Example 1: optimal solution

$$\begin{aligned} \max x_1 + x_2 \\ 3x_1 + 2x_2 \leq 13 \\ 0 \leq x_1 \leq 3 \\ 0 \leq x_2 \leq 5. \end{aligned}$$



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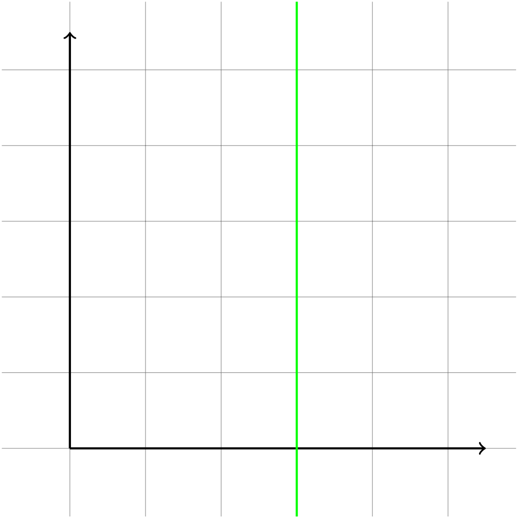

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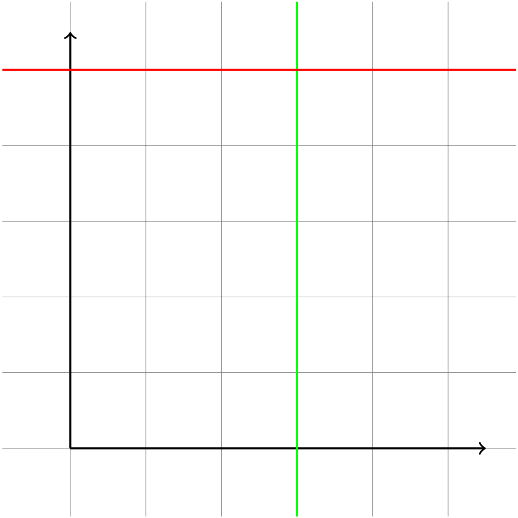

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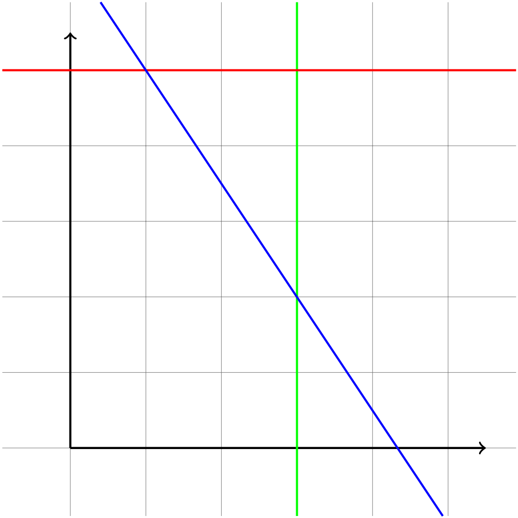

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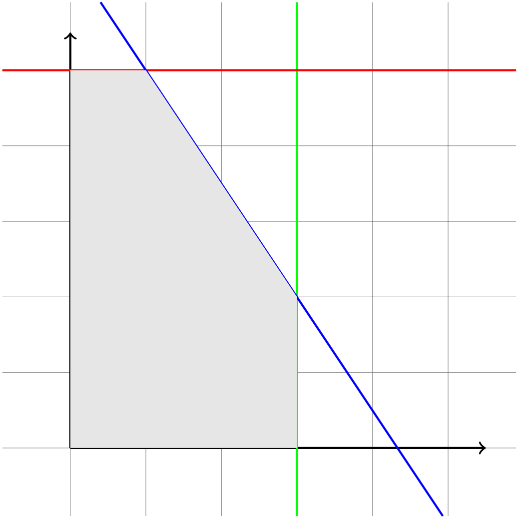

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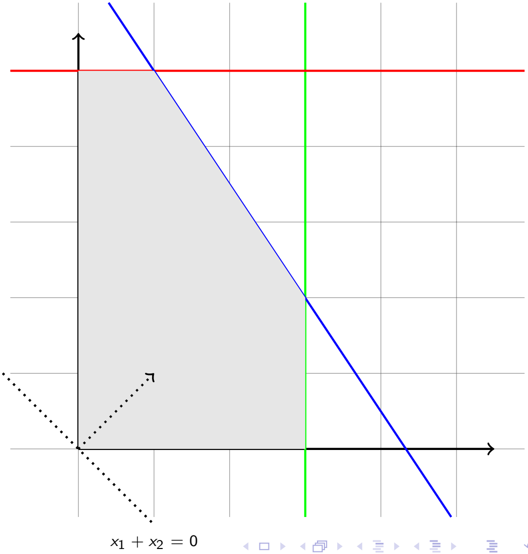

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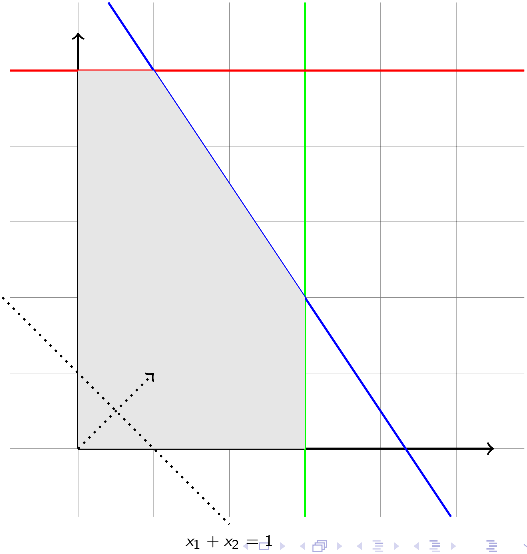

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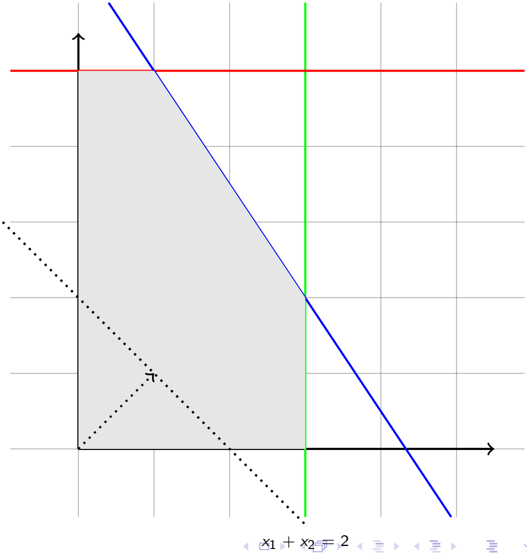

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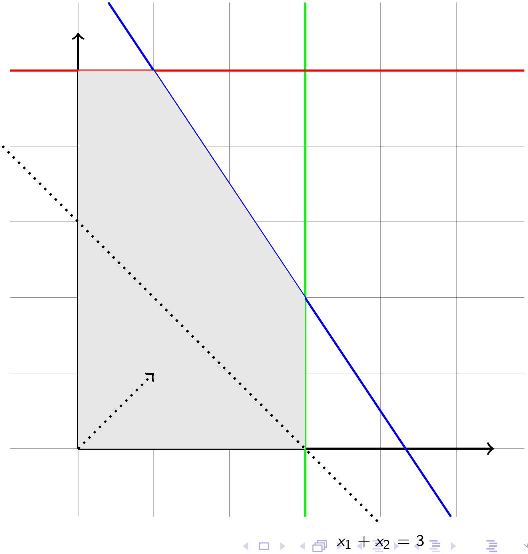

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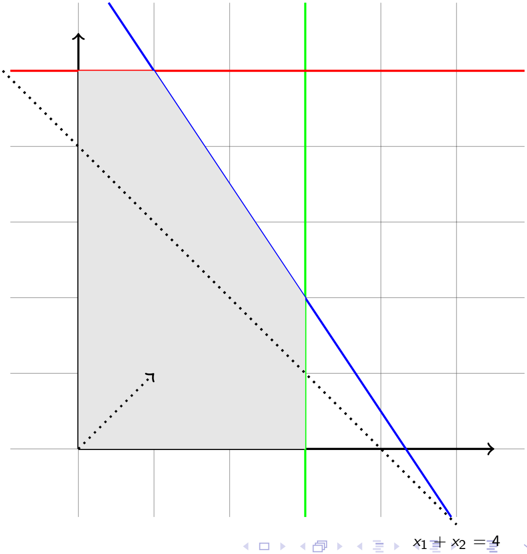

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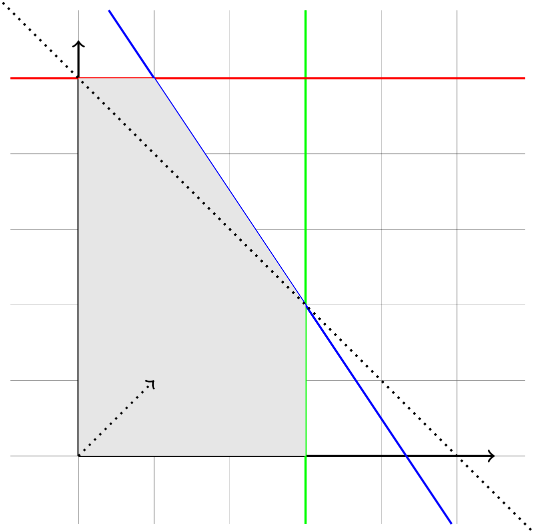

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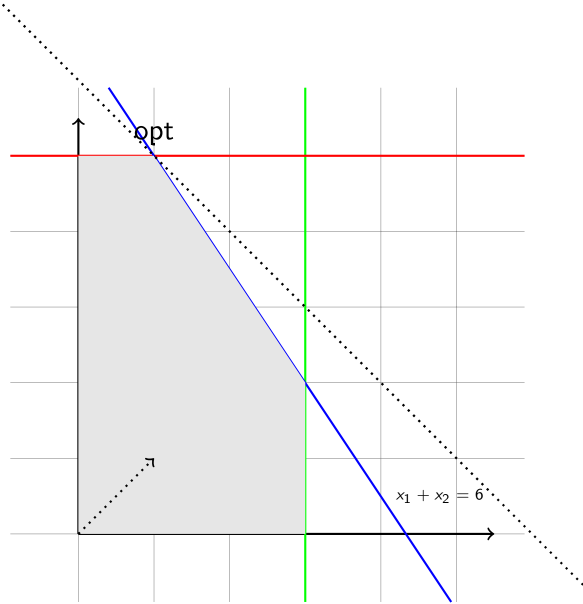

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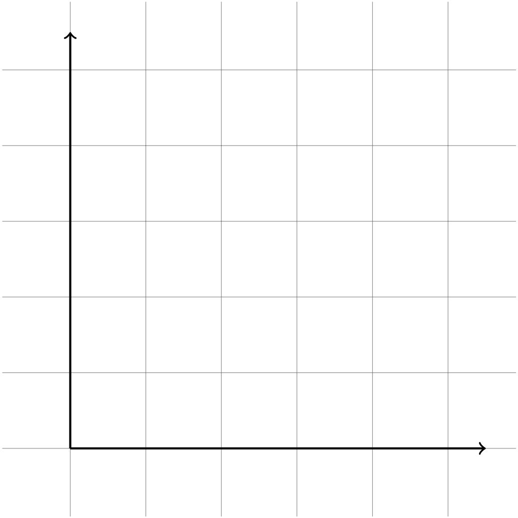

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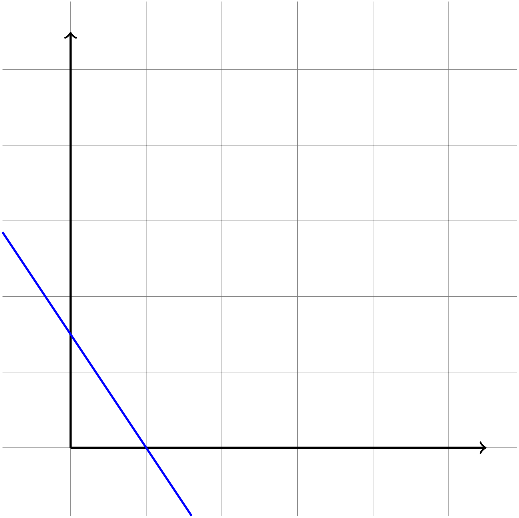

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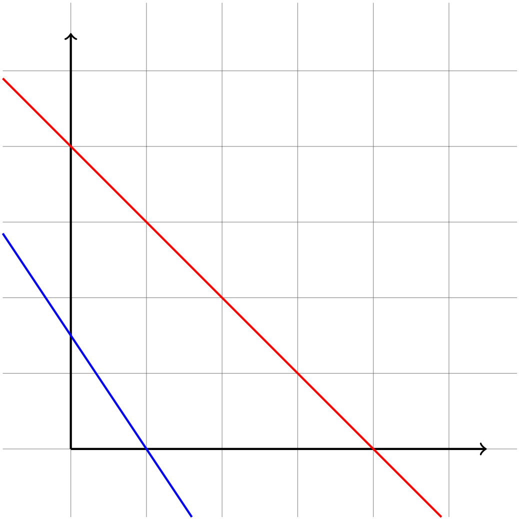

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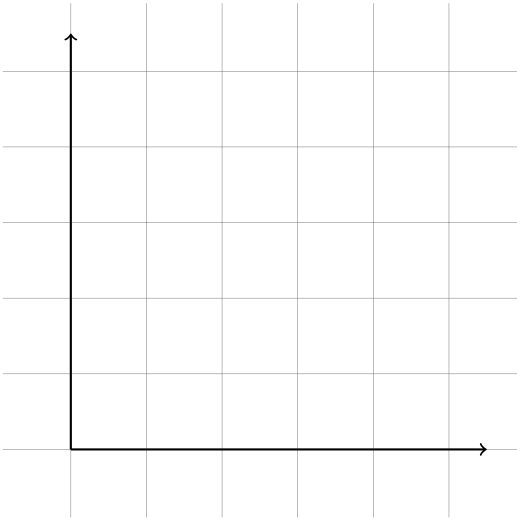

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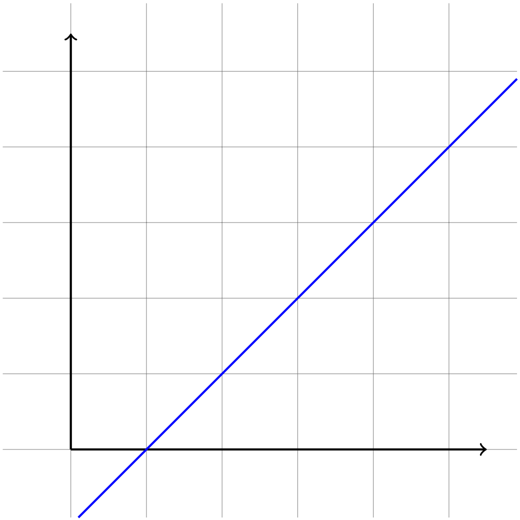

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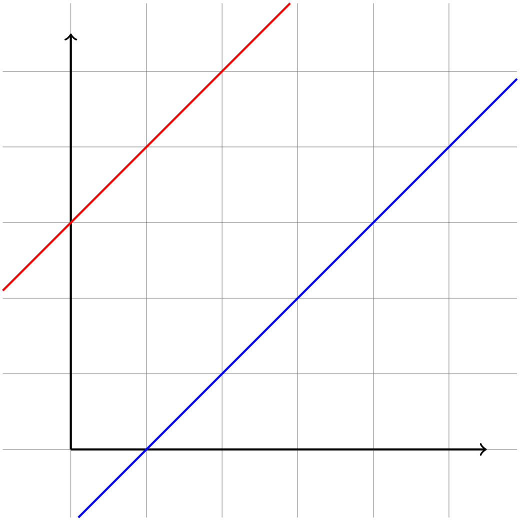

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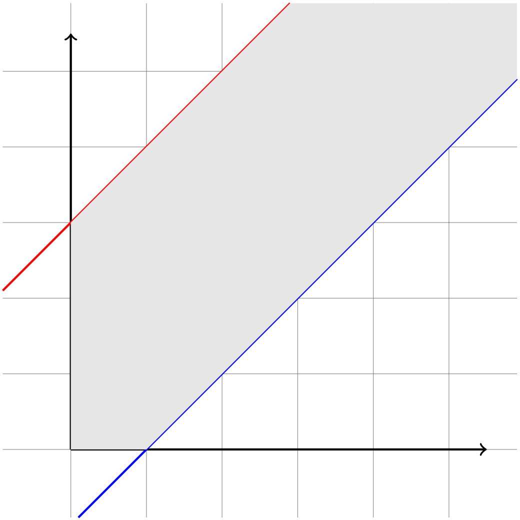

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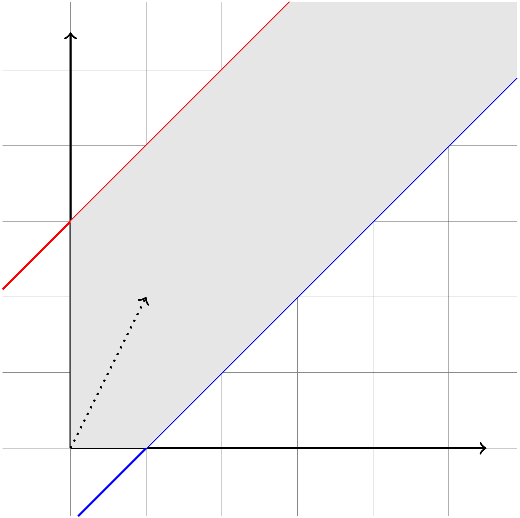

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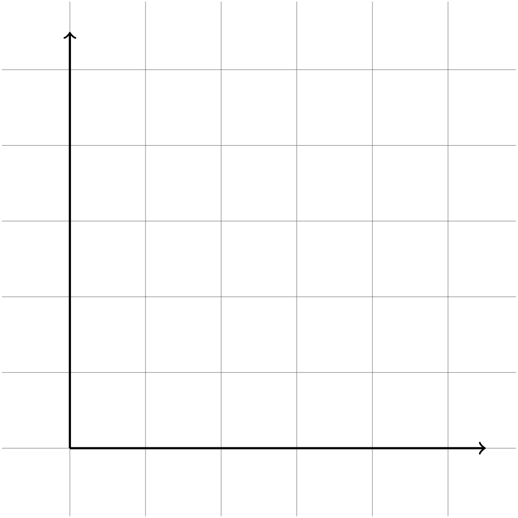

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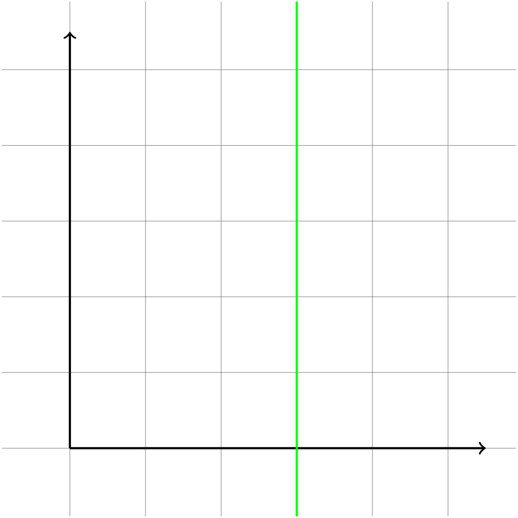

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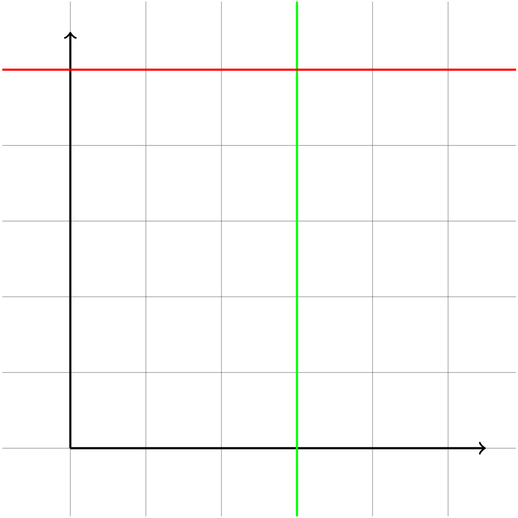

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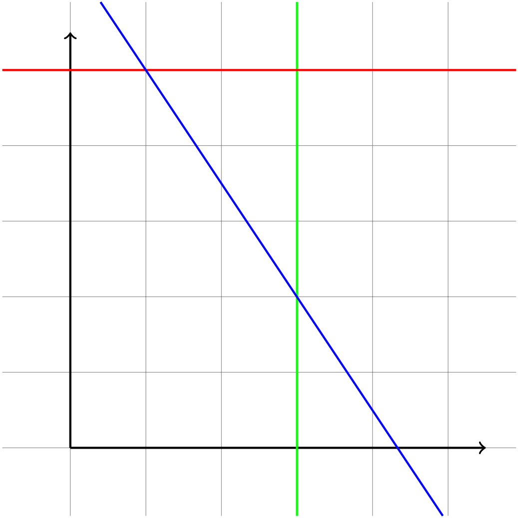

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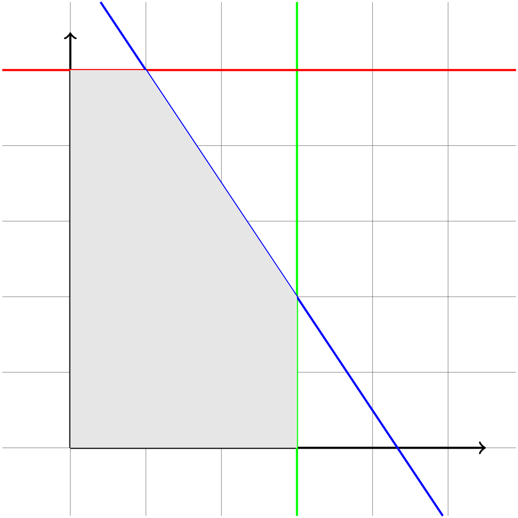

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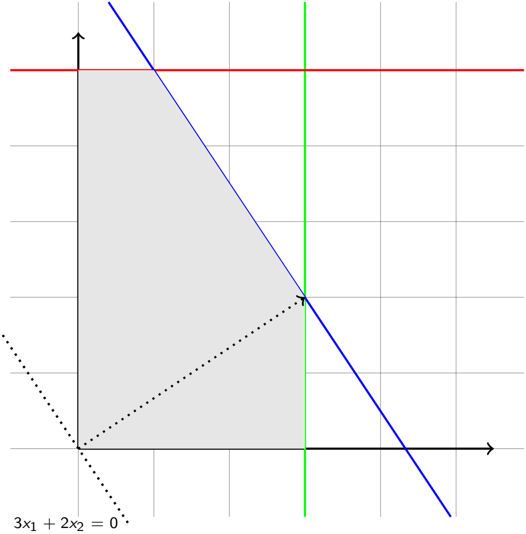

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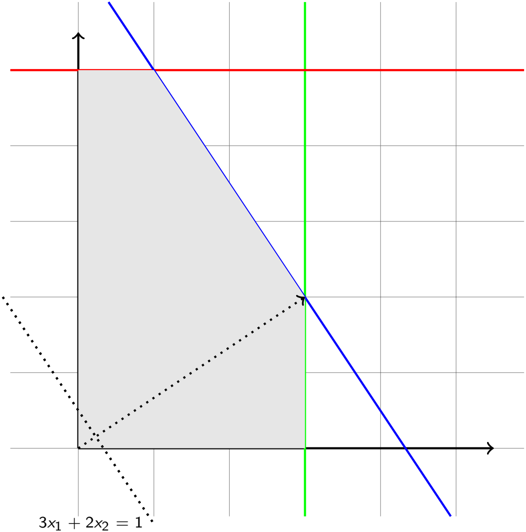

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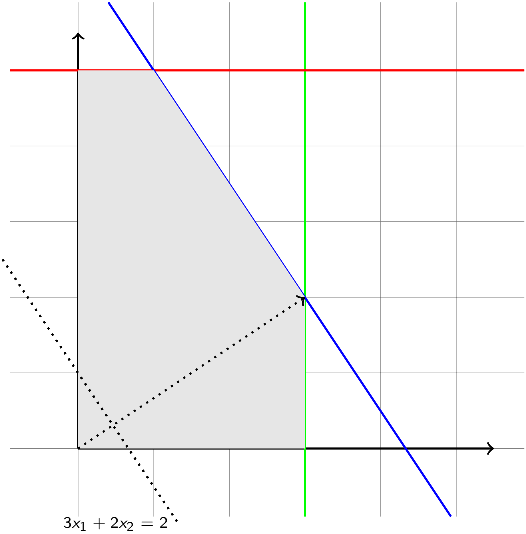

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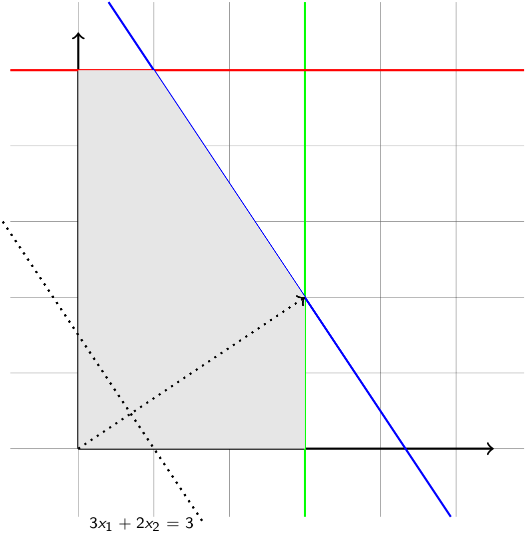

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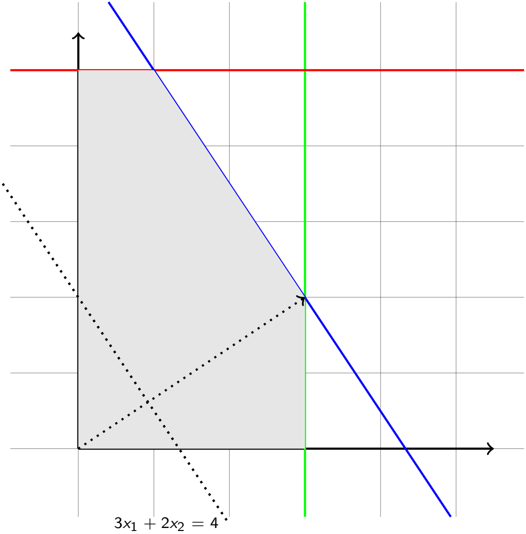

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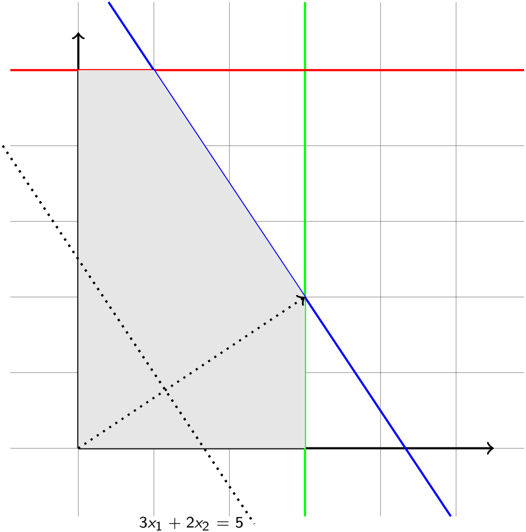

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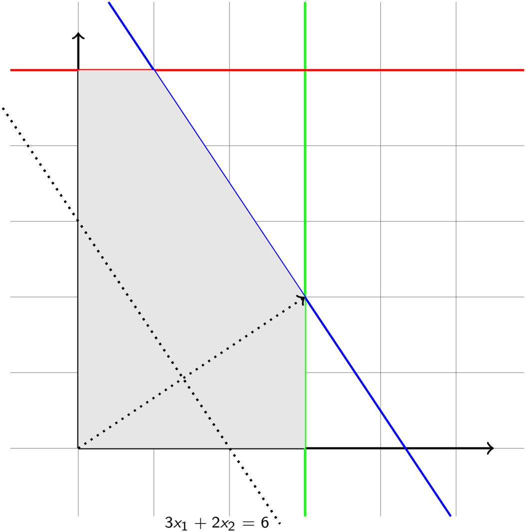

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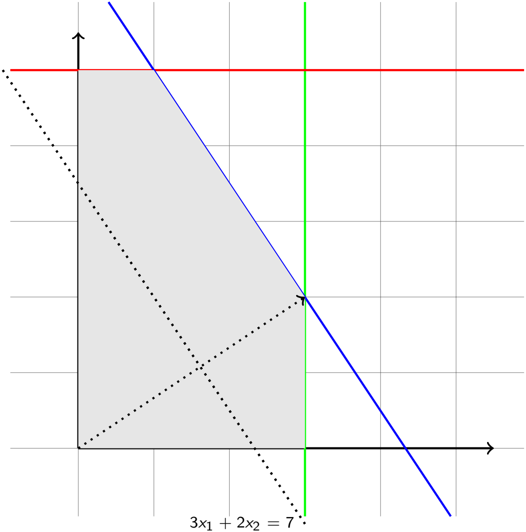

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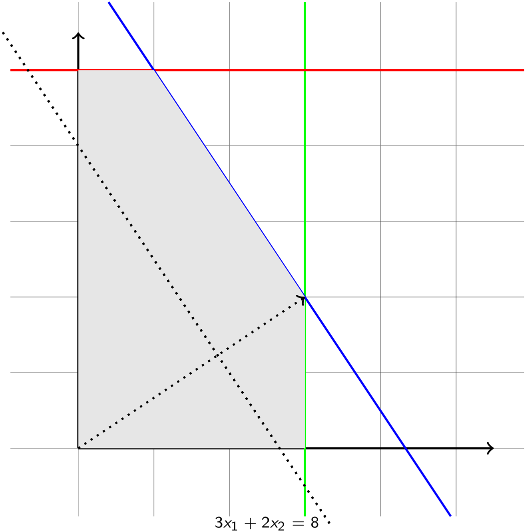

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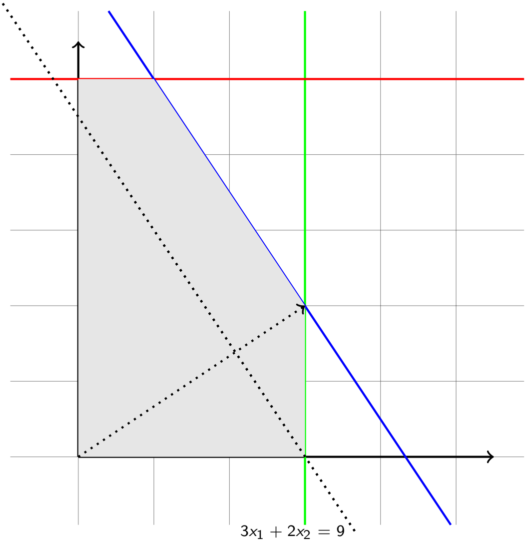

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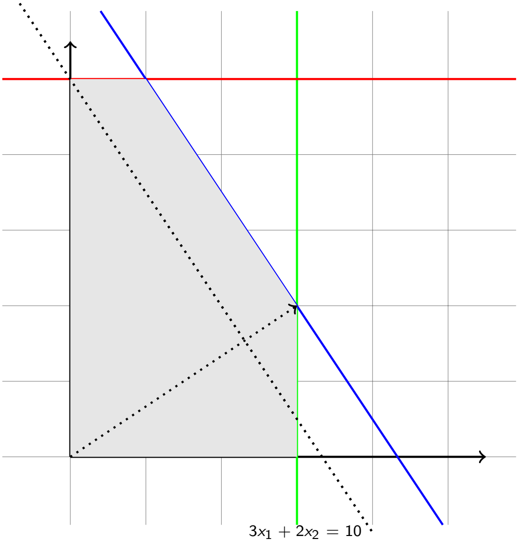

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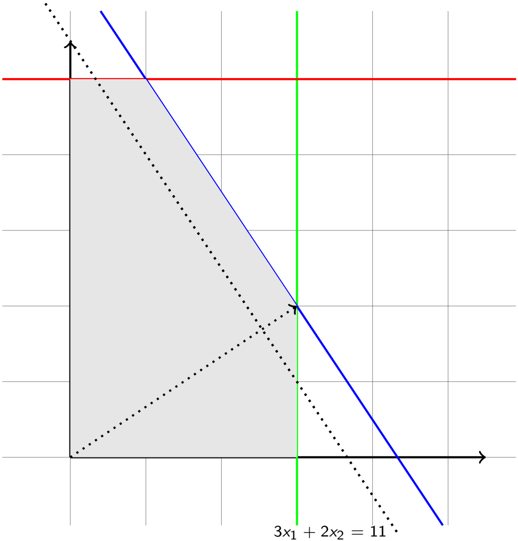

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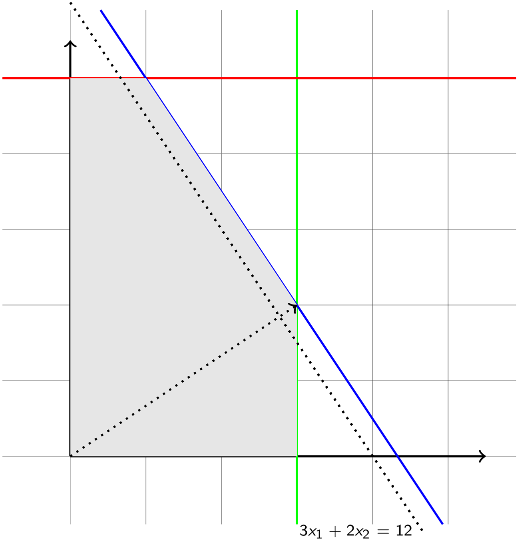

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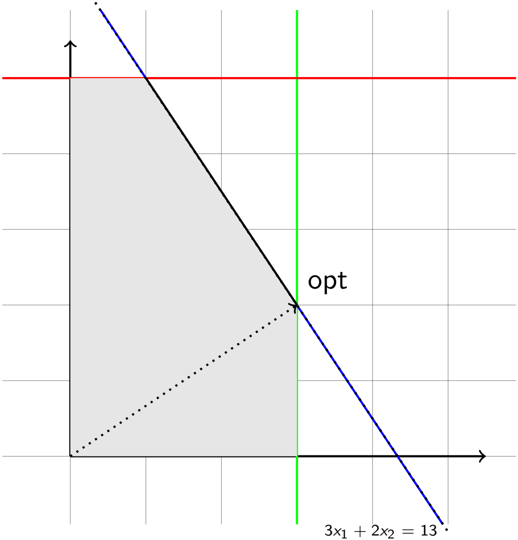

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$$\tilde{x} = \sum_{i=1}^{p} \alpha_i v^i \text{ with } \sum_{i=1}^{p} \alpha_i = 1, \alpha \geq 0.$$

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## Theorem

*Given a bounded LP* $\min\{c^\top x \mid Ax \le b\}$, *an optimal vertex does always exist.*

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Thus, as this excludes $c^\top v^j > c^\top \tilde{x}$, then $c^\top v^j = c^\top \tilde{x}$.

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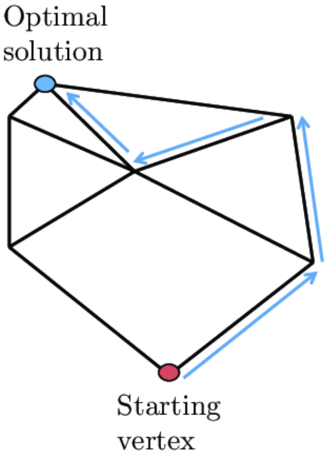
## Simplex Methods




From the Research Gate's page of by Laura Leal-Taixé

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Let $x = \sum_{i=1}^p \alpha_i v^i$ with $\sum_{i=1}^p \alpha_i = 1, \alpha \geq 0$.
Then, its cost is

$$c^\top x = c^\top \sum_{i=1}^p \alpha_i v^i = c^\top v^1 \sum_{i=1}^p \alpha_i = c^\top v^1$$

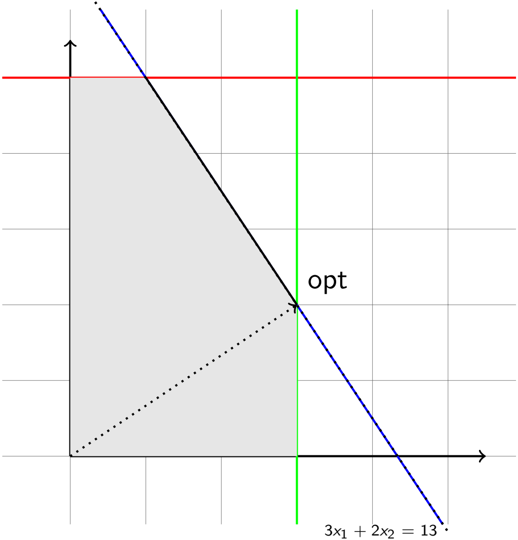
.

□

# Example 4: degenerate case

$$\max 3x_1 + 2x_2$$

$$\begin{aligned} 3x_1 + 2x_2 &\leq 13 \\ 0 \leq x_1 &\leq 3 \\ 0 \leq x_2 &\leq 5 \end{aligned}$$



# A few references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