Mixed Integer Non Linear Optimization: Methods and Applications

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Linear Programming

Claudia D'Ambrosio
dambrosio@lix.polytechnique.fr

Linear Programming problems

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\begin{aligned}
\min_{x} \quad & f(x) & & \\
g_i(x) \leq & \ 0 & & \forall i = 1, \ldots, m \\
\underline{x} \leq x \leq & \ \overline{x} & & \\
x_j \in & \ \mathbb{Z} & & \forall j \in Z
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\underline{x} \leq x \leq \overline{x} \quad & \rightarrow \quad \underline{x} \leq x \leq \overline{x} \\
x_j \in \mathbb{Z} \quad \forall j \in Z \quad & \rightarrow \quad \text{removed}
\end{aligned}
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LP problems

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LP problems

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Moreover, $\underline{x} \in [-\infty, +\infty)$ and $\overline{x} \in (-\infty, +\infty]$.

LPs characteristics

Feasible (solutions) set/region : $X = \{x \mid Ax \leq b, \underline{x} \leq x \leq \overline{x}\}$

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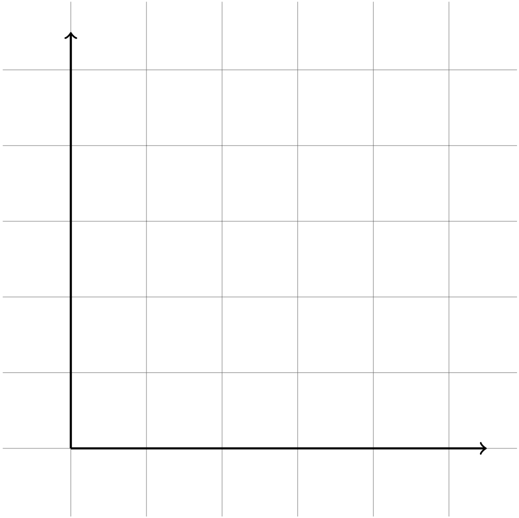
Geometrical interpretation of LPs

How to draw constraints and objective function

Example 1

$$\begin{aligned} \max x_1 + x_2 \\ 3x_1 + 2x_2 \leq 13 \\ 0 \leq x_1 \leq 3 \\ 0 \leq x_2 \leq 5. \end{aligned}$$



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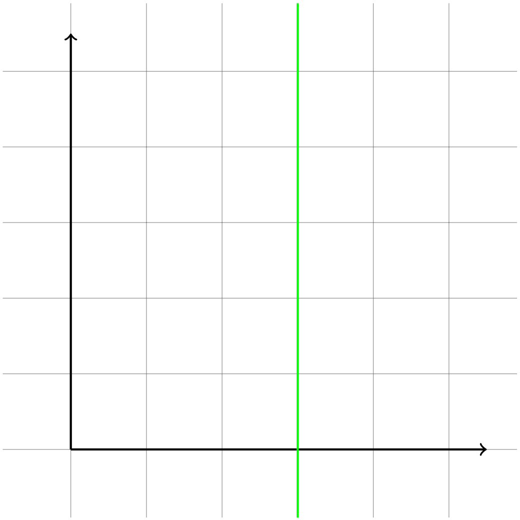

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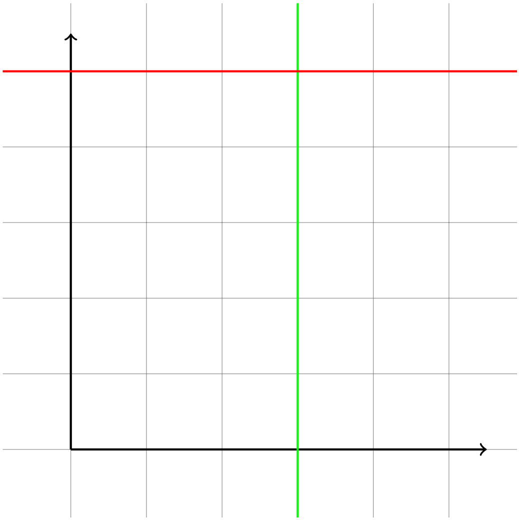

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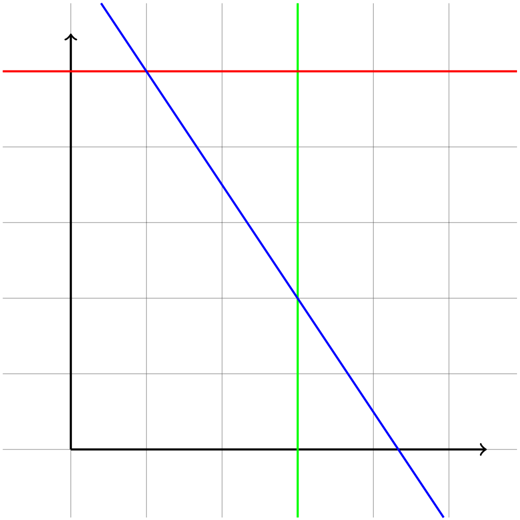

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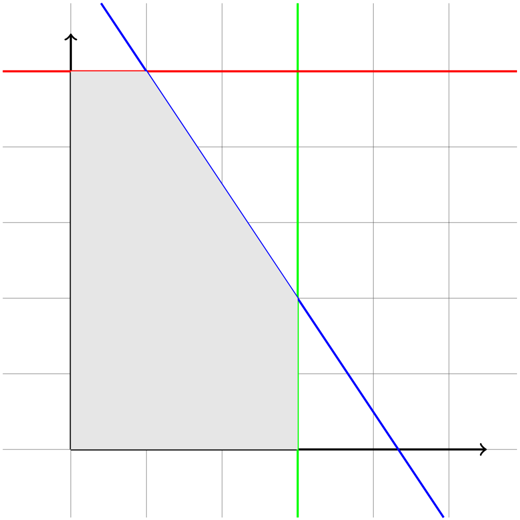
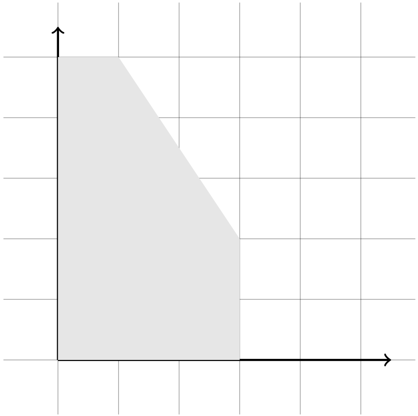


A few definitions



Consider some $i \in \{1, ..., m\}$

- Hyperplane : $\{x \in \mathbb{R}^n \mid A_i^\top x = b_i\}$

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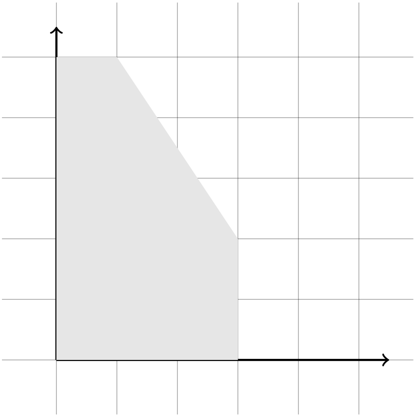
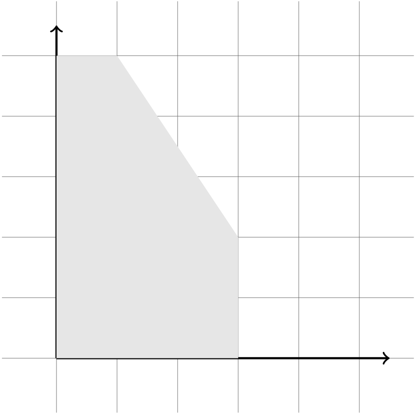


Consider some $i \in \{1, \ldots, m\}$

- **Hyperplane** : $\{x \in \mathbb{R}^n \mid A_i^\top x = b_i\}$
- **Half-space** : $\{x \in \mathbb{R}^n \mid A_i^\top x \leq b_i\}$ or $\{x \in \mathbb{R}^n \mid A_i^\top x \geq b\}$

A few definitions



- Polyhedron : $\{x \in \mathbb{R}^n \mid Ax \leq b\}$

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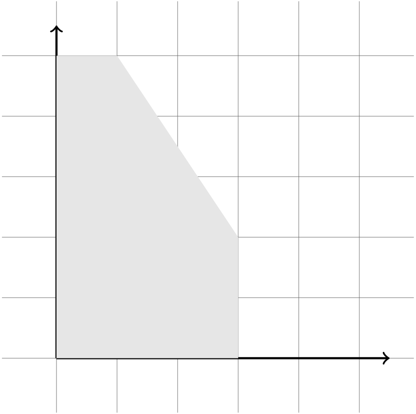

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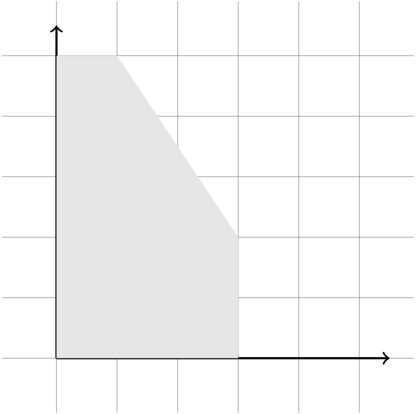

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Remark

The feasible region of a LP problem is a polyhedron (by definition).

Some properties and theorems

Definition

Given points $v^1, v^2, \ldots, v^p \in \mathbb{R}^n$, their **convex combination** is $z = \sum_{i=1}^{p} \alpha_i v^i$ s.t. $\sum_{i=1}^{p} \alpha_i = 1$ and $\alpha_i \geq 0$ for all $i = 1, \ldots, p$.

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Theorem

Every polyhedron $P \subseteq \mathbb{R}^d$ can be written as

$$P = conv\{v^1, \ldots, v^k\} + cone\{r^1, \ldots, r^\ell\}$$

with points $v^1, \ldots, v^k \in \mathbb{R}^d$ and rays $r^1, \ldots, r^\ell \in \mathbb{R}^d$

where
$\text{cone}\{r^1, \ldots, r^\ell\} = \{x \in \mathbb{R}^d \mid x = \mu_1 r^1 + \ldots \mu_\ell r^\ell, \mu_1, \ldots, \mu_\ell \geq 0\}$.

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Thus, a polytope can be **characterized/described** by a finite number of half-spaces (H-description) or its vertices (V-description).