Mixed Integer Non Linear Optimization: Methods and Applications Introduction to AMPL

Day 2

Exercises session

Claudia D'Ambrosio
dambrosio@lix.polytechnique.fr

Exercise 1:
The Uncapacitated Facility Location Problem

In the Uncapacitated Facility Location (UFL) Problem, we are given n facilities and m customers. We wish to choose which of the n facilities to open, so that the open facilities supply the demands from the customers. Our goal is to minimize the cost, which is composed of two parts:

- a fixed cost c_j to open facility j (for $j = 1, \dots, n$)
- a production cost d_{ij} which we pay if facility j supplies the demand of customer i (for $i = 1, \dots, m$ and $j = 1, \dots, n$).

We report in the following the formulation from the 1950s-1960s:

$$\min \sum_{j=1}^{n} c_j y_j + \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} x_{ij} \tag{1}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \forall i = 1, \dots, m \tag{2}$$

$$\sum_{i=1}^{m} x_{ij} \leq m y_j \qquad \forall j = 1, \dots, n \tag{3}$$

$$x_{ij} \in \{0, 1\} \qquad \forall i = 1, \dots, m; j = 1, \dots, n \tag{4}$$

$$y_j \in \{0, 1\} \qquad \forall j = 1, \dots, n \tag{5}$$

Now, we report an alternative formulation:

$$\min \sum_{j=1}^{n} c_j y_j + \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} x_{ij} \tag{6}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \forall i = 1, \dots, m \tag{7}$$

$$x_{ij} \leq y_j \qquad \forall i = 1, \dots, m; j = 1, \dots, n \tag{8}$$

$$x_{ij} \in \{0, 1\} \qquad \forall i = 1, \dots, m; j = 1, \dots, n \tag{9}$$

$$y_j \in \{0, 1\} \qquad \forall j = 1, \dots, n \tag{10}$$

Questions

- Which is the strongest formulation and why?

- Code and test the two formulations with AMPL (an instance is provided `UFL_1.dat`). Note: compare the optimal value of the LP relaxations.

Exercise 2: The Knapsack Problem

The knapsack problem is defined as follows. We are given a knapsack of capacity c (maximum weight). Given n available items, each of weight w_j and profit p_j (for all $j = 1, \ldots, n$), select the items to insert in the knapsack so as to respect the capacity and maximize the profit.

The decision variables are:

- $x_j = 1$ when item j is selected, 0 otherwise ($\forall j = 1, \ldots, n$).

We now present the 01-Knapsack Problem (KP) formulation:

$$\max \sum_{j=1}^{n} p_j x_j$$

$$\sum_{j=1}^{n} w_j x_j \leq c$$

$$x_j \in \{0, 1\} \qquad j = 1, \ldots, n$$

Let us define $N = \{1, 2, \ldots, n\}$.
We provide one instance `KP_*.dat` and an instance generator `instances_generator_KP.run` based on the book "The Knapsack Problem" by Martello & Toth.
The weights w_j are generated uniformly random in $[1, v]$ (where $v = \min\left(\sum_{j=1}^{m-1} w_j, c\right)$).
We consider the uncorrelated variant, thus, the profits are generated uniformly random in $[1, v]$. For c we could consider the following two options: $2v$ or $0.5 \sum_{j=1}^{n} w_j$. The latter is the one coded in the instance generator.

Question 1: the LP relaxation

Solve the LP relaxation of the instance with 10 items: just one value is fractional, why? Is this always the case (you can check by generating and solving other instances LP relaxations)?

Question 2: Valid Inequalities

A set $C \subseteq N$ is a *cover* if $\sum_{j \in C} w_j > c$. Moreover, a cover C is *minimal* if $C \setminus j$ is not a cover $\forall j \in C$.
We define *cover inequality* the equation

$$\sum_{j \in C} x_j \leq |C| - 1,$$

which is valid for the 01-KP.

Consider the file KP_JL.dat (inspired by an example by J. Linderoth])

- Add to the model the valid inequalities corresponding to the following covers:
  - {1,2,3}, {1,2,6}, {1,5,6},{3,4,5,6}. Solve the resulting formulation and compare it with the basic one (continuous relaxation)
- Add to the model the valid inequalities corresponding to the all the possible covers. Solve the resulting formulation and compare it with the basic one (continuous relaxation)

Question 3: Extended cover inequalities

Extended cover $E(C) = C \cup \{j \in N \mid w_j \geq w_i \ \forall i \in C\}$. The *extended cover inequality* is

$$\sum_{j \in E(C)} x_j \leq |C| - 1.$$

Example of cover and correspoding extended cover: $C = \{3, 4, 5, 6\}$, $E(C) = \{1, 2, 3, 4, 5, 6\}$

Consider the file KP_JL.dat (inspired by an example by J. Linderoth])

- Add to the model the extended cover below. Solve the resulting formulation and compare it with the basic one (continuous relaxation)

Exercise 3: The Assignment Problem

The Assignment Problem (AP) is defined as follows. Given n people available for being assigned to n tasks, we wish to find the minimum cost assignment, where the cost c_{ij} is inversely proportional to the suitedness of person i to task j (for each $i, j = 1, \ldots, n$).
Decision variables:

- $x_{ij} = 1$ when person i is assigned to task j, 0 otherwise ($\forall i = 1, \ldots, n; j = 1, \ldots, n$).

The AP formulation is the following:

$$\begin{aligned}
\min \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} & \\
\sum_{j=1}^{n} x_{ij} = 1 & \qquad \forall i = 1, \ldots, n \\
\sum_{i=1}^{n} x_{ij} = 1 & \qquad \forall j = 1, \ldots, n \\
x_{ij} \in \{0, 1\} & \qquad \forall i = 1, \ldots, n; j = 1, \ldots, n
\end{aligned}$$

Let us now consider the Generalized AP (GAP). Namely, we are given n items, each of which to be assigned to one of the available m bins. Each of the bin has a maximum capacity b_i (for $i = 1, \ldots, m$) and the weight w_{ij} of each item i depends on the bin j to which it is assigned (for $i = 1, \ldots, n$ and $j = 1 \ldots, m$).

Thus, we can formulate the GAP as follows:

$$\begin{aligned}
\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} & \\
\sum_{j=1}^{n} w_{ij} x_{ij} \leq b_i & \qquad \forall i = 1, \ldots, m \\
\sum_{i=1}^{m} x_{ij} = 1 & \qquad \forall j = 1, \ldots, n \\
x_{ij} \in \{0, 1\} & \qquad \forall i = 1, \ldots, m; j = 1, \ldots, n.
\end{aligned}$$

We provide an instance for each problem `AP_1.dat` and `generalized_AP_1.dat`, respectively. We provide as well an instance generator file for each problem, `instances_generator_AP.run` and `instances_generator_GAP.run`, respectively. The instances generators are based on the book "The Knapsack Problem" by Martello & Toth, Class (c).

- w_{ij} uniformly random in [5,25]
- c_{ij} uniformly random in [1,40]
- $b_i = 0.8 \sum_{j=1,\ldots,n} \frac{w_{il}}{m}$

Exercise 1: Questions

1. Code the AP model in AMPL.
2. Solve the provided instance. Solve now the LP relaxation (use the command "option relax_integrality 1;"). Compare the two solution and the CPU time.
3. Now generate larger instances thanks the the provided file instances_generator_AP.run and repeat the previous point.
4. Code the GAP model in AMPL.
5. Solve the provided instance. Solve now the LP relaxation (use the command "option relax_integrality 1;"). Compare the two solution and the CPU time.
6. Now generate larger instances thanks the the provided file instances_generator_AP.run and repeat the previous point.