# Optimizing the Design of Water Distribution Networks Using Mathematical Optimization

C. Bragalli, C. D'Ambrosio, J. Lee, A. Lodi, and P. Toth

## 1 Introduction

Decaying infrastructure in municipalities is becoming a problem of increasing importance as growing populations put increasing stress on all service systems. In tough economic times, renewing and maintaining infrastructure has become increasingly difficult. As an example, many municipal water networks were installed several decades ago and were designed to handle much smaller demand and additionally have decayed due to age.

For example, consider the case of Modena, a city northwest of Bologna in the Emilia-Romagna region of Italy. We can see in Figure 1 how the population has increased by more than a factor of 2.5 over the last century. The Modena water distribution network comprises 4 reservoirs, 272 junctions, and 317 pipes. Its complexity can be gleaned from Figure 2.

The aim is to replace all the pipes using the same network topology at minimum cost to achieve pressure demands at junctions of the network. Pipes are only available from commercial suppliers that produce pipes in a limited number of diameters. Reservoirs pressurize the network while most pressure is lost due to friction in pipes (some pressure is also lost at the junctions). Our

Cristiana Bragalli
DISTART, University of Bologna, 40136 Bologna, Italy
e-mail: cristiana.bragalli@mail.ing.unibo.it

Claudia D'Ambrosio
CNRS LIX, École Polytechnique, 91128 Palaiseau, France
e-mail: dambrosio@lix.polytechnique.fr

Jon Lee
IOE Department, University of Michigan, Ann Arbor, MI 48109-2117, USA
e-mail: jonxlee@umich.edu

Andrea Lodi, Paolo Toth
DEI, University of Bologna, 40136 Bologna, Italy
e-mail: {andrea.lodi,paolo.toth}@unibo.it

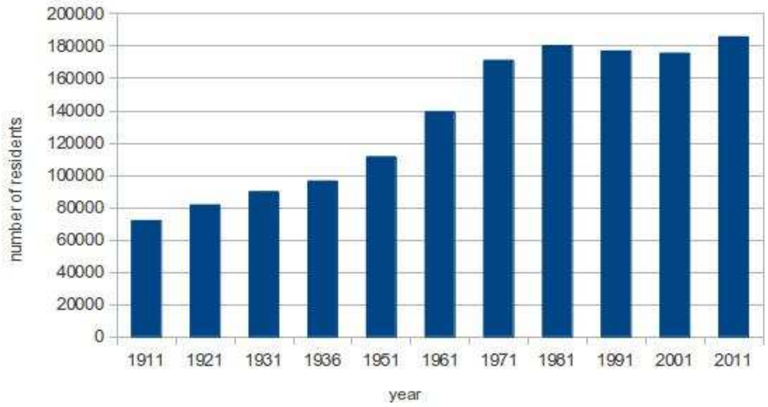


**Fig. 1** Population of Modena

**Fig. 2** Water distribution network of Modena

goal is to model the problem using continuous variables as flow rates in the pipes, and pressures at the junctions; and discrete variables for the diameters of the pipes. Noting that pressure loss due to friction behaves nonlinearly, this puts us in the domain of Mixed Integer Nonlinear Programming (MINLP). Thus, we will try to obtain a model which is tractable by standard MINLP solvers.

In fluid dynamics, *hydraulic head* equates the energy in an incompressible fluid with the height of an equivalent static column of that fluid. The hydraulic head of a fluid is composed of pressure head and elevation head. The **pressure head** is the internal energy of a fluid due to the pressure exerted on its container and the **elevation head** is the energy given by the gravitational force acting on a column of fluid, in this case water. As a convention, the unit of hydraulic, pressure, elevation head are meters, i.e., each unit represents the energy provided by a column of water of the height of one meter. The water in its way from the reservoir to the rest of the junctions loose energy. This loss is called **head loss**. Head loss is divided into two main categories, "major losses" associated with energy loss per length of pipe, and "minor losses" associated with bends and relatively small obstructions. For our purposes, considering that we are interested in networks covering relatively large geographic areas, it suffices to ignore minor losses.

## 2 Nomenclature

In this section we introduce the sets and parameters, i.e., the data input of our problem.

### *2.1 Sets*

The water network will be represented as a directed graph $G = (N, E)$ where $N$ is the set of node that represent the junctions and $E$ is the set of pipes. Moreover we define the set of reservoirs $S$ as a subset of $N$. Finally, for ease of notation, we define $\delta-(i)$ $(\delta+(i))$ as the sets of pipes with an head (tail) at junction $i$.

### *2.2 Parameters*

In the following, we introduce the notation for the parameters.
For each junction but the reservoirs $i \in N \setminus S$ we have:

elev(i) = physical elevation of juction $i$, i.e., the heigh of junction $i$ ($[m]$).
dem(i) = water demand at junction $i$ ($[m^3/s]$).
$\text{ph}_{min}(i)$ = minimum pressure head at junction $i$ ($[m]$).
$\text{ph}_{max}(i)$ = maximum pressure head at junction $i$ ($[m]$).

Moreover, for each reservoir $i \in S$, we have $h_s(i)$, i.e., the fixed hydraulic head at junction $i$.
Finally, for each pipe $e = (i, j) \in E$ the following parameters are needed:

l$(e)$ = length of pipe $e$ ($[m]$).
$v_{max}(e)$ = maximum velocity of water in pipe $e$ ($[m/s]$).
$k(e)$ = physical constant depending on the roughness of pipe $e$.
$\mathfrak{D}(e, r)$ = r-th diameter that can be choosen for pipe $e$ ($[m]$).
$\mathfrak{C}(e, r)$ = cost of the r-th diameter that can be choosen for pipe $e$ ($[€/m]$).

Note that all the parameters described above are data input and are used to define constraints and objective function of the model.

We proceed further, in describing the problem.

## 2.3 Decision Variables of the Problem

First, we specify the variables which are also the expected output of our problem:

$Q(e)$ = flow in pipe $e$, for all $e \in E$ $[m^3/s]$.
$D(e)$ = diameter of pipe $e$, for all $e \in E$ $[m]$.
$H(i)$ = hydraulic head of junction $i$, for all $i \in N$ $[m]$.

For modeling purposes, suppose that each pipe $e$ has a nominal orientation, and negative flow corresponds to water flow in the direction opposite to the nominal orientation of the pipe.

The goals of the problem are to chose, for each edge, the diameter of the pipe that has to be installed in order to implement the drinkable water distribution network by minimizing the installation costs and satisfying physical and operational constraints.

## 2.4 Constraints in the Problem

Each reservoir have a fixed hydraulic head that is specified $h_s(i)$.

Typically, there are inequalities imposed that bound the velocity of the water in each pipe. Because the water flow is given by the product between the cross-sectional area of the pipe $\pi(D(e)/2)^2$ and the velocity, for semplicity the velocity bounds can be written as bounds on the flow.

Flow-conservation equations for junctions that are not reservoirs.

Upper and lower limits on the pressure head at each junction that are not reservoirs.

Head loss along each pipe is modeled via a nonlinear function of the diameter of the pipe and the flow in the pipe. Typically, one uses the so-called

Hazen-Williams equation, which is an empirical formula relating the head loss caused by frictions to the physical properties of the pipe.
For each pipe $e$, the available diameters belong to a discrete set of $r_e$ elements.

### *2.5 Objective function*

Our objective to be minimized is represented by the installation cost, i.e, euros spent to install pipes of the selected diameters.

## 3 Example

In this section, we describe a real-world instance as an example. The data are taken from a neighborhood of Bologna called Fossolo. We have 37 junctions (of which 1 reservoir, identified by the index 37), and 58 pipes. The topology of the network is represented in Figure 3 and provided in details in Table 3, see Appendix.

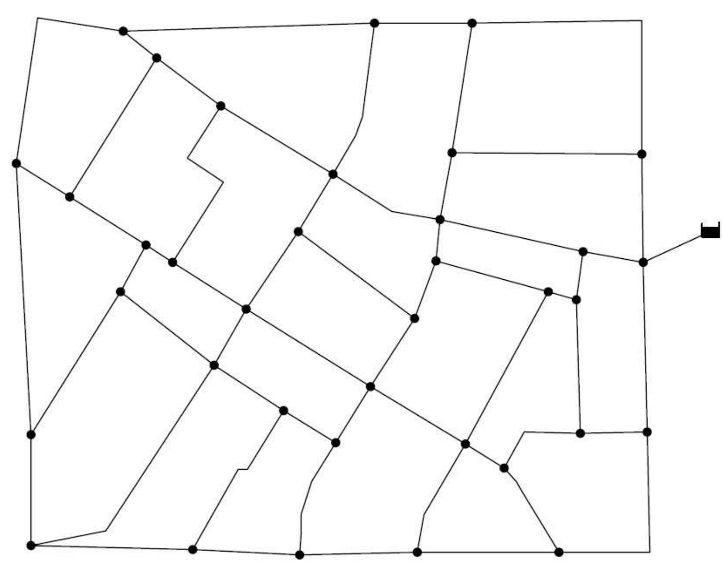

**Fig. 3** Fossolo network

The hydraulic head at the reservoir is always fixed. In our example it is fixed to 121.0 meters. The maximum speed of the water within the pipes is set to 1 $m^3/s$ and the roughness coefficient is set to 100. The lower bound on the pressure head $ph_{min}(i)$ is equal to 40 meters and the upper bound $ph_{max}(i)$ is equal to $121 - eval(i)$ meters for all $i \in N$. The values of the elevation and the demand for each junction besides the reservoir are reported in Table 1. The network topology and the length of the pipes are reported in Table 3.

Concerning the available diameters for each pipe, we have 13 different possibilities. Table 2 reports for each $e \in E$ the value of $(\mathfrak{D}(e,r), \mathfrak{C}(e,r))$ for all $r = 1, \ldots, r_e$.

## 4 Solution Found by the MINLP model

The solution found by `Bonmin` is depicted in Figure 4 where the size of each diameter is proportional to the thickness of the arc. The diameters are expressed in meters, and the diameter is equal to 0.06 for the pipes without explicit number, i.e., the minimum diameter permissible for this data set.

The analysis of this solution shows a configuration in which the size of the selected diameters decreases from the reservoir toward the parts of the network farther away from the inlet point. This characteristic of the allocation of diameters to pipes plays in favor of a correct hydraulic operation of the network and has a beneficial effect on water quality, see, e.g., the discussion in [2].

This characteristic could be noticed in the solution of MINLP for different instances, see Bragalli et al. [1] for details.

Note that the proposed solution is not guaranteed to be a global optimum of the problem. However, because of the intrinsic difficulty of the problem at hand, the proposed solution is, from a practical viewpoint, a very good quality feasible solution. Other approaches based on heuristic algorithms, mixed integer linear programming, or nonlinear programming models are not effective for medium/large instances. In this application, modeling in the most natural way the problem seems to be the most successful approach.

## 5 Conclusions

More details concerning our methodology and results can be found in [1].

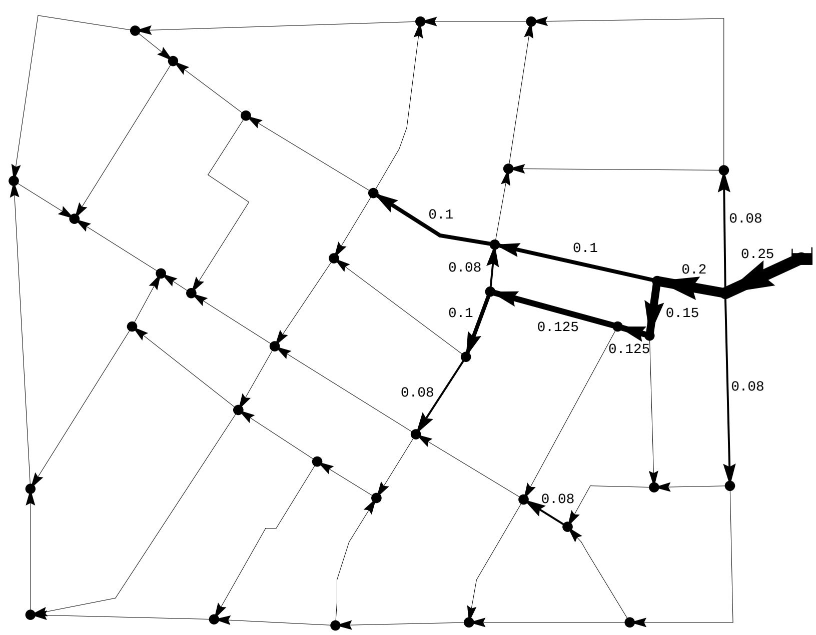


**Fig. 4** Solution for Fossolo network: the size of each diameter is proportional to the thickness of the arc. The diameter (expressend in meters) is equal to 0.06 m for the pipes without explicit number.

## References


1. C. Bragalli, C. D'Ambrosio, J. Lee, A. Lodi, and P. Toth. On the optimal design of water distribution networks: a practical minlp approach. *Optimization and Engineering*, 13:219–246, 2012.
2. M. Van Den Boomen, A. Van Mazijk, and R.H.S. Beuken. First evaluation of new design concepts for self-cleaning distribution networks. *Journal of Water Supply: Research and Technology AQUA*, 53(1):43–50, 2004.

# Appendix

| $i$ | $elev(i)$ | $dem(i)$ |
|---|---|---|
| 1 | 65.15 | 0.00049 |
| 2 | 64.40 | 0.00104 |
| 3 | 63.35 | 0.00102 |
| 4 | 62.50 | 0.00081 |
| 5 | 61.24 | 0.00063 |
| 6 | 65.40 | 0.00079 |
| 7 | 67.90 | 0.00026 |
| 8 | 66.50 | 0.00058 |
| 9 | 66.00 | 0.00054 |
| 10 | 64.17 | 0.00111 |
| 11 | 63.70 | 0.00175 |
| 12 | 62.64 | 0.00091 |
| 13 | 61.90 | 0.00116 |
| 14 | 62.60 | 0.00054 |
| 15 | 63.50 | 0.00110 |
| 16 | 64.30 | 0.00121 |
| 17 | 65.50 | 0.00127 |
| 18 | 64.10 | 0.00202 |

| $i$ | $elev(i)$ | $dem(i)$ |
|---|---|---|
| 19 | 62.90 | 0.00188 |
| 20 | 62.83 | 0.00093 |
| 21 | 62.80 | 0.00096 |
| 22 | 63.90 | 0.00097 |
| 23 | 64.20 | 0.00086 |
| 24 | 67.50 | 0.00067 |
| 25 | 64.40 | 0.00077 |
| 26 | 63.40 | 0.00169 |
| 27 | 63.90 | 0.00142 |
| 28 | 65.65 | 0.00030 |
| 29 | 64.50 | 0.00062 |
| 30 | 64.10 | 0.00054 |
| 31 | 64.40 | 0.00090 |
| 32 | 64.20 | 0.00103 |
| 33 | 64.60 | 0.00077 |
| 34 | 64.70 | 0.00074 |
| 35 | 65.43 | 0.00116 |
| 36 | 65.90 | 0.00047 |

**Table 1** Elevation (meters) and water demand ($m^3$/second) for instance Fossolo

| $r$ | $\mathfrak{D}(e,r)$ | $\mathfrak{C}(e,r)$ |
|---|---|---|
| 1 | 0.060 | 19.8 |
| 2 | 0.080 | 24.5 |
| 3 | 0.100 | 27.2 |
| 4 | 0.125 | 37.0 |
| 5 | 0.150 | 39.4 |
| 6 | 0.200 | 54.4 |
| 7 | 0.250 | 72.9 |
| 8 | 0.300 | 90.7 |
| 9 | 0.350 | 119.5 |
| 10 | 0.400 | 139.1 |
| 11 | 0.450 | 164.4 |
| 12 | 0.500 | 186.0 |
| 13 | 0.600 | 241.3 |

**Table 2** Diameter set (meters) and relative cost per meter (€/m) for instance Fossolo for each pipe

| $e=(i,j)$ | $i$ | $j$ | $l(e)$ |
|---|---|---|---|
| 1 | 1 | 17 | 132.76 |
| 2 | 17 | 2 | 374.68 |
| 3 | 2 | 3 | 119.74 |
| 4 | 3 | 4 | 312.72 |
| 5 | 4 | 5 | 289.09 |
| 6 | 5 | 6 | 336.33 |
| 7 | 6 | 7 | 135.81 |
| 8 | 7 | 24 | 201.26 |
| 9 | 24 | 8 | 132.53 |
| 10 | 8 | 28 | 144.66 |
| 11 | 28 | 9 | 175.72 |
| 12 | 9 | 36 | 112.17 |
| 13 | 36 | 1 | 210.74 |
| 14 | 1 | 31 | 75.41 |
| 15 | 31 | 10 | 181.42 |
| 16 | 10 | 11 | 146.96 |
| 17 | 11 | 19 | 162.69 |
| 18 | 19 | 12 | 99.64 |
| 19 | 12 | 4 | 52.98 |
| 20 | 2 | 18 | 162.97 |
| 21 | 18 | 10 | 83.96 |
| 22 | 10 | 32 | 49.82 |
| 23 | 32 | 27 | 78.5 |
| 24 | 27 | 16 | 99.27 |
| 25 | 16 | 25 | 82.29 |
| 26 | 25 | 8 | 147.49 |
| 27 | 3 | 11 | 197.32 |
| 28 | 11 | 26 | 83.3 |
| 29 | 26 | 15 | 113.8 |

| $e=(i,j)$ | $i$ | $j$ | $l(e)$ |
|---|---|---|---|
| 30 | 15 | 22 | 80.82 |
| 31 | 22 | 7 | 340.97 |
| 32 | 5 | 13 | 77.39 |
| 33 | 13 | 14 | 112.37 |
| 34 | 14 | 20 | 37.34 |
| 35 | 20 | 15 | 108.85 |
| 36 | 15 | 16 | 182.82 |
| 37 | 16 | 29 | 136.02 |
| 38 | 29 | 30 | 56.7 |
| 39 | 30 | 9 | 124.08 |
| 40 | 17 | 18 | 234.6 |
| 41 | 12 | 13 | 203.83 |
| 42 | 19 | 20 | 248.05 |
| 43 | 14 | 21 | 65.19 |
| 44 | 21 | 6 | 210.09 |
| 45 | 21 | 22 | 147.57 |
| 46 | 22 | 23 | 103.8 |
| 47 | 24 | 23 | 210.95 |
| 48 | 23 | 25 | 75.08 |
| 49 | 26 | 27 | 180.29 |
| 50 | 28 | 29 | 149.05 |
| 51 | 29 | 33 | 215.05 |
| 52 | 32 | 33 | 144.44 |
| 53 | 33 | 34 | 34.74 |
| 54 | 31 | 34 | 59.93 |
| 55 | 34 | 35 | 165.67 |
| 56 | 30 | 35 | 119.97 |
| 57 | 35 | 36 | 83.17 |
| 58 | 37 | 1 | 1.0 |

**Table 3** Network topology and length of the pipes (meters) for instance Fossolo