SUMMER SCHOOL ON ASPECTS OF OPTIMIZATION

Discrete Optimization

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Outline

1. Tailored cutting-plane methods
   - Introduction
   - Strong inequalities
   - Polytope Dimension
   - Facet and Convex Hull Proofs
   - Lifting
   - Separation procedure

2. Research Talk

Outline

Introduction

Inequalities:

- **valid**: for a specific problem, not all (M)ILPs

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Decomposition :
$X = X^1 \cap X^2 \rightarrow$ focus on a simpler set, e.g. X^2 because inequalities valid for X^2 will be valid for X.

Introduction

- Definition of strong inequalities

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Strong valid inequalities

Remark

Inequalities $\pi^\top x \leq \pi^0$ and $\lambda\pi^\top x \leq \lambda\pi^0$ are identical for any $\lambda > 0$.

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Definition

If $\pi^\top x \leq \pi^0$ and $\mu^\top x \leq \mu^0$ are two valid inequalities for $X \subseteq \mathbb{R}^n_+$, $\pi^\top x \leq \pi^0$ dominates $\mu^\top x \leq \mu^0$ if there exists $u > 0$ such that $\pi > u\mu$ and $\pi^0 \leq u\mu^0$.

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Definition

An inequality $\pi^\top x \leq \pi^0$ is redundant in the description of $X \subseteq \mathbb{R}^n_+$, if there exist $k \geq 2$ valid inequalities $(\pi^i)^\top x \leq \pi^{i0}$ for $i = 1, \ldots, k$ for X and weights $u_i > 0$ for $i = 1, \ldots, k$ such that $(\sum_{i=1}^k u_i \pi^i)^\top x \leq \sum_{i=1}^k u_i \pi^{i0}$ dominates $\pi^\top x \leq \pi^0$.

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- **Theoretically**: it is important to know which inequalities are nonredundant
- **Practically** : it is important to avoid using an inequality when one that dominates it is available

Valid inequalities for the 01-KP problem

Classical formulation

$$K = \left\{ x \in \{0,1\}^n \mid \sum_{j=1}^{n} w_j x_j \leq c \right\}$$

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The sets K and K^C coincide.

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Which formulation is stronger?

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$n = 3, w = (3, 3, 3), c = 5$, thus

$$K = \{x \in \{0, 1\}^n \mid 3x_1 + 3x_2 + 3x_3 \leq 5\}$$

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Summing up the 3 inequalities of K^C we have $2x_1 + 2x_2 + 2x_3 \leq 3$ which is implied by $x_1 + x_2 + x_3 \leq 1$, thus K is stronger than K^C (e.g. $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$).

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- Can we improve K^C? $\rightarrow$ strong valid inequalities
- K^C has an exponential number of constraints $\rightarrow$ no a-priori addition
- Use “some” constraints of K^C to improve K $\rightarrow$ separation procedure

Strong Valid Inequalities

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Theorem

If P is a full-dimensional polyhedron, it has a unique minimal description

$$P = \{x \in \mathbb{R}^n \mid (a^i)^\top x \leq b_i \text{ for } i = 1, \ldots, m\}$$

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where each inequality is unique within a positive multiple.

Thus, any other valid inequality is redundant.

Polyhedra, Face, Facets

Definition

The points $x^1, x^2, \ldots, x^k \in \mathbb{R}^n$ are affinely independent if the only solution to the linear system

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Definition

The dimension of P, $\dim(P)$, is one less than the maximum number of affinely independent points in P.

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Definition

- F defines a face of the polyhedron P if $F = \{x \in P \mid \pi^\top x = \pi^0\}$ for some valid inequality $\pi^\top x \leq \pi^0$ of P. The valid inequality $\pi^\top x \leq \pi^0$ is said to represent or define a face.

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- Alternatively, F is a face of the polyhedron P if there is a vector c for which F is the set of vectors attaining $\max\{c^\top x \mid x \in P\}$ provided that this maximum is finite.

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- Alternatively, F is a face of the polyhedron P if there is a vector c for which F is the set of vectors attaining $\max\{c^\top x \mid x \in P\}$ provided that this maximum is finite.
- F is a facet of P if F is a face of P of $\dim(F) = \dim(P) - 1$.

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If P is full-dimensional, a valid inequality $\pi^\top x \leq \pi^0$ is necessary in the description of P if and only if it defines a facet of P.

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Thus, for full-dimensional polyhedra, $\pi^\top x \leq \pi^0$ defines a facet of P if and only if there are n affinely independent points of P satisfying it at equality.

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Method 1:

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Method 2:

- Find a system $Ax = b$ of equations such that $S \subseteq \{x \in \mathbb{R}^n \mid Ax = b\}$ and $\text{rank}(A) = n - k$
- Prove that every equation $\alpha^\top x = \beta$ satisfied by all $x \in S$ is a linear combination of $Ax = b$ (i.e., there exists u s.t. $\alpha = uA$ and $\beta = ub$). Thus, $Ax = b$ is the affine hull of S, thus S has dimension k.

Polytope dimension

Example: 01-KP

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$S = \text{conv}(\{x \in \{0,1\}^n \mid w^\top x \leq c\})$

Proposition

The dimension of S is $n - |J|$, with $J = \{j \mid w_j > c \text{ for } j = 1, \dots n\}$.

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Facet and Convex Hull Proofs

Hp. conv(X) is bounded and full-dimensional.

- Given $X \in \mathbb{Z}^n_+$ and a valid inequality $\pi^\top x \leq \pi^0$ for X, how to prove that $\pi^\top x \leq \pi^0$ is a facet of conv(X)?

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Approach 1:

- Find n points in X that satisfy $\pi^\top x = \pi^0$ and prove they are affinely independent.

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Approach 2:

- Select $t \geq n$ points $x^1, \ldots, x^t \in X$ satisfying $\pi^\top x = \pi^0$. Suppose that all these points lie on a generic hyperplane $\mu^\top x = \mu^0$.
- Solve the linear equation system

$$\sum_{j=1}^{n} \mu_j x_j^k = \mu_0 \quad \text{for } k = 1, \ldots, t$$

  in the $n+1$ unknowns (μ, μ^0)
- If the only solution is $(\mu, \mu^0) = \alpha(\pi, \pi^0)$ for $\alpha \neq 0$, then the inequality $\pi^\top x = \pi^0$ is facet-defining.

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- Show that matrix A have a special structure guaranteeing $P = \text{conv}(X)$, like TUM.

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Approach 2:

- Show that points $x^* \in P$ with a fractional element are not extreme points of P.

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- Show that for all $c \in \mathbb{R}^n$ the linear program $\min\{c^\top x \mid Ax \leq b\}$ has an optimal solution $x^* \in X$.

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Approach 4:

- Show that if $\pi^\top x \leq \pi^0$ defines a facet of conv(X) then it must be identical to one of the inequalities $a_i^\top x \leq b_i$ defining P.

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Approach 5:

- Show that for any $c \in \mathbb{R}^n, c \neq 0$ the set of optimal solutions to the problem $\min\{c^\top x \mid x \in X\}$ lies in $\{x \mid a_i^\top x = b_i\}$ for some $i = 1, \ldots, m$.

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Etc.

Some remarks

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Computational results to show method effectiveness.

Strengthening Cover Inequalities: Extended Covers

Yesterday we have seen the Extended Cover Inequalities for the 01-KP problem.

Proposition

If C is a cover for X, the feasible set of the 01-KP problem, the extended cover inequality

$$\sum_{j \in E(C)} x_j \leq |C| - 1$$

is valid for X, where $E(C) = C \cup \{j \mid w_j \geq w_i \text{ for all } i \in C\}$.

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- The Extended Cover inequality dominates the Cover inequality
- The (Extended) Cover inequality can be strengthened by lifting

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Let C be a cover for K. The cover inequality associated with C is facet-defining for $P_C = \text{conv}(K) \cap \{x \in \mathbb{R}^n \mid x_j = 0 \ \forall j \in \{1, \ldots, n\} \setminus C\}$ if and only if C is a minimal cover.

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Given a cover C for the 01-KP problem, find the best possible values for α_j for $j \in \{1, \ldots, n\} \setminus C$ such that

$$\sum_{j \in C} x_j + \sum_{j \in \{1,\ldots,n\} \setminus C} \alpha_j x_j \leq |C| - 1$$

is valid for X.

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Iteratively solve the problem

$$\xi_t = \max \sum_{i=1}^{t-1} \alpha_{j_i} x_{j_i} + \sum_{j \in C} x_j$$

$$\sum_{i=1}^{t-1} \alpha_{j_i} x_{j_i} + \sum_{j \in C} w_j x_j \leq b - w_{j_t}$$

$$x \in \{0, 1\}^{|C|+t-1}$$

for t the current iteration and $j_1, \ldots, j_r$ be an ordering of $N \setminus C$.

Strengthening Cover Inequalities: Lifting

Given a cover C for the 01-KP problem, find the best possible values for α_j for $j \in \{1, \dots, n\} \setminus C$ such that $\sum_{j \in C} x_j + \sum_{j \in \{1,\dots,n\} \setminus C} \alpha_j x_j \leq |C| - 1$ is valid for X.

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Set $\alpha_{j_t} = |C| - 1 - \xi_t$ and iterate.

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The resulting Lift Cover Inequalities are **facet-defining**.

Strengthening Cover Inequalities: Lifting

Generalization

Strengthening Cover Inequalities: Lifting

Generalization

Proposition

Consider a set $S \subseteq \{0,1\}^n$ such that $S \cap \{x \mid x_n = 1\} \neq \emptyset$ and let $\sum_{j=1}^{n-1} \alpha_j x_j \leq \beta$ be a valid inequality for $S \cap \{x \mid x_n = 0\}$. Then

$$\alpha_n = \beta - \max \left\{ \sum_{j=1}^{n=1} \alpha_j x_j \mid x \in S, x_n = 1 \right\}$$

is the largest coefficient such that $\sum_{j=1}^{n-1} \alpha_j x_j + \alpha_n x_n \leq \beta$ is valid for S. Furthermore, if $\sum_{j=1}^{n-1} \alpha_j x_j \leq \beta$ defines a d-dimensional face of $\text{conv}(S) \cap \{x_n = 0\}$, then $\sum_{j=1}^{n} \alpha_j x_j \leq \beta$ defines a face of $\text{conv}(S)$ of dimension at least $d + 1$.

Outline

1. Tailored cutting-plane methods
   - Introduction
   - Strong inequalities
   - Polytope Dimension
   - Facet and Convex Hull Proofs
   - Lifting
   - Separation procedure

2. Research Talk

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Separation Procedure

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Potentially as difficult as solving the 01-KP problem $\rightarrow$ heuristics!

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