SUMMER SCHOOL ON ASPECTS OF OPTIMIZATION
Discrete Optimization
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Outline

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Mathematical Programming (MP)

$$\begin{aligned} \min_x \quad & f(x) \\ & g_i(x) \leq 0 \quad \forall i = 1, \ldots, m \\ & \underline{x} \leq x \leq \overline{x} \\ & x_j \in \mathbb{Z} \quad \forall j \in Z \end{aligned}$$

where

- x is an n-dimensional vector of the **decision variables**
- $\underline{x}$ and $\overline{x}$ are the given vectors of **lower and upper bounds** on the variables
- set $Z \subseteq \{1, 2, \ldots, n\}$ is the set of the indexes of the **integer variables**

$$
\begin{array}{rcll}
\min_x & f(x) & & \\
& g_i(x) \leq & 0 & \forall i = 1, \ldots, m \\
\underline{x} & \leq x \leq & \overline{x} & \\
& x_j \in & \mathbb{Z} & \forall j \in Z
\end{array}
$$

where $f(x)$ and $g_i(x)$ ($\forall i = 1, \ldots, m$):

- can be written in **closed form**
- are **twice continuously differentiable** functions of the variables

A few definitions:

- **Formulation** : a MP modeling an optimization problem
- An optimization problem can be modeled in different ways $\rightarrow$ **several formulations**
- **Instance** : when the expression of $f(x)$, $g(x)$ and the values of $\underline{x}$, $\overline{x}$, and Z are known. The set of instances of a MP problems is potentially infinite.

A few definitions:

- **Feasible solutions** :
  $X = \{x \mid g(x) \leq 0, \quad \underline{x} \leq x \leq \overline{x}, \quad x_j \in \mathbb{Z} \; \forall j \in Z\}$

- **Optimal solution** : $\arg\min_{x \in X} f(x)$

- **Heuristc solution** : a feasible solution (hopefully of good quality)

$$\begin{aligned} \min_x \quad & f(x) \\ g_i(x) \leq \; & 0 \quad \forall i = 1, \ldots, m \\ \underline{x} \leq x \leq \; & \overline{x} \\ x_j \in \; & \mathbb{Z} \quad \forall j \in Z \end{aligned}$$

- **Linear Programming** (LP): $f(x)$ and $g(x)$ are linear, $Z = \emptyset$
- **Integer (Linear) Programming** (ILP): $f(x)$ and $g(x)$ are linear, $Z = \{1, 2, \ldots, n\}$
- **Mixed Integer (Linear) Programming** (MILP): $f(x)$ and $g(x)$ are linear, $Z \subset \{1, 2, \ldots, n\}$
- **Mixed Integer Non Linear Programming** (MINLP): $f(x)$ and $g(x)$ are twice continuously differentiable, $Z \subset \{1, 2, \ldots, n\}$

Black Box Optimization: $f(x)$ or $g(x)$ $\rightarrow$ no closed form

$$
\begin{aligned}
\min_x \quad & f(x) \\
g_i(x) \leq \quad & 0 \quad \forall i = 1, \dots, m \\
\underline{x} \quad \leq x \leq \quad & \overline{x} \\
x_j \in \quad & \mathbb{Z} \quad \forall j \in Z
\end{aligned}
$$

Linear Programming (LP) problem:

$$
\begin{aligned}
\min_x f(x) \quad & \rightarrow \quad \min_x c^\top x \\
g(x) \leq 0 \quad & \rightarrow \quad Ax \leq b \\
\underline{x} \leq x \leq \overline{x} \quad & \rightarrow \quad \underline{x} \leq x \leq \overline{x} \\
x_j \in \mathbb{Z} \quad \forall j \in Z \quad & \rightarrow \quad \text{removed}
\end{aligned}
$$

LP problems

$$\min_x c^\top x$$
$$\begin{aligned} Ax &\leq b \\ \underline{x} \leq x &\leq \overline{x} \end{aligned}$$

W.l.o.g. because

$$\max \tilde{c}^\top x \rightarrow -\min -\tilde{c}^\top x$$

For some i, $\tilde{A}_i x \geq \tilde{b}_i \rightarrow -\tilde{A}_i x \leq -\tilde{b}_i$

For some i, $\tilde{A}_i x = \tilde{b}_i \rightarrow -\tilde{A}_i x \leq -\tilde{b}_i$ and $\tilde{A}_i x \leq \tilde{b}_i$

Moreover, $\forall j \in 1, \ldots, n$ $\underline{x}_j \in [-\infty, +\infty)$ and $\overline{x}_j \in (-\infty, +\infty]$.

LPs characteristics

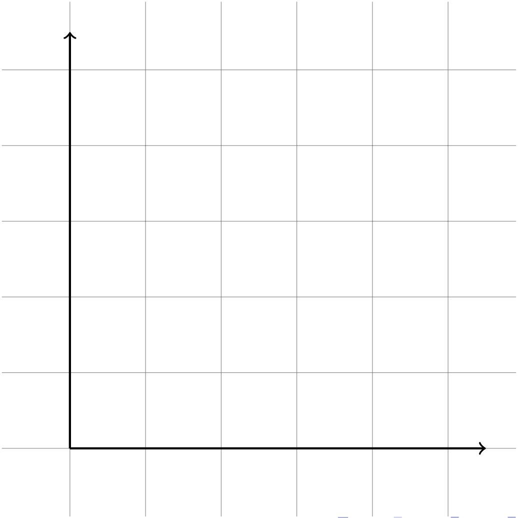
Feasible (solutions) set/region : $X = \{x \mid Ax \leq b, \underline{x} \leq x \leq \overline{x}\}$

- **optimal**: when $X \neq \emptyset$, bounded. In this case, an optimal solution is found, i.e., a feasible point x^* s.t. $c^\top x^* \leq c^\top x$ for all feasible $x \in X$
- **infeasible**: when $X = \emptyset$
- **unbounded**: when the $\min\{c^\top x \mid x \in X\} = -\infty$

Geometrical intuition of LPs

Example 1

$$\max x_1 + x_2$$
$$3x_1 + 2x_2 \leq 13$$
$$0 \leq x_1 \leq 3$$
$$0 \leq x_2 \leq 5.$$



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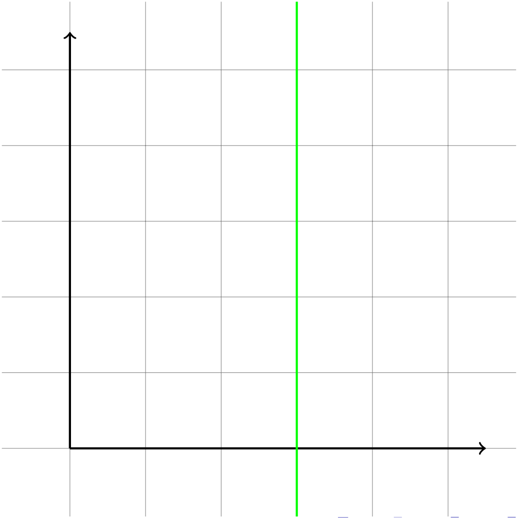

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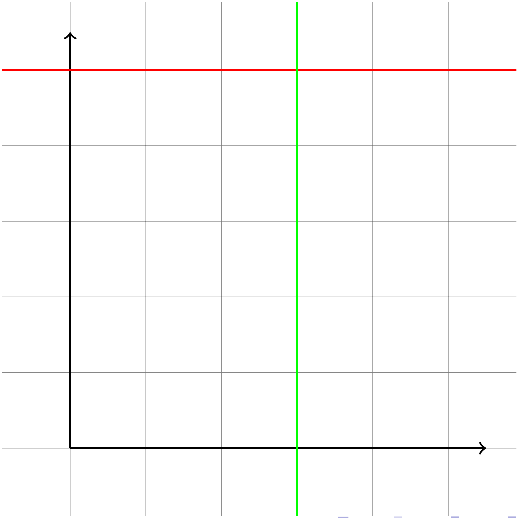

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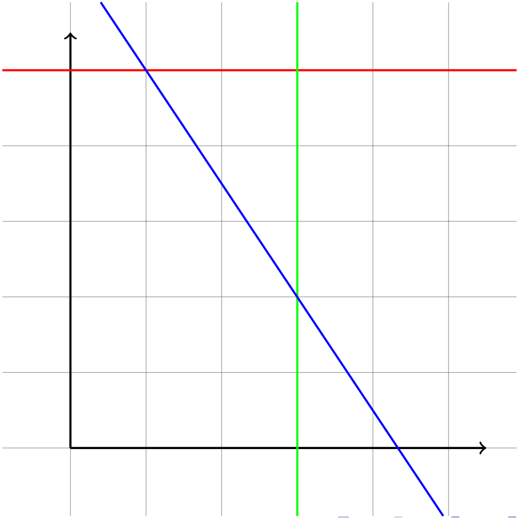

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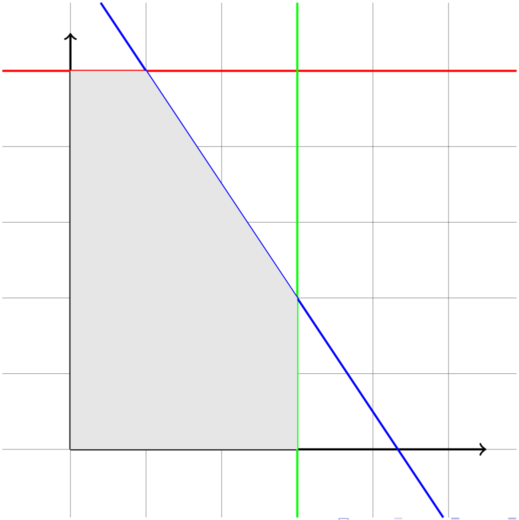

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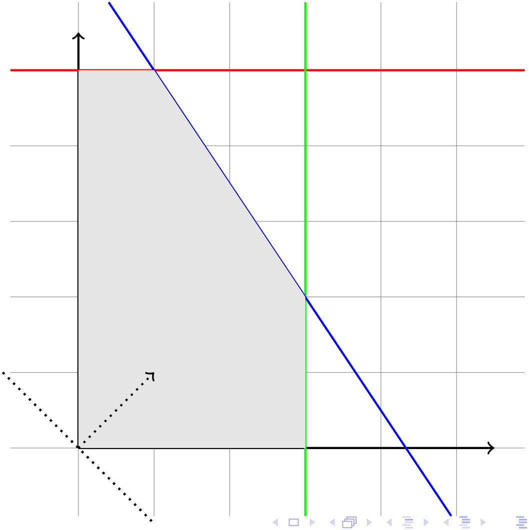

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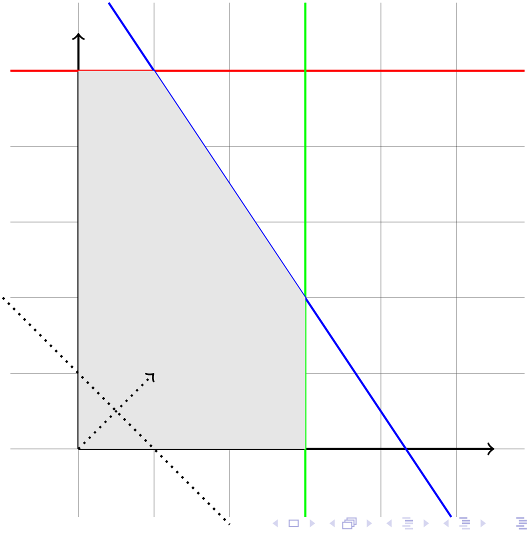

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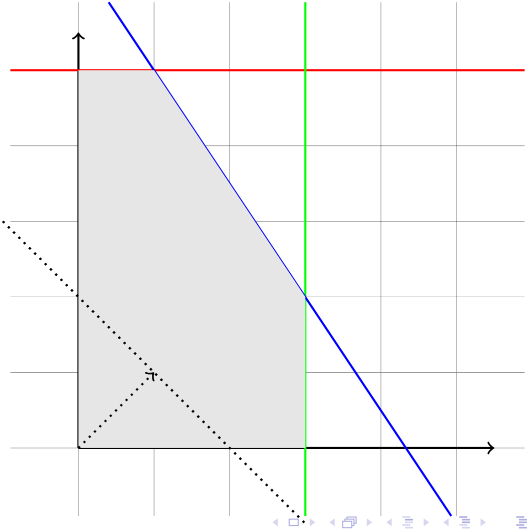

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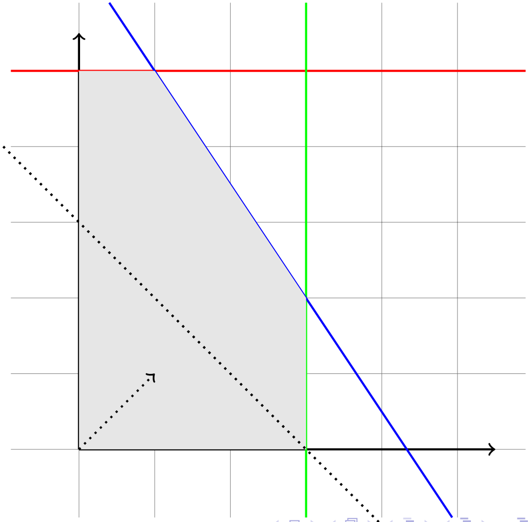

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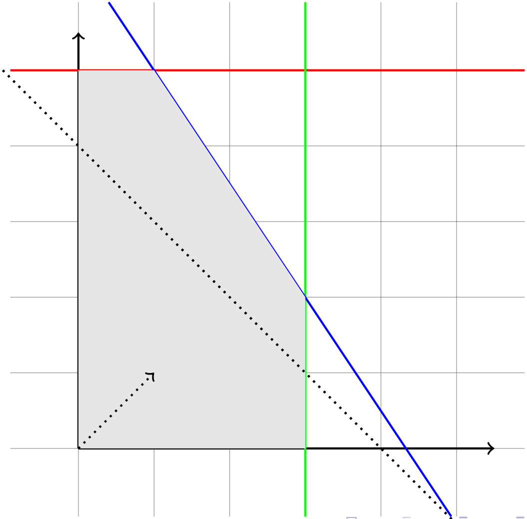

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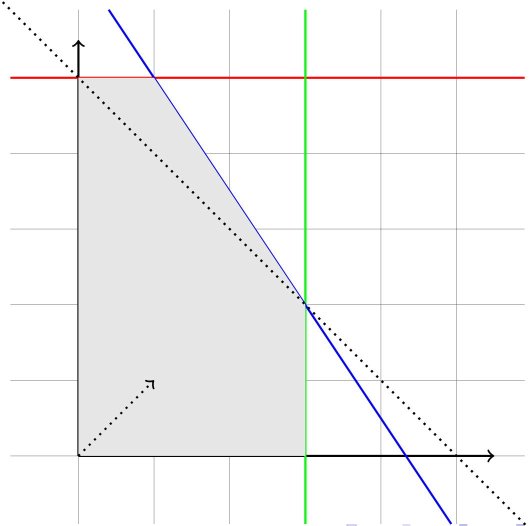

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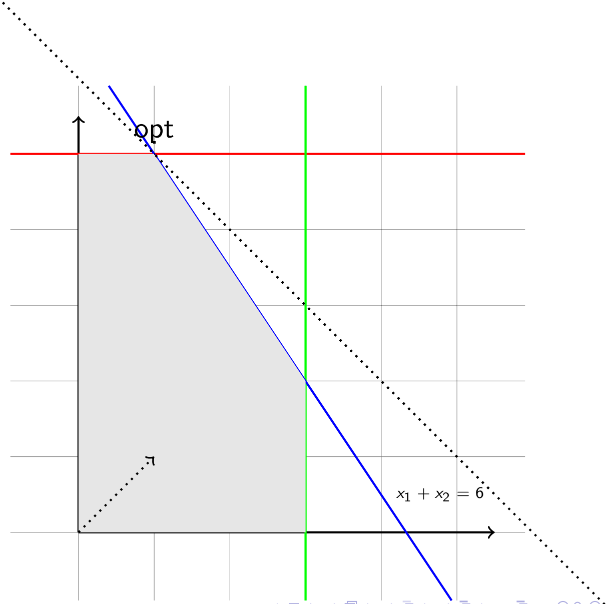
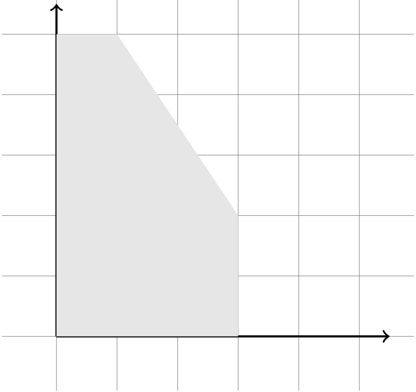


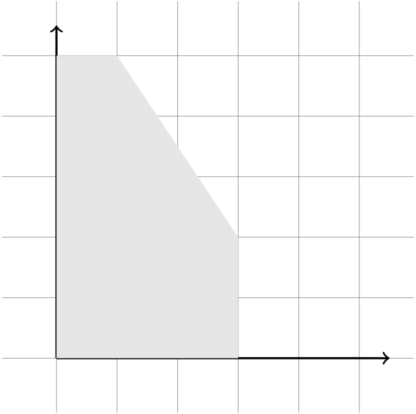
A few definitions



Consider some $i \in \{1, \ldots, m\}$

- Hyperplane : $\{x \in \mathbb{R}^n \mid A_i^\top x = b_i\}$
- Half-space : $\{x \in \mathbb{R}^n \mid A_i^\top x \leq b_i\}$ or $\{x \in \mathbb{R}^n \mid A_i^\top x \geq b\}$

A few definitions



- Polyhedron : $\{x \in \mathbb{R}^n \mid Ax \leq b\}$
- Polytope : a bounded polyhedron

Remark

The feasible region of a LP problem is a polyhedron (by definition).

Some properties and theorems

Definition

Given points $v^1, v^2, \dots, v^p \in \mathbb{R}^n$, their **convex combination** is $z = \sum_{i=1}^{p} \alpha_i v^i$ s.t. $\sum_{i=1}^{p} \alpha_i = 1$ and $\alpha_i \geq 0$ for all $i = 1, \dots, p$.

Theorem

Every polyhedron $P \subseteq \mathbb{R}^d$ can be written as

$$P = conv\{v^1, \dots, v^k\} + cone\{r^1, \dots, r^\ell\}$$

with points $v^1, \dots, v^k \in \mathbb{R}^d$ and rays $r^1, \dots, r^\ell \in \mathbb{R}^d$

where $\text{cone}\{r^1, \dots, r^\ell\} = \{x \in \mathbb{R}^d \mid x = \mu_1 r^1 + \dots \mu_\ell r^\ell, \mu_1, \dots, \mu_\ell \geq 0\}$.

Some properties and theorems

Theorem

Each point of a polytope is a convex combination of its vertices.

Theorem

Each convex combination of the vertices of a polytope is a point of the polytope.

Theorem

A vertex is not a strict convex combination of two distinct points of the polytope.

Thus, a polytope can be **characterized/described** by a finite number of half-spaces (H-description) or its vertices (V-description).

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Mixed Integer Linear Programming

$$\min_x c^\top x$$
$$Ax \leq b$$
$$\underline{x} \leq x \leq \overline{x}$$
$$x_j \in \mathbb{Z} \qquad \forall j \in Z$$

where

- x is an n-dimensional vector of the decision variables,
- $\underline{x}$ and $\overline{x}$ are the given vectors of lower and upper bounds on the variables,
- c is the cost vector, A the constraints matrix, and b the right-hand-side vector,
- the set Z includes the indexes of the integer variables.

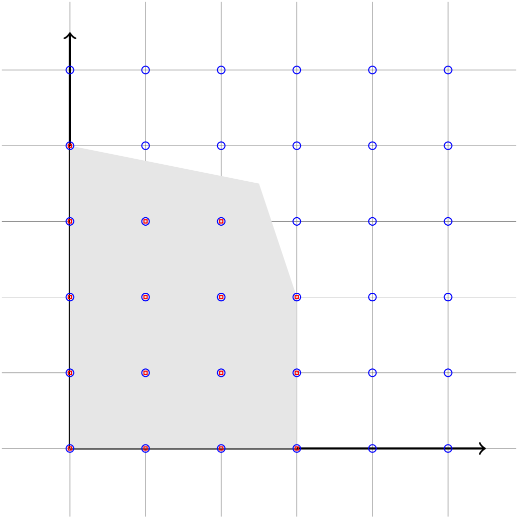

Figure: Lattice points (in blue), feasible region of the continuous relaxation (in gray), and their intersection (in red).

Examples: the Assignment Problem (AP)

- n people available for n tasks.
- Cost c_{ij} is invers. proportional to the suitedness of person i to task j.
- Find the minimum cost assignment.

Variables: $x_{ij} = 1$ when person i is assigned to task j, 0 otherwise $(\forall i = 1, \ldots, n; j = 1, \ldots, n)$

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad \forall i = 1, \ldots, n$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad \forall j = 1, \ldots, n$$

$$x_{ij} \in \{0, 1\} \qquad \forall i = 1, \ldots, n; j = 1, \ldots, n$$

Examples: the 01-Knapsack Problem (KP)

- Knapsack capacity c (maximum weight).
- n available items
- w_j weight of item j, p_j profit given by item j
- Select the items so as to respect the capacity and maximize the profit.

Variables: $x_i = 1$ when item j is selected, 0 otherwise $(\forall j = 1, \ldots, n)$

$$\max \sum_{j=1}^{n} p_j x_j$$

$$\sum_{j=1}^{n} w_j x_j \leq c$$

$$x_j \in \{0, 1\} \qquad j = 1, \ldots, n$$

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Applications

- **Finance**, e.g., robust portfolio selection
- **Power systems** , e.g., unit commitment, optimal power flow
- **Air traffic management** , e.g., aircraft conflicts detection and resolution
- **Transportation** , e.g., vehicle routing problem
- **etc.**

Motivation

- **Discrete domain** of one (or more) variables
- Domain **discontinuity**
- **Conditional** constraints
- **Fixed cost**
- **Disjunctive** constraints
- **Absolute value** of a variable

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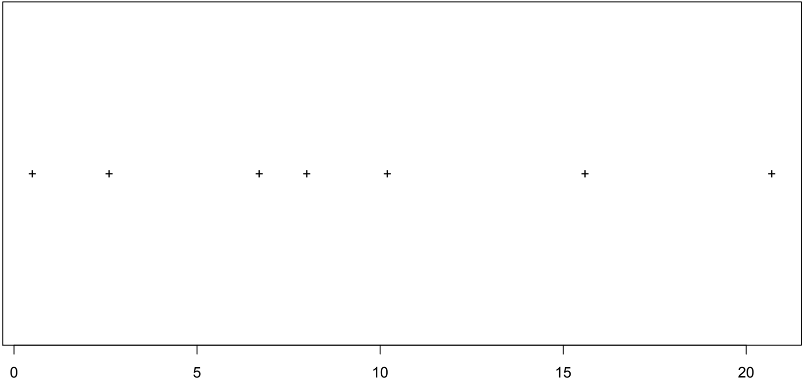
Examples of modeling with integer/binary variables: Discrete domain of one (or more) variables

How to model the condition: $x \in \mathbb{R}$ and $x \in \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{\tilde{k}}\}$ where $\tilde{x}_k \in \mathbb{R}$ for $k = 1, \ldots, \tilde{k}$ within a MILP?



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- Additional **binary variables**: $y \in \{0, 1\}^{\tilde{k}}$

$$x = \sum_{k=1}^{\tilde{k}} \tilde{x}_k y_k$$

$$\sum_{k=1}^{\tilde{k}} y_k = 1.$$

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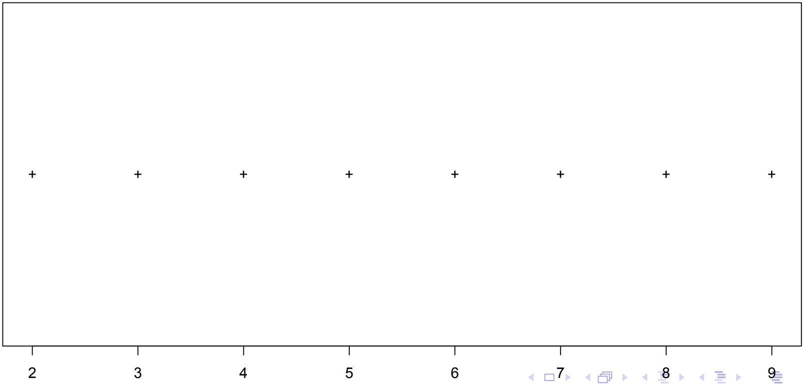
When $\tilde{x}_1 \in \mathbb{Z}$ and $\tilde{x}_k - \tilde{x}_{k-1} = 1$ for $k = 2, \ldots, \tilde{k}$



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When $\tilde{x}_1 \in \mathbb{Z}$ and $\tilde{x}_k - \tilde{x}_{k-1} = 1$ for $k = 2, \ldots, \tilde{k}$

- An **integer variable** x

$$
\begin{aligned}
\tilde{x}_1 \leq x &\leq \tilde{x}_{\tilde{k}} \\
x &\in \text{integer.}
\end{aligned}
$$

Examples of modeling with integer/binary variables: Discrete domain of one (or more) variables

How to model the condition: $x \in \mathbb{R}$ and $x \in \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{\tilde{k}}\}$ where $\tilde{x}_k \in \mathbb{R}$ for $k = 1, \ldots, \tilde{k}$ within a MILP?

When $\tilde{x}_k - \tilde{x}_{k-1} = 1$ for $k = 2, \ldots, \tilde{k}$, another alternative

- Additional **binary variables**: $y \in \{0,1\}^{(\ell+1)}$ (ℓ is the smallest integer such that $\tilde{x}_{\tilde{k}} - \tilde{x}_1 < 2^{\ell+1}$)

$$
\begin{aligned}
x &= \tilde{x}_1 + \sum_{i=0}^{\ell} 2^i y_i \\
\tilde{x}_1 \le x &\le \tilde{x}_{\tilde{k}} \\
y_i &\in \{0,1\} \quad \forall i = 0, \ldots, \ell
\end{aligned}
$$

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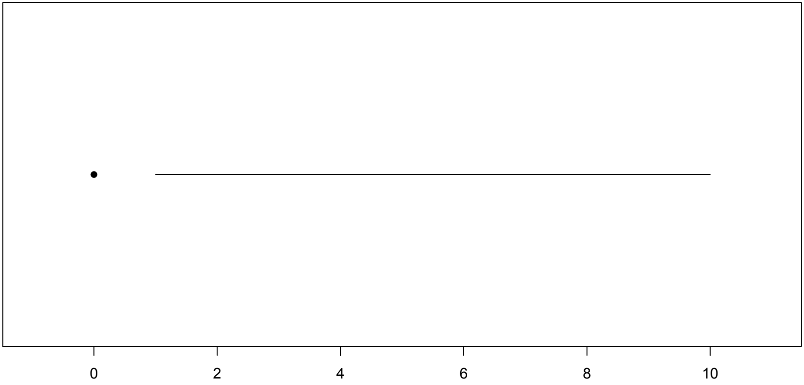
Examples of modeling with integer/binary variables: Domain discontinuity

Discontinuous domain $x \in \{0\} \cup [\underline{x}, \overline{x}]$



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Discontinuous domain $x \in \{0\} \cup [\underline{x}, \overline{x}]$

MILP formulation:

$$\underline{x}y \leq x \leq \overline{x}y$$
$$y \in \{0, 1\}.$$

For $y = 0$, $x = 0$

For $y = 1$, $x \in [\underline{x}, \overline{x}]$.

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Examples of modeling with integer/binary variables: Conditional constraints

Impose a constraint $a_i^\top x \leq b_i$ only **under certain conditions** .

For example: If $x_1 \geq \tilde{x}_1$ then $a_i^\top x \leq b_i$.

- Additional **binary variable**, say $y_i \in \{0, 1\}$, allows to activate or deactivate both the condition and the conditional constraint

$$\begin{aligned} \tilde{x}_1(1 - y_i) \leq x_1 &\leq \tilde{x}_1 + (1 - y_i)(\overline{x}_1 - \tilde{x}_1) \\ a_i^\top x &\leq b_i + My_i \end{aligned}$$

where M is the so-called *big-M* , i.e., a large enough parameter (hp. $x_1 \geq 0$).

$y_i = 0 \rightarrow x_1 \geq \tilde{x}_1 \rightarrow a_i^\top x \leq b$

$y_i = 1$ the constraint is deactivated and $x_1 \geq 0$, $x_1 \leq \tilde{x}_1$.

How to set the value of the big-M?

Examples of modeling with integer/binary variables: Conditional constraints

Example

If $x_1 \geq 5$, then $10x_2 + 5x_3 \leq 25$

$$10x_2 + 5x_3 \leq 25 + My_1$$
$$y_1 \in \{0, 1\}.$$

Hp: $x_2 \leq 10$ and $x_3 \leq 10$

Value of M:
Max LHS : $10 \cdot 10 + 5 \cdot 10 = 150$
$M \geq 150 - 25 = 125.$

Tricky to set big-M value. Overestimate (valid model)

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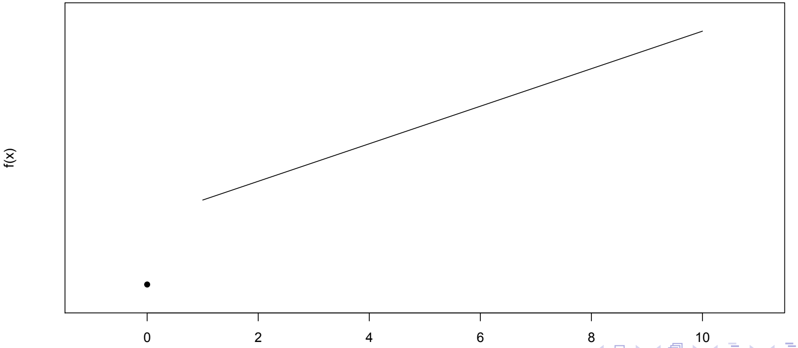
Examples of modeling with integer/binary variables: Fixed cost

A cost: composed of a **fixed part** and a variables part (**discontinuous**):

$$f(x) = \begin{cases} cx + d & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$



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MILP modeling:

$$\begin{aligned} \min f(x) &= cx + dy \\ x &\leq \bar{x}y \\ y &\in \{0, 1\} \end{aligned}$$

If $x > 0$, $y = 1$, thus $f(x) = cx + d$.
If $x = 0$, $y = 0$ (because of min of the obj function)

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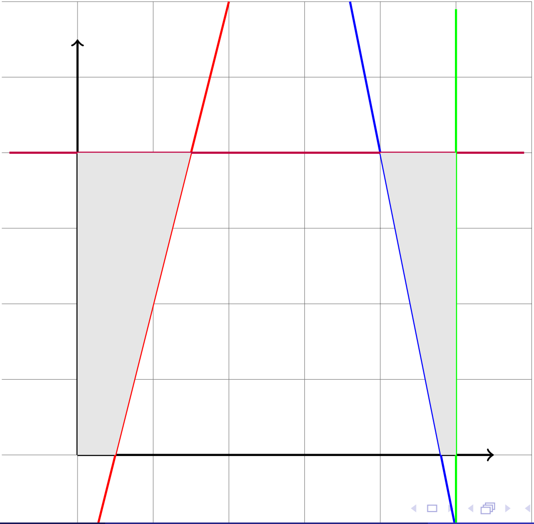
Examples of modeling with integer/binary variables: Disjunctive constraints

Disjunction : satisfy $a_i^\top x \leq b_i$ or $a_k^\top x \leq b_k$.



Examples of modeling with integer/binary variables: Disjunctive constraints

Disjunction : satisfy $a_i^\top x \leq b_i$ or $a_k^\top x \leq b_k$.

$$
\begin{aligned}
a_i^\top x &\leq b_i + M_i y_i \\
a_k^\top x &\leq b_k + M_k y_k \\
y_i + y_k &\leq 1 \\
y_i &\in \{0, 1\} \\
y_k &\in \{0, 1\}
\end{aligned}
$$

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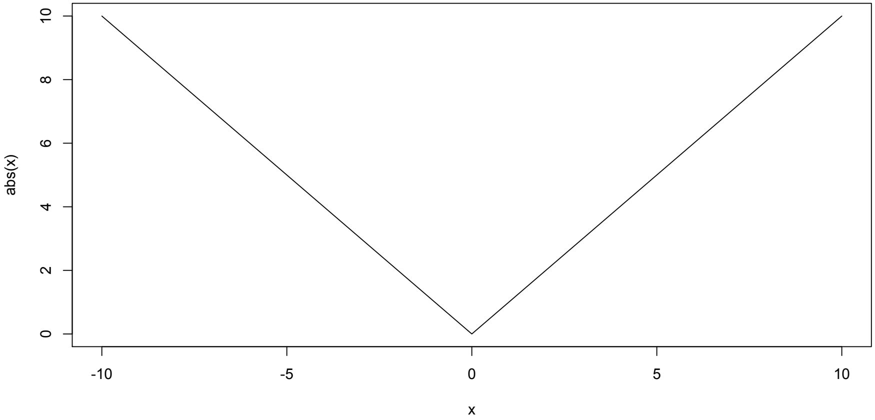
Examples of modeling with integer/binary variables: Absolute value of a variable

MILP modeling of $|x|$?



Examples of modeling with integer/binary variables: Absolute value of a variable

MILP modeling of $|x|$?

$$
\begin{aligned}
|x| &= x^+ + x^- \\
x &= x^+ - x^- \\
0 \le x^+ &\le \overline{x} y \\
0 \le x^- &\le -\underline{x}(1-y) \\
y &\in \{0,1\}.
\end{aligned}
$$

If $x \le 0$, $y = 0$, $x^+ = 0$, and $x^- \in [0, -\underline{x}]$.
If $x \ge 0$, $y = 1$, $x^- = 0$, and $x^+ \in [0, \overline{x}]$.

Where $\underline{x} \le x \le \overline{x}$, hp. $\underline{x} < 0$ w.l.o.g.

Mixed Integer Linear Programming

$$\min_x c^\top x$$
$$Ax \leq b$$
$$\underline{x} \leq x \leq \overline{x}$$
$$x_j \in \mathbb{Z} \qquad \forall j \in Z$$

where

- x is an n-dimensional vector of the decision variables,
- $\underline{x}$ and $\overline{x}$ are the given vectors of lower and upper bounds on the variables,
- c is the cost vector, A the constraints matrix, and b the right-hand-side vector,
- the set Z includes the indexes of the integer variables.

Mixed Integer Linear Programming

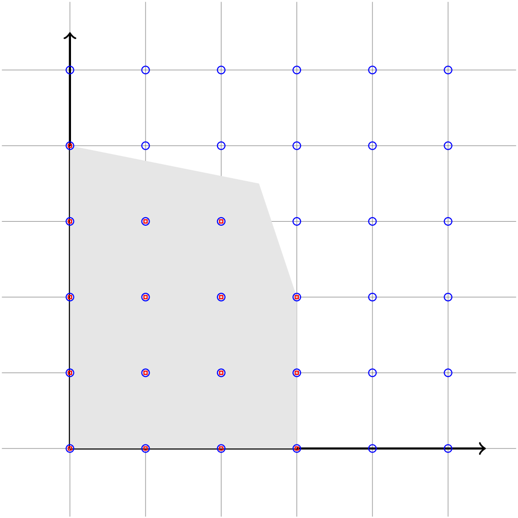


Figure: Lattice points (in blue), feasible region of the continuous relaxation (in gray), and their intersection (in red).

Exercises

Formulate the following as mixed integer linear programs:

1. $u = \min\{x_1, x_2\}$, assuming that $0 \leq x_j \leq C$ for $j = 1, 2$.
2. $v = \|x_1 - x_2\|_\infty$ with $0 \leq x_j \leq C$ for $j = 1, 2$.
3. the set $X \setminus \{x^*\}$ where $X = \{x \in \mathbb{Z}^n \mid Ax \leq b\}$ and $x^* \in X$.

Outline

Outline

Formulations/Reformulations

- An optimization problem could be modeled in several, different ways
- Each of the possible MP models is a **formulation** of the same problem
- A MP formulation $\mathcal{Q}$ is a **reformulation** of another MP formulation $\mathcal{P}$ if they are different formulations of the same optimization problem
- Reformulating a problem is interesting when
  - The reformulation shows nicer **mathematical properties**
  - The reformulation is more **tractable**

Reformulation Examples: equivalent forms of LPs

General form

$$
\begin{array}{lrcll}
\min c^\top x & & & & \\
& a_i^\top x & = & b_i & i \in M \\
& a_i^\top x & \geq & b_i & i \in \overline{M} \\
& x_j & \geq & 0 & j \in N \\
& x_j & \lesseqgtr & 0 & j \in \overline{N}
\end{array}
$$

Canonical form

$$
\begin{array}{lrcl}
\min c^\top x & & & \\
& Ax & \geq & b \\
& x & \geq & 0
\end{array}
$$

Standard form

$$
\begin{array}{lrcl}
\min c^\top x & & & \\
& Ax & = & b \\
& x & \geq & 0
\end{array}
$$

Reformulation Examples: equivalent forms of LPs

$$\max c^\top x \quad \rightarrow \quad -\min(-c^\top x)$$

$$a_i^\top x \geq b_i \quad \rightarrow \quad \begin{cases} a_i^\top x - s_i = b_i \\ s_i \geq 0 \end{cases}$$

$$a_i^\top x \leq b_i \quad \rightarrow \quad \begin{cases} a_i^\top x + s_i = b_i \\ s_i \geq 0 \end{cases}$$

$$a_i^\top x = b_i \quad \rightarrow \quad \begin{cases} a_i^\top x \geq b_i \\ a_i^\top x \leq b_i \end{cases}$$

$$x_j \lesseqqgtr 0 \quad \rightarrow \quad \begin{cases} x_j = x_j^+ - x_j^- \\ x_j^+ \geq 0 \\ x_j^- \geq 0 \end{cases}$$

Relaxation

Definition

A problem R: $z^R = \min\{f(x) \mid x \in T \subseteq \mathbb{R}^n\}$ is a relaxation of problem P: $z = \min\{c^\top x \mid x \in X \subseteq \mathbb{R}^n\}$ if:

- $X \subseteq T$ and
- $f(x) \leq c^\top x \ \forall x \in X$.

Proposition

If R is a relaxation of P, then $z^R \leq z$.

Classical relaxation:

- **continuous**: integrality requirements are relaxed (also known as LP relaxation in the (MI)LP context)

Linear Programming Relaxation

Definition

For an integer program $\min\{c^\top x \mid x \in P \cap \mathbb{Z}^n\}$ with formulation $P = \{x \in \mathbb{R}^n_+ \mid Ax \leq b\}$, the linear programming relaxation is the linear program $z^{LP} = \min\{c^\top x \mid x \in P\}$.

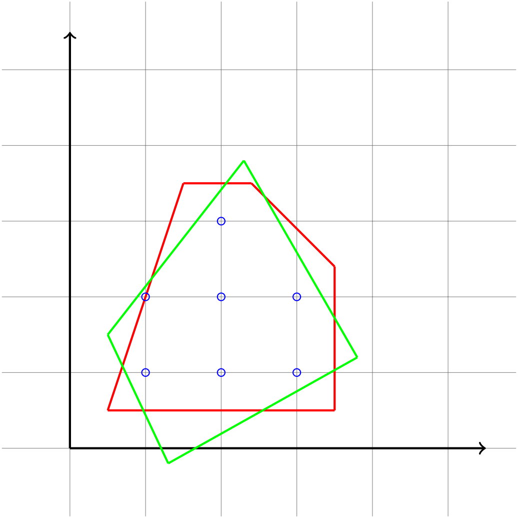
Relationship between the solution of a relaxation and of the original problem

Proposition

1. If a relaxation R is infeasible, the original problem P is infeasible.
2. Let x^* be an optimal solution of R. If $x^* \in X$ and $f(x^*) = c^\top x^*$, then x^* is an optimal solution of P.

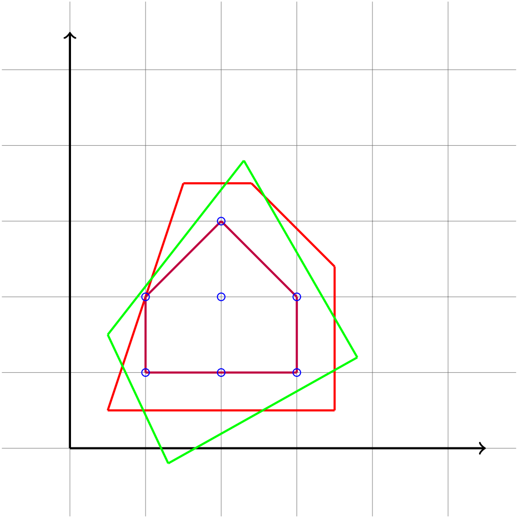
Formulations of the same IP problem

Example inspired by Example 1.2, "Integer Programming", Wolsey.
$X = \{(1,1),(2,1),(3,1),(1,2),(2,2),(3,2),(2,3)\}$



Ideal formulation

Example inspired by Example 1.2, "Integer Programming", Wolsey.
$X = \{(1,1),(2,1),(3,1),(1,2),(2,2),(3,2),(2,3)\}$



Comparing formulations

Convex hull

$\text{conv}(X) = \{x \mid x = \sum_{i=1}^{t} \lambda_i x^i, \sum_{i=1}^{t} \lambda_i = 1, \lambda_i \geq 0 \ \forall i = 1, \ldots, t$ for every subset $\{x^1, \ldots, x^t\}$ of $X\}$.

Proposition

$\text{conv}(X)$ is a polytope.

Proposition

The extreme points of $\text{conv}(X)$ all lie in X.

Thus, $\{\min c^\top x \mid x \in X\}$ is equivalent to $\{\min c^\top x \mid x \in \text{conv}(X)\}$.

Usually no simple characterization of $\text{conv}(X)$ (exponential number of inequalities).

Comparing formulations

Given two formulations P_1 and P_2 for X, is one better than the other?

Definition

Given a set $X \subseteq \mathbb{R}^n$ and two formulations P_1 and P_2 for X, P_1 is a better formulation than P_2 if $P_1 \subset P_2$.

Proposition

Suppose P_1 and P_2 are two formulations for the integer program $\min\{c^\top x \mid x \in X \subseteq \mathbb{Z}^n\}$ with P_1 a better formulation than P_2 ($P_1 \subset P_2$). If $z_i^{LP} = \min\{c^\top x \mid x \in P_i\}$ ($i = 1, 2$) are the values of the associated linear programming relaxations, then $z_1^{LP} \geq z_2^{LP}$.

$$z_1 = \min\{c^\top x \mid x \in P_1\} \geq z_2 = \min\{c^\top x \mid x \in P_2\}$$

Comparing formulations

P_1 and P_2 two formulations for X, where $P_1 \subset P_2$.

$z_1 = \min\{c^\top x \mid x \in P_1\} \geq z_2 = \min\{c^\top x \mid x \in P_2\}$

$z^* = \min\{c^\top x \mid x \in \text{conv}(X)\} \geq \min\{c^\top x \mid x \in P_1\}$

z^*, z_1, z_2 are **Lower Bounds (LB)** of $\min\{c^\top x \mid x \in X\}$

Actually $z^* = \min\{c^\top x \mid x \in X\}$!

Ideal formulation : if we solve its LP relaxation, then we solve the IP as well (each extreme point is integer).

Outline

Outline

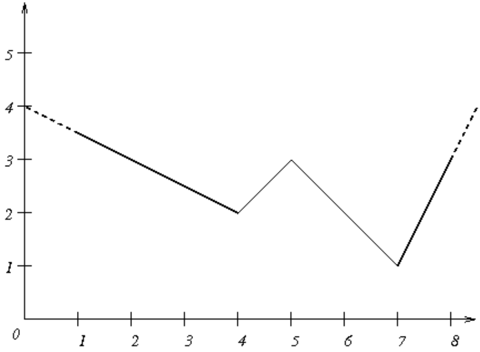
An example of formulations comparison: the 01-KP problem

$$X = \{(0,0,0,0),(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1),(0,1,0,1),(0,0,1,1)\}$$

$P_1 = \{x \in \mathbb{R}^4 : 0 \leq x \leq 1, 83x_1 + 61x_2 + 49x_3 + 20x_4 \leq 100\}$
$P_2 = \{x \in \mathbb{R}^4 : 0 \leq x \leq 1, 4x_1 + 3x_2 + 2x_3 + 1x_4 \leq 4\}$

$$\begin{aligned} P_3 = \{x \in \mathbb{R}^4 : & \\ 4x_1 + 3x_2 + 2x_3 + 1x_4 &\leq 4 \\ 1x_1 + 1x_2 + 1x_3 &\leq 1 \\ 1x_1 \qquad\qquad + 1x_4 &\leq 1 \\ 0 \leq x \leq 1\} & \end{aligned}$$

An example of formulations comparison: Piecewise Linear Functions



- Breakpoints $(\tilde{x}_k, \tilde{z}_k)$ for $k = 0, \ldots, s$
- Slope of k-th segment $\sigma_k = \frac{\tilde{z}_k - \tilde{z}_{k-1}}{\tilde{x}_k - \tilde{x}_{k-1}}$ for $k = 1, \ldots, s$

An example of formulations comparison: Piecewise Linear Functions

Three Different Formulations of PWL functions:

- Convex Combination (**CC**) Formulation
- Multiple Choice (**MC**) Formulation
- Incremental (**Inc**) Formulation
- ....

An example of formulations comparison: Piecewise Linear Functions

Convex Combination Formulation

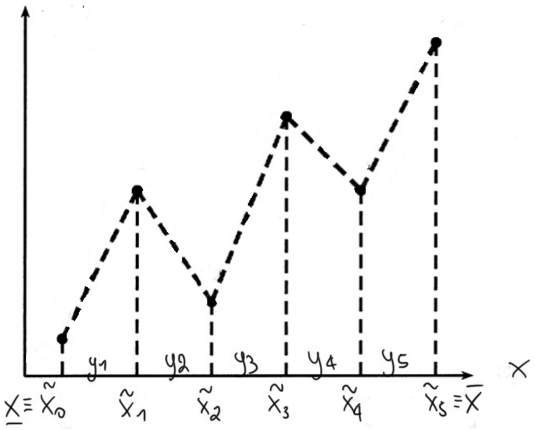


Figure: Source: Padberg, 2000 (modified)

An example of formulations comparison: Piecewise Linear Functions

Convex Combination Formulation

- Each piece is a convex combination of two consecutive breakpoints $(\tilde{x}_{k-1}, \tilde{z}_{k-1})$ and $(\tilde{x}_k, \tilde{z}_k)$ for all $k = 1, \ldots, s$

$$
\begin{aligned}
z &= \sum_{k=1}^{s} (\mu_k \tilde{z}_{k-1} + \lambda_k \tilde{z}_k) \\
x &= \sum_{k=1}^{s} (\mu_k \tilde{x}_{k-1} + \lambda_k \tilde{x}_k) \\
\mu_k + \lambda_k &= y_k \qquad \forall k = 1, \ldots, s \\
\sum_{k=1}^{s} y_k &= 1 \\
\mu, \lambda &\geq 0 \\
y_k &\in \{0, 1\} \qquad \forall k = 1, \ldots, s.
\end{aligned}
$$

An example of formulations comparison: Piecewise Linear Functions

Multiple Choice Formulation

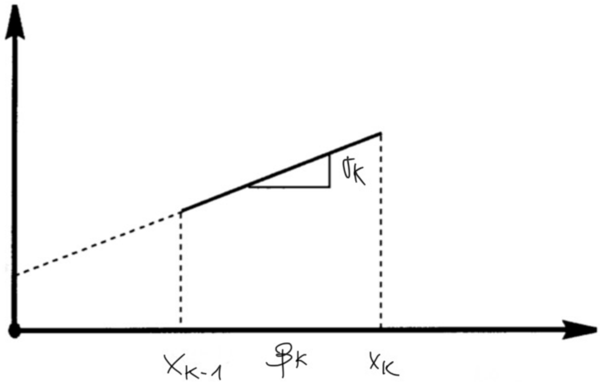


Figure: Source: Croxton et al., 2003

An example of formulations comparison: Piecewise Linear Functions

Multiple Choice Formulation

- x (and, consequently $\tilde{z}$) can only lie on one of the s intervals of the piecewise linear approximation

$$
\begin{aligned}
z &= \sum_{k=1}^{s} (\tilde{z}_{k-1} y_k + \sigma_k(\beta_k - \tilde{x}_{k-1} y_k)) \\
x &= \sum_{k=1}^{s} \beta_k \\
\tilde{x}_{k-1} y_k \le \beta_k &\le \tilde{x}_k y_k \qquad \forall k = 1, \ldots, s \\
\sum_{k=1}^{s} y_k &= 1 \\
y_k &\in \{0, 1\} \qquad \forall k = 1, \ldots, s
\end{aligned}
$$

An example of formulations comparison: Piecewise Linear Functions

Incremental Formulation

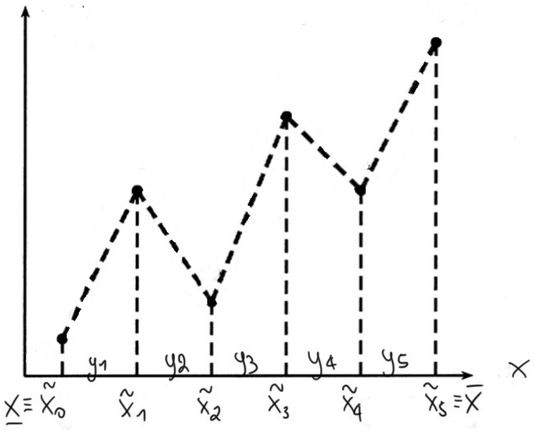


Figure: Source: Padberg, 2000 (modified)

An example of formulations comparison: Piecewise Linear Functions

Incremental Formulation

- One more than one binary variable y could take the value one. In particular, they observe the following order $1 \geq y_1 \geq y_2 \geq \cdots \geq y_s \geq 0$

$$
\begin{aligned}
z &= \tilde{z}_0 + \sum_{k=1}^{s} \sigma_k \delta_k \\
x &= \tilde{x}_0 + \sum_{k=1}^{s} \delta_k \\
\delta_k &\leq (\tilde{x}_k - \tilde{x}_{k-1}) y_k && \forall k = 1, \ldots, s \\
\delta_k &\geq (\tilde{x}_k - \tilde{x}_{k-1}) y_{k+1} && \forall k = 1, \ldots, s \\
y_k &\in \{0, 1\} && \forall k = 1, \ldots, s
\end{aligned}
$$

where the additional variable y_{s+1} is set to 0.

An example of formulations comparison: Piecewise Linear Functions

Croxton et al. (2003) analyzed the continuous relaxations of the 3 formulations.

Aim of the analysis: identifying the **strongest formulation** (continuous relaxation is the closest to the formulation itself).

Proposition 1 (Croxton et al. (2003)) The LP relaxations of the incremental, multiple choice, and convex combination formulations are **equivalent**, in the sense that any feasible solution of one LP relaxation corresponds to a feasible solution to the others with the same cost.

An example of formulations comparison: Piecewise Linear Functions

Exercises:

1. Prove the equivalence of the Multiple Choice and Convex Combination formulations
2. Prove the values of z are the same
3. Prove the equivalence of the Incremental and Multiple Choice formulations (implies the equivalence of the Incremental and Convex Combination formulations)

An example of formulations comparison: Piecewise Linear Functions

Recall: a formulation is **ideal** if all vertices of its continuous relaxation are integer.

- Lee and Wilson (2001) and Padberg (2000) showed that a variant of CC is not locally ideal.
- Vielma et al. (2010) showed that the other formulation are locally ideal.
- Jeroslow and Lowe (1984): a formulation P of S **sharp** when its projection is exactly the convex hull of S.
- Vielma et al. (2010): Any locally ideal formulation is sharp.
- Vielma et al. (2010): All formulations presented are sharp.

Sharpness weaker property than being locally ideal.

Sharpness is sufficient to consider a formulation strong.

An example of formulations comparison: Piecewise Linear Functions

Formulations Size

Number of constraints, additional variables, binaries for the three formulations

Model	Constraints	Continuous	Binaries
CC	2+s	2s	s
MC	2+2s	s	s
Inc	2+2s	s	s(+1)

Outline

Outline

When is it sufficient to solve the LP relaxation to get the IP solution?

- When the compact description of the convex hull $\text{conv}(X)$ is known.

Definition

A convex set $P \subseteq \mathbb{R}^n$ is integral if $P = \text{conv}(P \cap \mathbb{Z}^n)$.

Theorem (Hoffman and Kruskal)

Let A be an $m \times n$ matrix. The polyhedron $\{x \mid Ax \leq b, x \in \mathbb{R}^n_+\}$ is integral for every vector $b \in \mathbb{Z}^m$ if and only if A is totally unimodular.

Definition

A matrix A is totally unimodular if every square submatrix of A has determinant $+1$, 0, or -1.

TU Matrix

Theorem (Sufficient condition)

A matrix A is TU if

1. $a_{ij} \in \{-1, 0, +1\}$ *for all* i, j
2. *Each column contains at most two nonzero coefficients* $(\sum_{i=1}^{m} |a_{ij}| \leq 2)$
3. *There exists a partition* (M_1, M_2) *of the rows set such that each column* j *containing two nonzero coefficients satisfies* $\sum_{i \in M_1} a_{ij} = \sum_{i \in M_2} a_{ij}$ *(i.e., if the two non-zero entries have the same sign they are in different sets, if the two non-zero entries have a different sign they are in the same set).*

TU Matrix

Proposition (Poincaré)

Let $A \in \{-1, 0, +1\}^{m \times n}$. If every column of A has at most one 1 and at most one -1, then A is TU.

Corollary

The AP matrix is TU, thus solving the LP relaxation of AP provides its optimal solution.

Outline

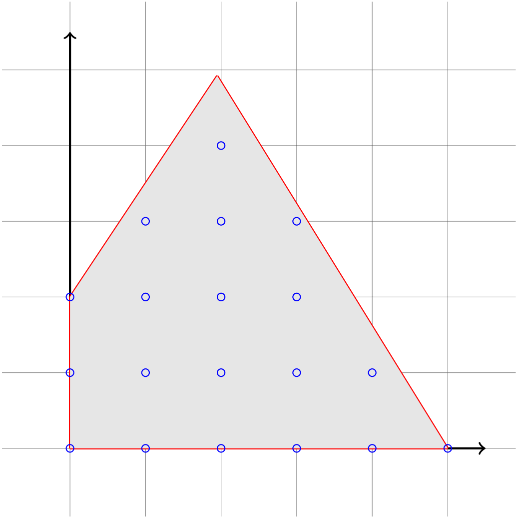
Outline

Rounding the LP solution?

Example from "Integer Programming", Wolsey.

$$\max 1.00x_1 + 0.64x_2$$
$$50x_1 + 31x_2 \leq 250$$
$$3x_1 - 2x_2 \geq -4$$
$$x_1, x_2 \geq 0 \text{ and integer.}$$

LP solution: $(\frac{376}{193}, \frac{950}{193})$



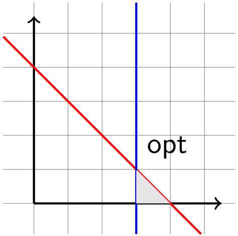
Rounding of the LP solution: $(2, 5)$ IP solution: $(5, 0)$

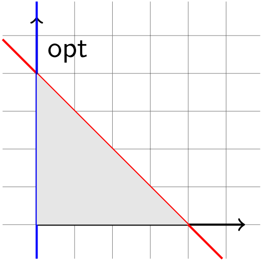
Restriction

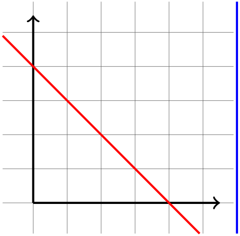
- The feasible region of the restriction is a **subset of the feasible region** of the original problem (when mapped in the same space).
- The restrictions are useful to obtain an **upper bound** on the optimal value (feasible solutions) of the original problem.

Restriction: example

$$
\begin{aligned}
\max x_1 + 2x_2 + 10x_3 & \\
x_1 + x_2 & \leq 4 \\
-x_1 + 3x_3 & \leq 0 \\
x_1, x_2 & \geq 0 \\
x_3 & \in \{0, 1, 2\}
\end{aligned}
$$







Complete enumeration?

Any purely binary program can be solved by considering all the 2^n potential solutions.

As n grows, the time needed to compute all the 2^n potential solutions grows exponentially in n.

n	2^n
10	1,024
100	1.26765060022823e+30
1,000	1.07150860718627e+301

Not an applicable approach in practice.

Which methods are used in practice?

Main ingredients

Ingredients for solving MILPs:

- Lower bound(s)

- Upper bound(s)

If LB $=$ UB, then we found an optimal solution of the (M)ILP.

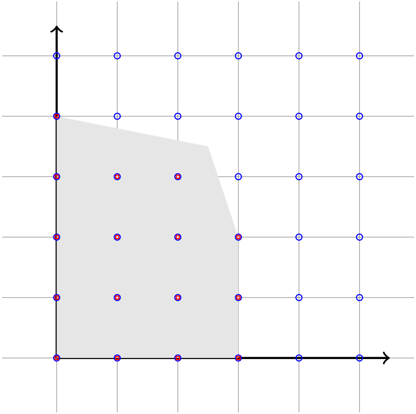
Otherwise: improve LB and UB.

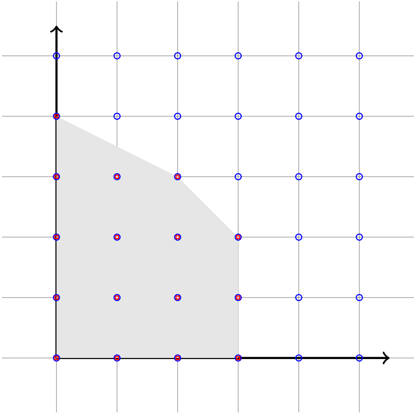
We focus on how to improve the LB.

Outline

Outline

MILP Methods





Complete enumeration?

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10	1,024
100	1.26765060022823e+30
1,000	1.07150860718627e+301

Not an applicable approach in practice.

Which methods are used in practice?

Convex hull : given a set $S \subseteq \mathbb{R}^n$, conv(S) is the smallest convex set containing S.

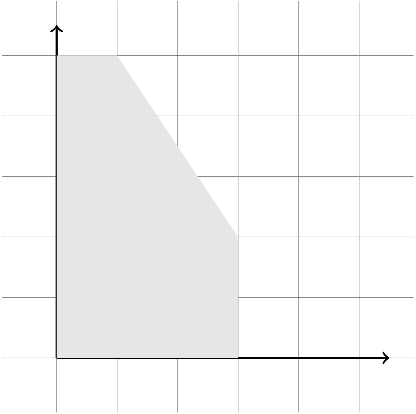
When S is the set of solutions of an IP, **Conv(S) is a polyhedron** whose vertices are **integer points** .

Ideal formulation of S : $\{x \in \mathbb{R}^n \mid \tilde{A}x \leq \tilde{b}, \underline{x} \leq x \leq \overline{x}\} = \text{conv}(S)$.

The ideal formulation is usually **very difficult** to find or can include an **exponential** number of constraints.

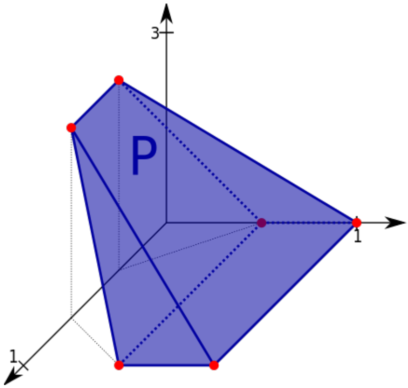
Good approximation of conv(X)?

A few definitions



- Supporting hyperplane : $\{x \mid d^\top x = \delta\}$ s.t. d a nonzero vector and $\delta = \min\{d^\top x \mid Ax \leq b\}$
- Face : subset of polyhedron s.t. $F = P$ or $F = P \cap H$ where H is some supporting hyperplane

A few definitions



Source: `https://en.wikipedia.org/wiki/Convex_polytope`

- Facet of P: bounded face of dimension $n-1$ (where n is the dimension of P).
- Edge of P: bounded face of dimension 1.
- Vertex of P: bounded face of dimension 0.

Definition

Given a polyhedron P, $d^\top x \leq \delta$ is called *valid* inequality for P if it holds for any $x \in P$.

Which are useful valid inequalities? How can we use them in trying to solve a particular instance?

Cutting plane (R. E. Gomory, 1958)

Based on **continuous relaxation strengthening** through valid and non trivial inequalities which **cut** iteration after iteration part of the feasible region of the relaxation (but no feasible point of the MILP problems).

Outline

Outline

Cutting Plane

- Iteratively adding to an initial formulation **valid, non trivial inequalities**
- Called cuts because they **cut fractional solutions**
- **Ideally** , CP would add the cuts characterizing the convex hull (continuous relaxation with integer vertices)
- Very **challenging** in general

Cutting Plane

MILP problem P:

$$z^* = \min\{c^\top x \mid x \in X\}$$

with $X = \{x \mid Ax \le b, \underline{x} \le x \le \overline{x}, x_j \in \mathbb{Z}\ \forall j \in Z\} \subseteq \mathbb{R}^n$.

LP relaxation R^0:

$$z^0 = \min\{c^\top x \mid x \in X^0\}$$

with $X^0 = \{x \mid Ax \le b, \underline{x} \le x \le \overline{x}\}$.

When the solution of R^0 $x^0 \in X$, then it is an optimal solution of P.

Otherwise, find α, β such that:

- $\alpha^\top x \le \beta$ for $x \in X$
- $\alpha^\top x^0 > \beta$

Relaxation $R^1 : X^1 = X^0 \cup \{x \mid \alpha^\top x \le \beta\}$

Cutting Plane

MILP problem P:

$$z^* = \min\{c^\top x \mid x \in X\}$$

with $X = \{x \mid Ax \leq b, \underline{x} \leq x \leq \overline{x}, x_j \in \mathbb{Z} \; \forall j \in Z \subseteq \mathbb{R}^n\}$.

LP relaxation R^0:

$$z^0 = \min\{c^\top x \mid x \in X^0\}$$

with $X^0 = \{x \mid Ax \leq b, \underline{x} \leq x \leq \overline{x}\}$.

Relaxation $R^1 : X^1 = X^0 \cup \{x \mid \alpha^\top x \leq \beta\}$

Since $X \subseteq X^1 \subseteq X^0$, R^1 **stronger than** R^0.

Aim of the CP :
Generate a sequence of stronger relaxations converging to P.

Require: a MILP problem P (let R^0 be its continuous relaxation)
  $i = 0$
  solve R^i and let x^i be its optimal solution
  while x^i is non-integer **do**
    solve the **separation problem** of x^i from P and let $\alpha^\top x \leq \beta$ be the resulting cut
    add $\alpha^\top x \leq \beta$ to R^i and obtain R^{i+1}
    $i = i + 1$
    solve R^i and let x^i be its optimal solution
  end while
  return x^i

Separation problem :

identifying α and β such that

- $\alpha^\top x \leq \beta \quad \forall x \in X$
- $\alpha^\top x^i > \beta$

Tradeoff between time spent to find the cut vs. quality of the cut.

Cut $\alpha^\top x \leq \beta$ should be easily identified for any (M)ILP problem.

Cutting Plane

Separation problem :

identifying α and β such that

- $\alpha^\top x \leq \beta \quad \forall x \in X$
- $\alpha^\top x^i > \beta$

The CP method could be **generic** .

General-purpose solvers and the cuts added are of several types but all of them are generic.

Cutting Plane

If the problem has some mathematical properties or specific characteristics $\rightarrow$ a **tailored cutting plane** method.

In this case, **separation procedure** and cut $\alpha^\top x \leq \beta$ specific (valid for that class of problems).

Last lecture.

Example of generic separation problem and cuts.

Valid inequalities for LP problems.

Proposition

$\pi^\top x \leq \pi_0$ is valid for $Y = \{x \mid Ax \leq b, x \geq 0\} \neq \emptyset$ if and only if:

- there exists $u \geq 0$, $v \geq 0$ such that $u^\top A - v = \pi$ and $u^\top b \leq \pi_0$ or, alternatively,
- there exists $u \geq 0$ such that $u^\top A \geq \pi$ and $u^\top b \leq \pi_0$.

Valid inequalities for IP problems.

Proposition

Let $Y = \{x \in \mathbb{Z}^1 \mid x \leq b\}$, then the inequality $x \leq \lfloor b \rfloor$ is valid for Y.

Cutting Plane

Numerical example for IP (from Wolsey):

$$\begin{aligned} 7x_1 - 2x_2 &\leq 14 \\ x_2 &\leq 3 \\ 2x_1 - 2x_2 &\leq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

1. By combining the three constraints with the following nonnegative weight $(\frac{2}{7}, \frac{37}{63}, 0)$ we obtain the valid inequality:

$$2x_1 + \frac{1}{63}x_2 \leq \frac{121}{21}.$$

2. Round down the coefficient of x_2: $2x_1 + 0x_2 \leq \frac{121}{21}$.
3. Round down the RHS of x_2: $2x_1 \leq \lfloor \frac{121}{21} \rfloor = 5$.

Numerical example for IP (from Wolsey):

$$\begin{aligned} 7x_1 - 2x_2 &\leq 14 \\ x_2 &\leq 3 \\ 2x_1 - 2x_2 &\leq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Valid inequality:

$$2x_1 \leq 5$$

If we iterate with a weight of $\frac{1}{2}$, we get

$$x_1 \leq \lfloor \frac{5}{2} \rfloor = 2,$$

stronger!

Cutting Plane

General procedure for IP:

1. The inequality
$$\sum_{j=1}^{n} u^\top a_j x_j \le u^\top b$$
is valid for X as $u \ge 0$ and $\sum_{j=1}^{n} a_j x_j \le b$
2. The inequality
$$\lfloor \sum_{j=1}^{n} u^\top a_j \rfloor x_j \le u^\top b$$
is valid for X as $x \ge 0$.
3. The inequality
$$\lfloor \sum_{j=1}^{n} u^\top a_j \rfloor x_j \le \lfloor u^\top b \rfloor$$
is valid for X as x is integer, thus $\lfloor \sum_{j=1}^{n} u^\top a_j \rfloor x_j$ is integer.

Chvátal Inequalities

- $P : \min\{c^\top x \mid Ax \leq b, x \text{ integer}\}$
- $R^0 : \min\{c^\top x \mid Ax \leq b\}$
- x^0 be the optimal solution of the continuous relaxation of R^0 (fractional solution)
- Chvátal inequality:
  $\alpha^\top x \leq \beta$ with $\alpha = \lfloor u^\top A \rfloor$ and $\beta = \lfloor u^\top b \rfloor$ for some $u \geq 0$

Separation problem : find $u \in \mathbb{R}^m_+$ such that $\lfloor u^\top A \rfloor x^0 > \lfloor u^\top b \rfloor$.

Chvátal Inequalities

Theorem

Every valid inequality for X can be obtained by applying the Chvátal procedure a finite number of times.

Chvátal Inequalities

Properties:

$\lfloor u^\top A \rfloor x \leq \lfloor u^\top b \rfloor \quad \forall x \in X$

Given a fractional solution $x^0 \in R^0$, it is always possible to

- Find a $u \in \mathbb{R}^m_+$ such that $\lfloor u^\top A \rfloor x^0 > \lfloor u^\top b \rfloor$
- Separate x^0
- Find a cut to be added to R^0 that strengthen it

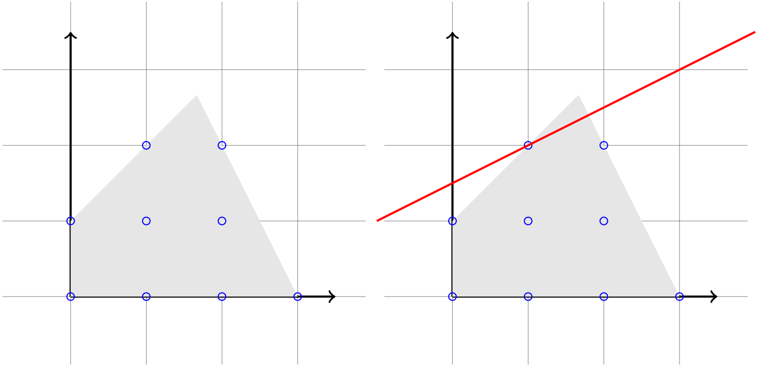
$\rightarrow$ CP with Chvátal inequalities is an exact method for solving (M)ILPs.

Chvátal Inequalities: example

Let us consider the following IP:

$$
\begin{aligned}
\max x_1 + x_2 & & \\
2x_1 + x_2 & \leq & 6 \\
-x_1 + x_2 & \leq & 1 \\
x_1, x_2 & \geq & 0 \\
x_1, x_2 & & \text{integer}
\end{aligned}
$$

Chvátal Inequalities: example



Other valid inequalities for (M)ILP

- Split inequalities (generalization of Chvátal Inequalities)
- Gomory's Mixed Integer cuts
- Lift-and-project inequalities

For details, cf. Chapter 5 of Conforti, Cornuéjols, Zambelli

The cutting plane game

http://www.columbia.edu/~gm2543/cpgame.html

Outline

Outline

Branch and Bound (A. H. Land & A. G. Doig, 1960)

Based on **upper and lower bounds** on the optimal solution value and on **branching** which divide iteration after iteration the feasible region in **smaller subproblems**.

In general **exponential worst case performance**.

 & 

Smaller subproblems easier to solve.

Proposition (Wolsey)

Let $X = X_1 \cup \dots X_K$ be a decomposition of X into smaller sets and let $z_k = \min\{c^\top x \mid x \in X_k\}$ for $k = 1, \dots, K$. Then, $z = \min_k z_k$.

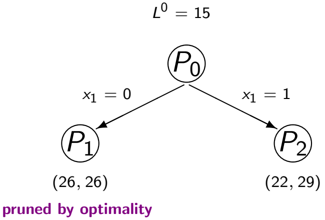
Proposition (Wolsey)

Let $X = X_1 \cup \dots X_K$ be a decomposition of X into smaller sets and let $z_k = \min\{c^\top x \mid x \in X_k\}$ for $k = 1, \dots, K$. Let $\underline{z}_k$ be a lower bound on z_k and $\overline{z}_k$ be an upper bound on z_k. Then, $\underline{z} = \min_k \underline{z}_k$ is a lower bound on z and $\overline{z} = \min_k \overline{z}_k$ is an upper bound on z.

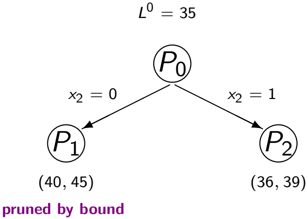
Branch and Bound

- **Bounding** and **branching** phases
- Solve the continuous relaxation of the problem (**bounding**)
- If it solution is fractional, **branch** to obtain two smaller subproblems and which do not contain the fractional solution
- explore implicitly all the subproblems and continue branching if necessary
- The subproblems could
  - be infeasible
  - have an optimal solution x^* which is integer feasible (no further branching). Upper bound is the best between x^* and the best integer feasible solution found so far x^{UB}
  - have an optimal solution x^* which is fractional
    - If $c^\top x^* < c^\top x^{\text{UB}}$ then branch
    - Otherwise continue

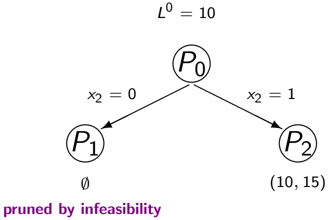
Example of pruning by optimality



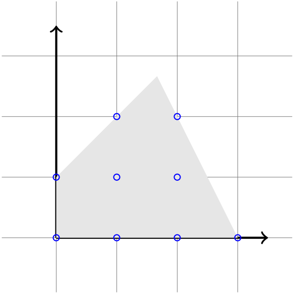
Example of pruning by bound



Example of pruning by infeasibility



$$
\begin{aligned}
-\min -x_1 - x_2 & & \\
2x_1 + x_2 &\leq 6 \\
-x_1 + x_2 &\leq 1 \\
x_1, x_2 &\geq 0 \\
x_1, x_2 & \quad \text{integer}
\end{aligned}
$$



$L^0 = \lceil -\frac{13}{3} \rceil = -4$

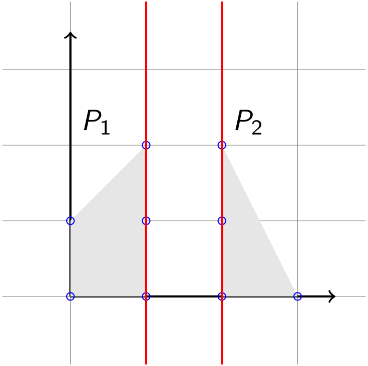
P_0

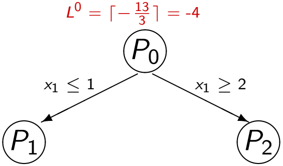
Branch and Bound: example

$x^* = (\frac{5}{3}, \frac{8}{3})$, $c^\top x^* = -\frac{13}{3}$

Branch on x_1:

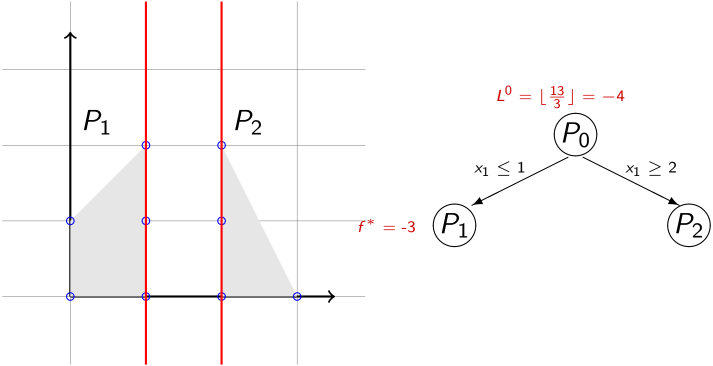
- Subproblem P_1: $P_0 \cap \{x \mid x_1 \leq \lfloor \frac{5}{3} \rfloor = 1\}$
- Subproblem P_2: $P_0 \cap \{x \mid x_1 \geq \lfloor \frac{5}{3} \rfloor + 1 = 2\}$





Branch and Bound: example

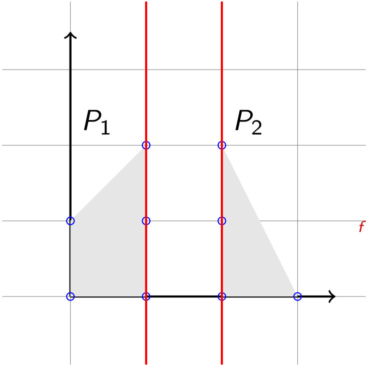
Explore P_1: optimal solution $(1, 2)$ of value -3. No further branching, upper bound of -3, corresponding to the solution $x^{\text{UB}} = (1, 2)$.

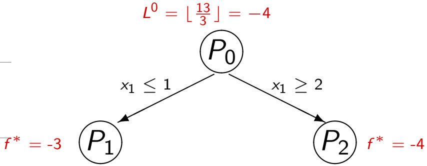


Branch and Bound: example

Explore P_1: optimal solution $(1, 2)$ of value -3. No further branching, upper bound of -3, corresponding to the solution $x^{\text{UB}} = (1, 2)$.

Explore P_2: optimal solution is $(2, 2)$ of value -4. No further branching, upper bound of -4, corresponding to the solution $x^{\text{UB}} = (2, 2)$.



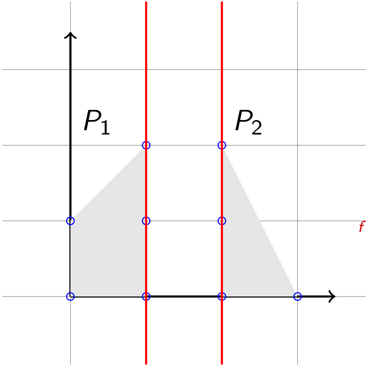


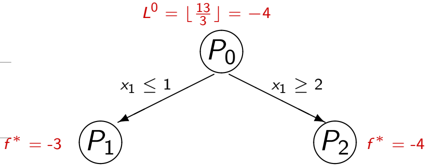
Branch and Bound: example

Explore P_1: optimal solution $(1, 2)$ of value -3. No further branching, upper bound of -3, corresponding to the solution $x^{\text{UB}} = (1, 2)$.

Explore P_2: optimal solution is $(2, 2)$ of value -4. No further branching, upper bound of -4, corresponding to the solution $x^{\text{UB}} = (2, 2)$.

No subproblems left to explore $\rightarrow$ optimal solution $(2, 2)$ of value -4.

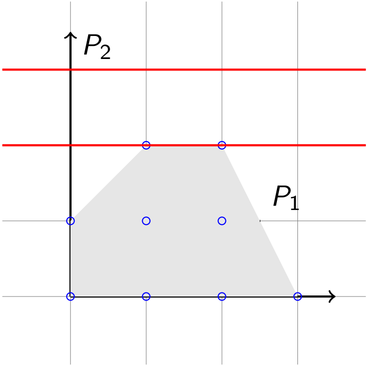


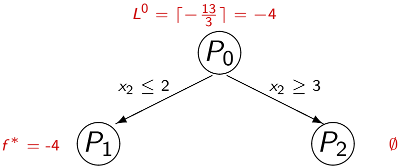


The selection of i. the **branching variable** and ii. the **next subproblem to explore** influence highly the exploration of the feasible region.

Example: **branch on** x_2

- $x_2 \leq \lfloor \frac{8}{3} \rfloor = 2$
- $x_2 \geq \lfloor \frac{8}{3} \rfloor + 1 = 3$





Require: a MILP problem P (P_0 is its continuous relaxation)

$i = 1$, $f^{UB} = +\infty$, $\Pi = \{P_0\}$

while $\Pi \neq \emptyset$ **do**

select a subproblem in Π, say P_k and remove it from Π

solve P_k, let x^* be its optimal solution and f^* be its value

if P_k is infeasible or $f^* > f^{UB}$ **then**

continue

end if

if x^* is non-integer **then**

select a variable, say x_j, with a fractional value x_j^*

define $P_i = P_k \cap \{x \mid x_j \leq \lfloor x_j^* \rfloor\}$ and $P_{i+1} = P_k \cap \{x \mid x_j \geq \lfloor x_j^* \rfloor + 1\}$

let $L_i = f^*$ and $L_{i+1} = f^*$

$\Pi = \Pi \cup \{P_i, P_{i+1}\}$

$i = i + 2$

else

$f^{UB} = f^*$, $x^{UB} = x^*$

remove from Π any P_ℓ with $L_\ell > f^{UB}$

end if

end while

return x^{UB}

Branch and Bound

Key ingredients:

- Formulation (small gap at root node)
- Heuristics (improve upper bound)
- Branching
- Node selection.

Outline

Outline

Branch and Cut

Require: a MILP problem P (P_0 is its continuous relaxation)

$i = 1$, $f^{UB} = +\infty$, $\Pi = \{P_0\}$
while $\Pi \neq \emptyset$ **do**
  select a subproblem in Π, say P_k and remove it from Π
  solve P_k, let x^* be its optimal solution and f^* be its value
  if P_k is infeasible or $f^* > f^{UB}$ **then**
    continue
  end if
  if x^* is non-integer **then**
    if Branch? **then**
      select a variable, say x_j, with a fractional value x_j^*
      define $P_i = P_k \cap \{x \mid x_j \leq \lfloor x_j^* \rfloor\}$ and $P_{i+1} = P_k \cap \{x \mid x_j \geq \lfloor x_j^* \rfloor + 1\}$
      let $L_i = f^*$ and $L_{i+1} = f^*$
      $\Pi = \Pi \cup \{P_i, P_{i+1}\}$
      $i = i + 2$
    else
      Strenthen P_k by adding cutting planes
      $\Pi = \Pi \cup P_k$
    end if
  else
    $f^{UB} = f^*$, $x^{UB} = x^*$
    remove from Π any P_ℓ with $L_\ell > f^{UB}$
  end if
end while
return x^{UB}

“Homeworks”

- Modeling exercises
- Proofs of equivalence of the continuous relaxation of the PWL formulations
- Install AMPL `https://www.lix.polytechnique.fr/~dambrosio/teaching/`. For instructions see `https://ampl.com/ampl-course-install/`
- Bring your laptop with you tomorrow!

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