SUMMER SCHOOL ON ASPECTS OF OPTIMIZATION
Discrete Optimization
Research Talk

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ÉCOLE POLYTECHNIQUE
IP PARIS

"Comparing perspective reformulations
for piecewise-convex optimization"

R.S. Trindade, C. D'Ambrosio, A. Frangioni, C. Gentile

Outline

1. The class of MINLP problems
2. General Framework
   - Lower Bounding problem
   - Previous theoretical results and hypothesis
3. Computational Results
   - Non linear knapsack problem
   - Uncapacitated Facility Location problem
4. Theoretical Results
5. Conclusions and Future Directions

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The class of MINLP problems

$$\min \sum_{j \in N} c_j x_j \quad (1)$$

$$f_i(x) + \sum_{j \in H(i)} g_{ij}(x_j) \leq 0 \qquad i \in M \quad (2)$$

$$l_j \leq x_j \leq u_j \qquad j \in N \quad (3)$$

$$x_j \in \mathbb{Z} \qquad j \in I. \quad (4)$$

where:

- $f_i : \mathbb{R}^n \to \mathbb{R}$ are convex functions $\forall i \in M$,
- $g_{ij} : \mathbb{R} \to \mathbb{R}$ are non convex univariate function $\forall i \in M, \forall j \in H(i)$,
- $H(i) \subseteq N \;\; \forall i \in M$,
- $I \subseteq N$, and
- l_j and u_j are finite $\forall i \in M, j \in H(i)$

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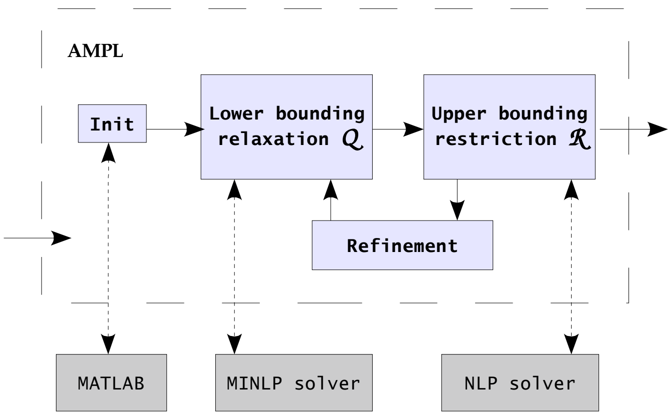
General Framework

Global optimization algorithm proposed in D'A., Lee, and Wächter (2009, 2012).



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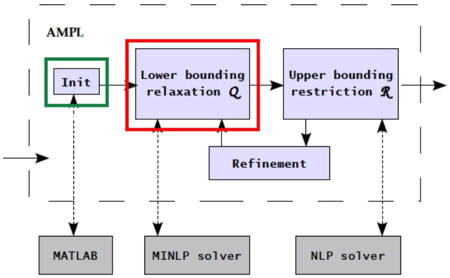

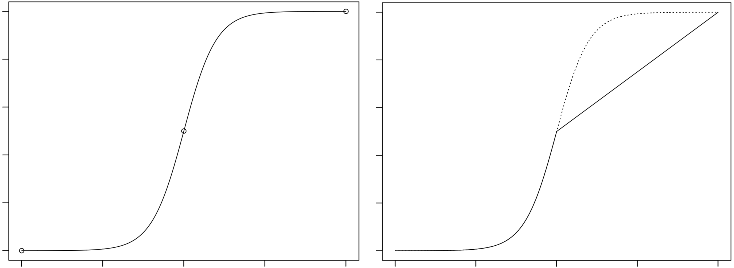

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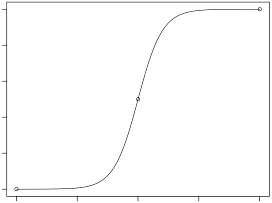
The Lower Bounding problem: step 1

For simplicity, let us consider, for a given pair i, j, the univariate nonconvex function $g(x_j)(:= g_{ij}(x_j))$:



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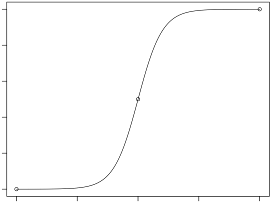

Automatically detect the **concavity/convexity intervals** or piecewise definition ($l_{ij}^1 = l_j$ and $l_{ij}^{s(ij)} = u_j$):
$[l_{ij}^s, l_{ij}^{s+1}] :=$ the s-th subinterval of the domain of g ($s \in \{1 \dots s(ij) - 1\}$);

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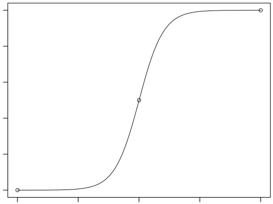

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$\check{S}(ij) :=$ the set of indices of **subintervals** on which g is **convex**;

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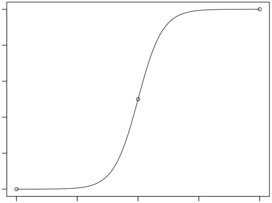

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$\hat{S}(ij)$:= the set of indices of **subintervals** on which g is **concave**.

The Lower Bounding problem: step 2

Reformulate the lower bounding problems as a piecewise defined problem, i.e., separating the convex and the concave intervals.

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Reformulate the lower bounding problems as a piecewise defined problem, i.e., separating the convex and the concave intervals.

Adapt the following piecewise linear formulations (see Croxton et al., 2003):

- Convex combination (CC)
- Multiple choice (MC)
- Incremental (Inc)

The formulations

$$\min \sum_{j\in N} c_j x_j$$

$$\bar{f}_i(x) + \sum_{j\in H(i)} \sum_{s\in \check{S}(ij)} z_{ij}^s \leq 0 \qquad i \in M$$

$$y_{ij}^s \in \{0,1\} \qquad s \in S(ij),\ j \in H(i),\ i \in M$$

$$x_j \in \mathbb{Z} \qquad j \in I$$

Multiple Choice Formulation

$$z_{ij}^s \geq [g_{ij}(x_{ij}^s) - g_{ij}(0)] \qquad s \in \check{S}(ij),\ j \in H(i),\ i \in M$$

$$x_j = \sum_{s\in S(ij)} x_{ij}^s \qquad j \in H(i), i \in M$$

$$l_{ij}^s y_{ij}^s \leq x_{ij}^s \leq l_{ij}^{s+1} y_{ij}^s \qquad s \in S(ij),\ j \in H(i),\ i \in M$$

$$\sum_{s\in S(ij)} y_{ij}^s = 1 \qquad i \in M,\ j \in H(i)$$

The formulations

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\begin{aligned}
\min \ & \textstyle\sum_{j \in N} c_j x_j \\
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\end{aligned}
$$

Incremental Formulation

$$
\begin{aligned}
& z_{ij}^s \geq [g_{ij}(l_{ij}^s + x_{ij}^s) - g_{ij}(l_{ij}^s)] && s \in \check{S}(ij),\ j \in H(i),\ i \in M \\
& x_j = l_j + \textstyle\sum_{s \in S(ij)} x_{ij}^s && j \in H(i), i \in M \\
& (l_{ij}^{s+1} - l_{ij}^s) y_{ij}^{s+1} \leq x_{ij}^s \leq (l_{ij}^{s+1} - l_{ij}^s) y_{ij}^s && s \in S(ij),\ j \in H(i),\ i \in M
\end{aligned}
$$

What is a perspective reformulation?

- Given a convex function $h(x)$, its perspective function $yh(x/y)$ describes its convex envelope when restricted to the mixed-integer set $\{(x,y) : 0 \leq x \leq uy\ ,\ y \in \{0,1\}\}$

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- The continuous relaxation of (PR), the Perspective Relaxation ($\underline{\text{PR}}$) of ($\underline{\text{P}}$), provides tighter lower bounds to the optimal value of ($\underline{\text{P}}$) than the continuous relaxation of the standard formulation

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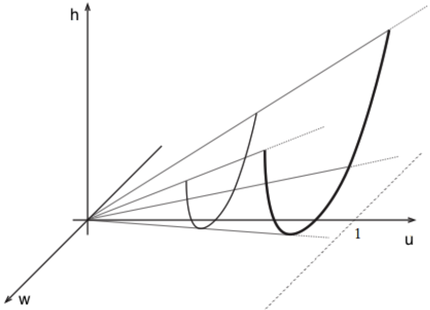

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$$(l_{ij}^{s+1} - l_{ij}^s)y_{ij}^{s+1} \le x_{ij}^s \le (l_{ij}^{s+1} - l_{ij}^s)y_{ij}^s \qquad s \in S(ij),\ j \in H(i),\ i \in M$$

The Lower Bounding problem: step 3

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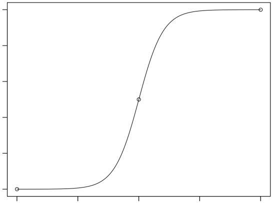
Use piecewise linear approximation for the concave intervals:



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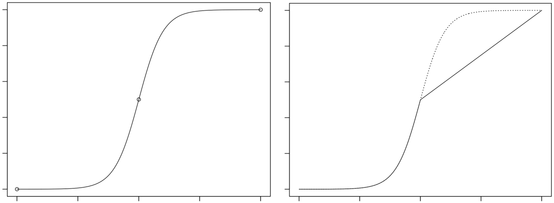

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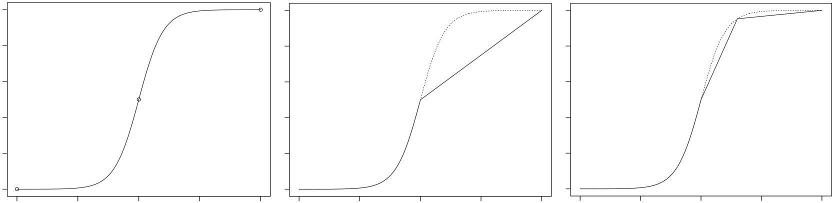

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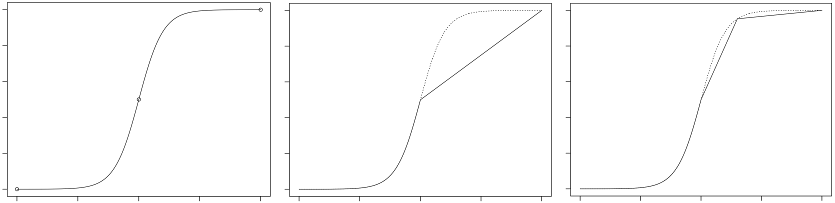

Piecewise linear formulation for the approximation (see CC, MC, Inc)

Previous theoretical results and hypothesis

Theorem (Croxton et al., 2003)

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What about the piecewise convex case?

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- We focus in two different problems:
  - Non linear knapsack problem;
  - Uncapacitated Facility Location problem.
- We tested our approach, based on separation of Perspective Cuts (PC) implemented using CPLEX .

Non linear knapsack problem

- The non linear knapsack problem is the same considered in D'A. et al., 2009:

$$\max \sum_{j \in N} p_j$$
$$p_j \leq g_j(x_j) \qquad j \in N$$
$$\sum_{j \in N} w_j x_j \leq C$$
$$0 \leq x_j \leq u_j \qquad j \in N$$

For each value of $|N| \in \{10, 20, 50, 100, 200, 500, 1000\}$ we randomly generated 10 instances where $w_j \in [1, 100]$.

- $g_j(x_j) = \frac{c_j}{1+b_j \exp(-a_j(x_j+d_j))}$, whith $a_j \in [0.1, 0.2]$, $b_j \in [0, 100]$, $c_j \in [0, 100]$, and $d_j \in [-100, 0]$
- $g_j(x_j) = 7.5 \sin(\pi \left(\frac{x_j - 10}{40}\right) - 15 \cos(\pi \left(\frac{x_j - 10}{80}\right)) + 19.5$

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We fixed $u_j = 100$ for all $j \in N$ and $C = 50 \sum_{j \in N} w_j$

Non linear knapsack problem

Table: Computational results for Non-linear Continuous Knapsack problem

INST.		INC			MC			INC RELAX.			MC RELAX.		
Int.	*Size*	*Sol.*	*Time*	*Cuts*	*Sol.*	*Time*	*Cuts*	*Gap*	*Time*	*Cuts*	*Gap*	*Time*	*Cuts*
2	10	305.04	0.02	114.70	305.04	0.03	105.60	0.48	0.01	50.30	0.48	0.01	50.30
2	20	594.57	0.03	187.40	594.57	0.03	179.80	0.18	0.01	92.20	0.18	0.02	92.20
2	50	1659.96	0.05	448.20	1659.96	0.05	448.20	0.02	0.02	246.10	0.02	0.02	246.10
2	100	3398.18	0.09	759.00	3398.18	0.09	759.50	0.00	0.04	499.00	0.00	0.05	499.00
2	200	6798.08	0.21	1614.50	6798.08	0.22	1635.90	0.00	0.09	989.40	0.00	0.08	989.40
2	500	17211.06	0.45	3293.90	17211.06	0.45	3202.20	0.00	0.22	2504.50	0.00	0.22	2504.50
2	1000	34562.94	1.12	5949.60	34562.94	1.00	5896.30	0.00	0.45	5039.40	0.00	0.43	5039.40
4	10	278.36	0.06	348.40	278.36	0.04	239.70	1.31	0.01	108.10	0.17	0.01	78.80
4	20	555.64	0.09	533.90	555.64	0.04	325.90	1.00	0.02	225.40	0.03	0.02	155.10
4	50	1417.20	0.41	1546.10	1417.20	0.16	886.70	0.80	0.04	501.90	0.01	0.04	360.30
4	100	2817.61	0.91	2332.30	2817.61	0.26	1416.30	0.83	0.10	1058.90	0.00	0.07	733.50
4	200	5618.12	3.10	4171.30	5618.12	0.54	2369.80	0.85	0.22	2255.20	0.00	0.15	1481.00
4	500	14123.84	20.34	8931.70	14123.84	2.40	5141.40	0.80	0.69	5611.90	0.00	0.42	3613.30
4	1000	28215.47	174.67	18249.10	28215.47	4.51	8480.70	0.83	1.92	11029.20	0.00	1.20	7363.50

Uncapacitated Facility Location problem

- The Uncapacitated Facility Location problem is the same considered in D'A. et al., 2009, i.e.:

$$\min \sum_{k \in K} C_k y_k + \sum_{t \in T} \sum_{k \in K} s_{kt}$$

$$\begin{aligned} & a_{kt}(\sin(b_{kt} w_{kt}) + c_{kt} w_{kt})^2 - s_{kt} \leq 0 && t \in T, k \in K \\ & \sum_{k \in K} w_{kt} = 1 && t \in T \\ & 0 \leq w_{kt} \leq y_k && t \in T, k \in K \\ & y_k \in \{0, 1\} && k \in K \end{aligned}$$

- For each costumer of $T \in \{6, 12, 24\}$ and facility $K \in \{12, 24, 48\}$ we randomly generated instances, where $C_k \in [1, 100]$, $a_{kt} \in \{-12, -25\}$, $b_{kt} \in [2, 13]$, $c_{kt} \in [1, 13]$. We generated 3 different sizes of instances: $(|K|.|T|) = (6, 12), (12, 24), (24, 48)$.

Uncapacitated Facility Location problem

Table: Computational results for Non-linear UFL problem

	Inc					MC				
Inst.	*Sol.*	*Time*	Gap	*Cuts*	*#O*	*Sol.*	*Time*	Gap	*Cuts*	*#O*
6x12x1	5419.439	0.60	0.00	1772.40	10	5419.439	0.46	0.00	1618.60	10
6x12x2	37807.512	0.45	0.00	1963.90	10	37807.512	0.32	0.00	1860.80	10
6x12x3	12403.535	7254.68	2.49	33355.70	4	12401.188	4449.73	0.73	14565.40	7
12x24x1	5614.138	3.68	0.00	10745.20	10	5614.138	3.30	0.00	10316.50	10
12x24x2	52806.983	1148.31	0.16	23916.80	9	52806.983	196.46	0.00	15677.20	10
12x24x3	19096.744	10000.08	20.66	128509.20	0	18616.806	10000.03	12.52	45311.10	0
24x48x1	6029.599	123.30	0.00	65840.30	10	6029.598	98.89	0.00	67110.90	10
24x48x2	69256.249	10000.04	4.71	104814.20	0	69082.252	10000.04	3.03	83943.70	0

Table: Computational results for the continuous relaxation of Non-linear UFL problem

	Inc Relax.			MC Relax.		
Inst.	*Gap*	*Time*	*Cuts*	*Gap*	*Time*	*Cuts*
6x12x1	7.79	0.05	802.50	5.13	0.05	808.70
6x12x2	4.28	0.07	1082.20	0.44	0.07	954.20
6x12x3	92.76	0.23	2490.20	14.10	0.14	1301.70
12x24x1	8.96	0.25	3330.40	8.33	0.23	3358.50
12x24x2	8.34	0.32	3993.30	3.48	0.31	3782.00
12x24x3	99.80	1.26	5978.30	18.30	1.09	5062.30
24x48x1	15.04	1.91	15377.00	14.81	1.78	15401.00
24x48x2	12.29	1.95	15345.30	6.85	1.84	14753.70

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Theoretical Results

$$g(x) = \begin{cases} g^k(x^k) + c^k & \text{if } x^k \in \mathcal{P}^k \text{ and } x^h = 0 \ \forall h \in K \setminus \{k\} \\ 0 & \text{if } x = 0 \\ +\infty & \text{otherwise} \end{cases}$$

Theorem (Frangioni et al., 2020)

The convex envelope of g can be described as follows:

$$\min\{\sum_{k \in K} \delta^k g^k(x^k/\delta^k) \mid \sum_{k \in K} \delta^k \leq 1. A^k x^k \leq b^k \delta^k, \delta^k \geq 0 \ \forall k \in K\}$$

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Corollary

The MC formulation constraints describe the convex envelope of each function g_j

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Theorem

The MC formulation is stronger than the Inc one.

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Example

$$\min p$$

$$p \geq -7.5\sin(2\pi\left(\frac{0.7x+20}{100}\right) - 15\cos(2\pi\left(\frac{0.7x+20}{100}\right))$$

$$x \leq C$$

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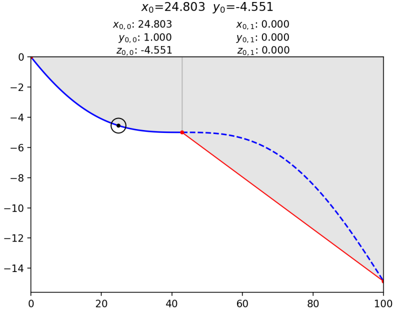

Figure: Integer optimal solution of the problem.

Theoretical Results

Example

$$
\begin{aligned}
& \min p \\
p & \geq -7.5\sin(2\pi\left(\frac{0.7x+20}{100}\right) - 15\cos(2\pi\left(\frac{0.7x+20}{100}\right)) \\
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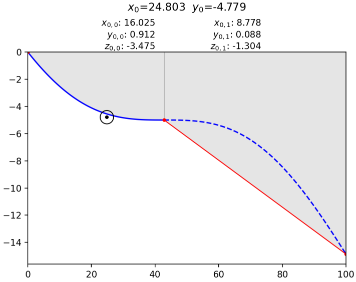

Figure: Multiple Choice solution.

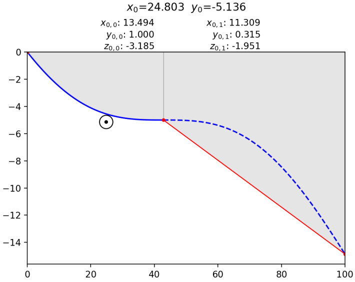

Figure: Incremental solution.

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Proposition

Suppose that a function $g = g_{ij}$ has a domain partitionable in two subsets $[l^1, l^2]$ and $[l^2, l^3]$ and that g is concave in $[l^1, l^2]$ and convex in $[l^2, l^3]$. Then, MC and Inc applied to g are equivalent.

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Outline

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Thanks for your attention!

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