Introduction to Mathematical Optimization

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Outline

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Decision Theory

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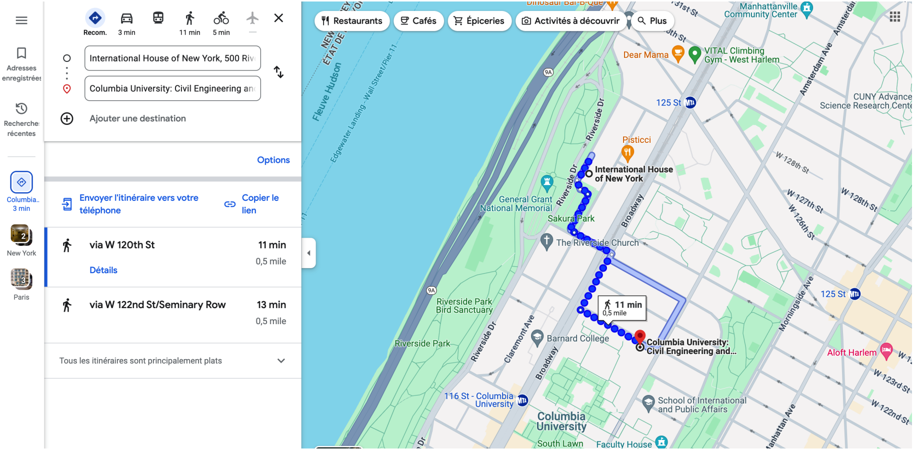

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- **Normative** vs. Descriptive Decision Theory
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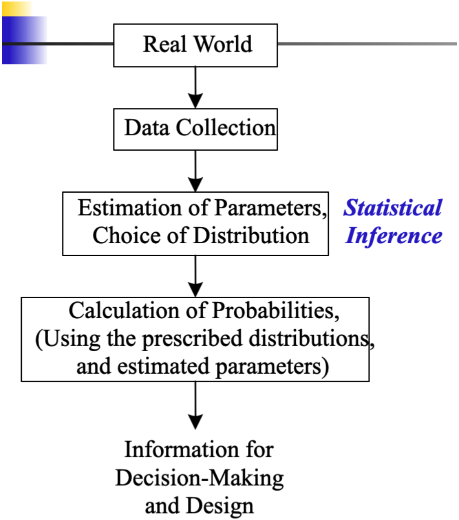
- **Interdisciplinary field** : computer scientists, mathematicians, economists, engineers, statisticians, ...
- **Operations Research** : analytical methods to help better decisions

The role of Decision-Making Tools



Electrification of the Transportation System

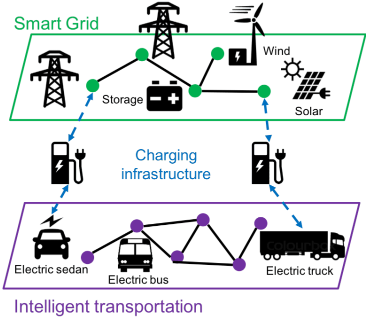


Figure: Source: https://skliotsc.um.edu.mo/power-and-transportation-nexus/

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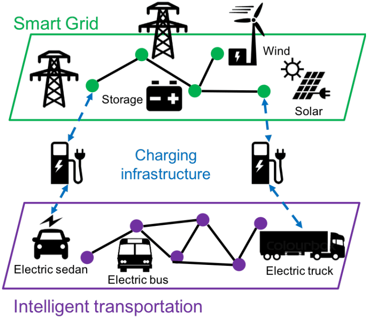

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Crucial problem involving the Transportation and Energy Systems

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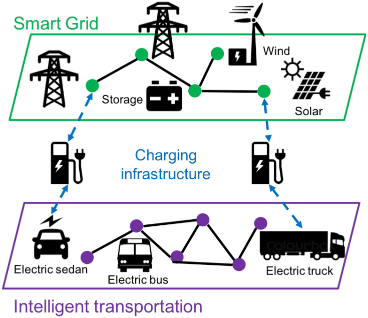

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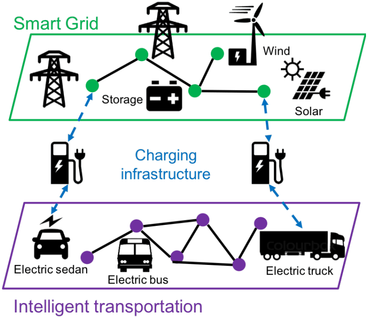

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- private and public transportation vehicles replaced with electric vehicles
- hugely affect transportation and energy systems

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https://ec.europa.eu/newsroom/horizon2020/document.cfm?doc_id=46368

https://www.uber.com/us/en/about/reports/spark-partnering-to-electrify-europe/

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Mathematical Optimization (MO) or Mathematical Programming (MP)

- Abstract and formal language

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- Aim: modeling (formulate) optimization problems

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- Abstract and formal language
- Aim: modeling (formulate) optimization problems
- **Formulate-and-solve** paradigm
- Available general-purpose solvers

Mathematical Optimization (MO) or Mathematical Programming (MP)

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\begin{array}{rrll}
\min_x & f(x) & & \\
& g_i(x) \leq & 0 & \forall i = 1, \ldots, m \\
\underline{x} & \leq x \leq & \overline{x} & \\
& x_j \in & \mathbb{Z} & \forall j \in Z
\end{array}
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where

- x is an n-dimensional vector of the **decision variables**

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- set $Z \subseteq \{1, 2, \ldots, n\}$ is the set of the indexes of the **integer variables**

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where

- $f(x)$ and $g(x)$ can be written in **closed form**
- $f(x)$ and $g_i(x)$ are given **twice continuously differentiable** functions of the variables $(\forall i = 1, \ldots, m)$

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Objective function? $\max x_1 + x_2$

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- An optimization problem can be modeled in different ways $\rightarrow$ **several formulations**
- **Instance** : when the expression of $f(x)$, $g(x)$ and the values of $\underline{x}$, $\overline{x}$, and Z are known. The set of instances of a MO problems is potentially infinite.

Mathematical Optimization

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$W = 13$, $L = \{1, 2\}$, $w^\top = (3, 2)$, $p^\top = (1, 1)$, $\overline{x}^\top = (3, 5)$ is an **instance** of the above formulation.

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Optimal solution(s) ?

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- Others?

Optimal solution(s) ? We will see later...

Classes of MO problems

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- **Mixed Integer Non Linear Programming** (MINLP): $f(x)$ and $g(x)$ are twice continuously differentiable, $Z \subset \{1, 2, \ldots, n\}$

Classes of MO problems

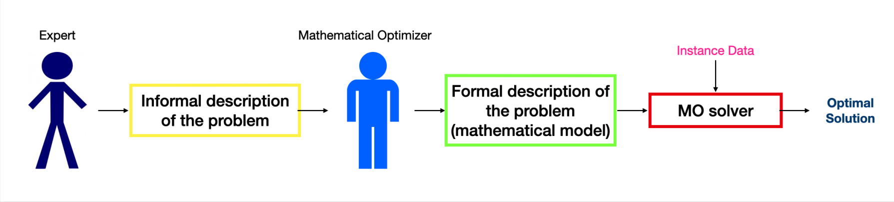
$$\begin{aligned} \min_{x} \quad & f(x) \\ & g_i(x) \leq 0 \quad \forall i = 1, \ldots, m \\ \underline{x} \leq & \; x \leq \overline{x} \\ & x_j \in \mathbb{Z} \quad \forall j \in Z \end{aligned}$$

- **Linear Programming** (LP): $f(x)$ and $g(x)$ are linear, $Z = \emptyset$
- **Integer Linear Programming** (ILP): $f(x)$ and $g(x)$ are linear, $Z = \{1, 2, \ldots, n\}$
- **Mixed Integer Linear Programming** (MILP): $f(x)$ and $g(x)$ are linear, $Z \subset \{1, 2, \ldots, n\}$
- **Mixed Integer Non Linear Programming** (MINLP): $f(x)$ and $g(x)$ are twice continuously differentiable, $Z \subset \{1, 2, \ldots, n\}$

Black Box Optimization: $f(x)$ or $g(x)$ $\rightarrow$ no closed form

The Mathematical Optimizer's Job

Mathematical Optimisation is a knowledge-based approach



Outline

Linear Programming problems

$$
\begin{aligned}
\min_{x} \quad & f(x) \\
& g_i(x) \leq 0 \quad \forall i = 1, \dots, m \\
\underline{x} \leq\ & x \leq \overline{x} \\
& x_j \in \mathbb{Z} \quad \forall j \in Z
\end{aligned}
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Linear Programming (LP) problem:

$$\min_x f(x) \quad \rightarrow \quad \min_x c^\top x$$

Linear Programming problems

$$
\begin{array}{rcll}
\min_x & f(x) & & \\
& g_i(x) \leq & 0 & \forall i = 1, \dots, m \\
\underline{x} & \leq x \leq & \overline{x} & \\
& x_j \in & \mathbb{Z} & \forall j \in Z
\end{array}
$$

Linear Programming (LP) problem:

$$
\begin{array}{rcl}
\min_x f(x) & \rightarrow & \min_x c^\top x \\
g(x) \leq 0 & \rightarrow & Ax \leq b
\end{array}
$$

Linear Programming problems

$$
\begin{aligned}
\min_x \quad & f(x) \\
g_i(x) \leq \quad & 0 \quad \forall i = 1, \ldots, m \\
\underline{x} \quad \leq x \leq \quad & \overline{x} \\
x_j \in \quad & \mathbb{Z} \quad \forall j \in Z
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\min_x \quad & f(x) \\
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x_j \in \; & \mathbb{Z} \quad \forall j \in Z
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\min_x f(x) \quad & \rightarrow \quad \min_x c^\top x \\
g(x) \leq 0 \quad & \rightarrow \quad Ax \leq b \\
\underline{x} \leq x \leq \overline{x} \quad & \rightarrow \quad \underline{x} \leq x \leq \overline{x} \\
x_j \in \mathbb{Z} \quad \forall j \in Z \quad & \rightarrow \quad \text{removed}
\end{aligned}
$$

LP problems

$$\min_x c^\top x$$
$$Ax \leq b$$
$$\underline{x} \leq x \leq \overline{x}$$

LP problems

$$\begin{aligned} \min_x \; & c^\top x \\ Ax & \leq b \\ \underline{x} \leq x & \leq \overline{x} \end{aligned}$$

W.l.o.g. because

$$\max \tilde{c}^\top x \rightarrow$$

LP problems

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For some i, $\tilde{A}_i x \geq \tilde{b}_i \rightarrow$

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For some i, $\tilde{A}_i x = \tilde{b}_i \rightarrow$

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For some i, $\tilde{A}_i x = \tilde{b}_i \rightarrow -\tilde{A}_i x \leq -\tilde{b}_i$ and $\tilde{A}_i x \leq \tilde{b}_i$

LP problems

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Moreover, $\underline{x} \in [-\infty, +\infty)$ and $\overline{x} \in (-\infty, +\infty]$.

LPs characteristics

Feasible (solutions) set/region : $X = \{x \mid Ax \leq b, \underline{x} \leq x \leq \overline{x}\}$

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Geometrical interpretation of LPs

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Geometrical interpretation of LPs

How to draw constraints and objective function

Outline

LP problems and methods

$$\begin{aligned} \min_x c^\top x & \\ Ax &\leq b \\ \underline{x} \leq x &\leq \overline{x} \end{aligned}$$

LP problems and methods

$$\min_x c^\top x$$
$$Ax \leq b$$
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Methods

- primal or dual simplex algorithm
- interior point method
- barrier method
- ...

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- ...

In this course: **graphical solution** of LPs and intuition on the **primal simplex method**

Outline

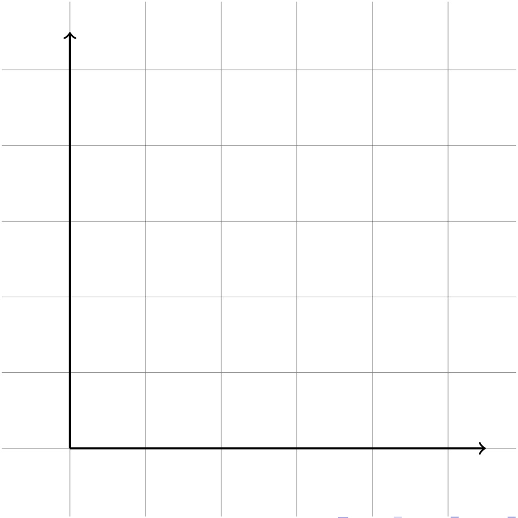
LP problems and methods

Possible outcomes:

- **optimal**: when $X = \{x \mid Ax \le b, \underline{x} \le x \le \overline{x}\} \neq \emptyset$, bounded. In this case, an optimal solution is found, i.e., a feasible point x^* s.t. $c^\top x^* \le c^\top x$ for all feasible $x \in X$
- **infeasible**: when $X = \{x \mid Ax \le b, \underline{x} \le x \le \overline{x}\} = \emptyset$
- **unbounded**: when the $\min\{c^\top x \mid Ax \le b, \underline{x} \le x \le \overline{x}\} = -\infty$

Example 1: optimal solution

$$\max x_1 + x_2$$
$$3x_1 + 2x_2 \leq 13$$
$$0 \leq x_1 \leq 3$$
$$0 \leq x_2 \leq 5.$$



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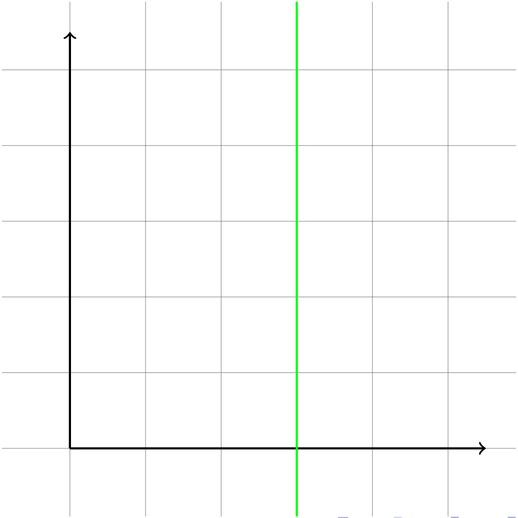

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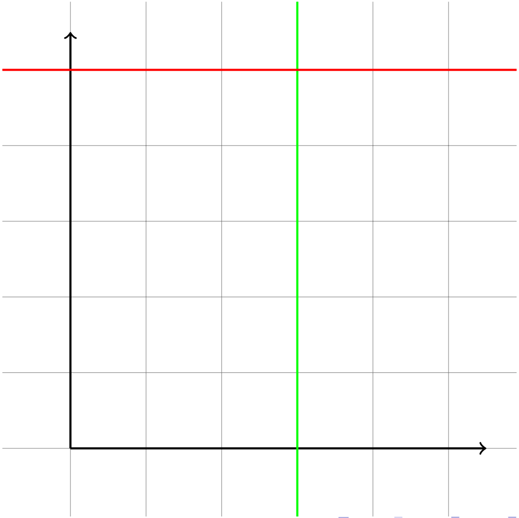

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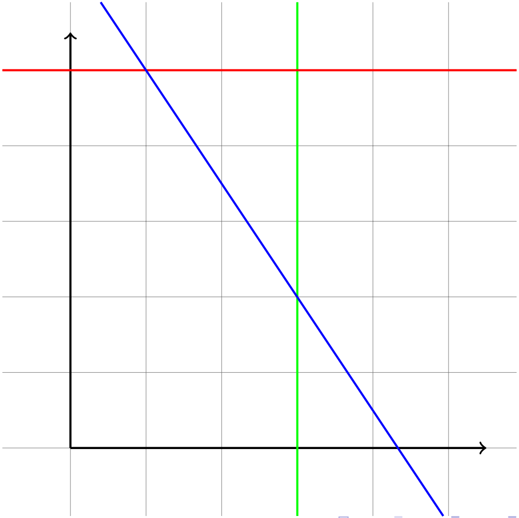

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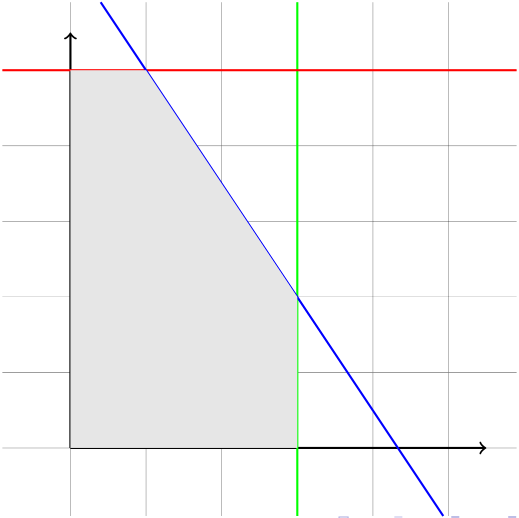

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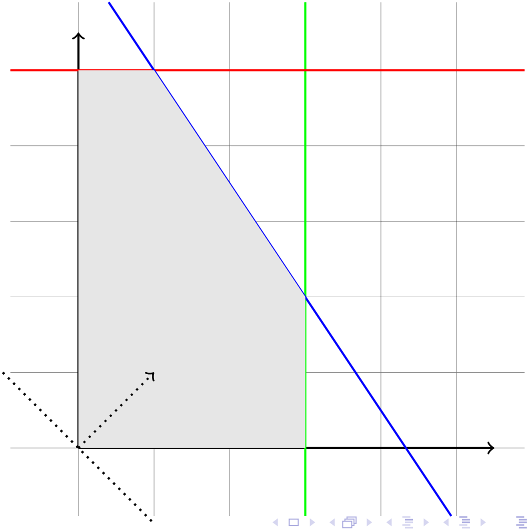

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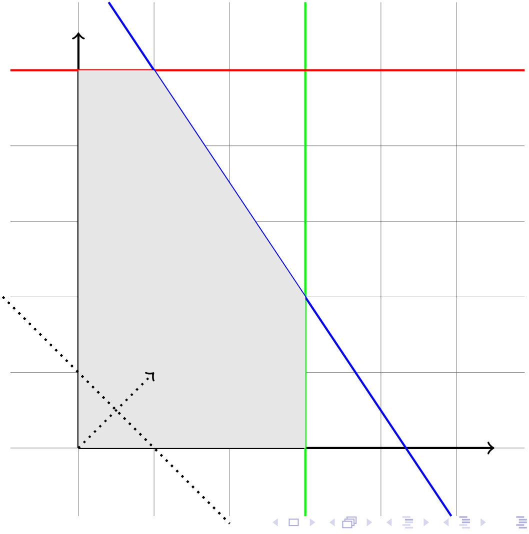

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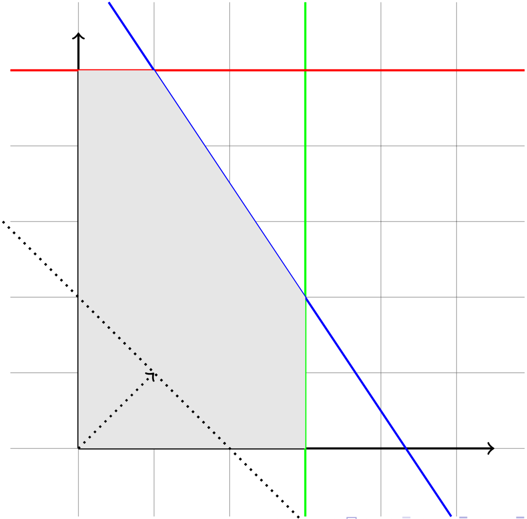

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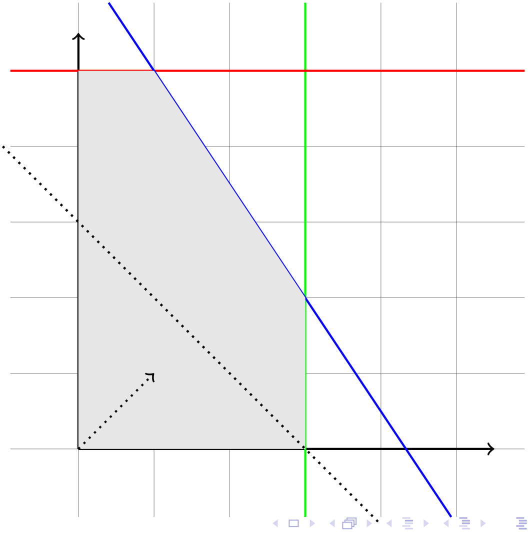

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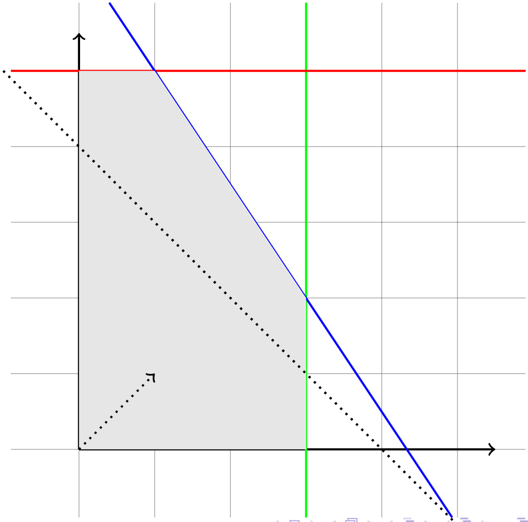

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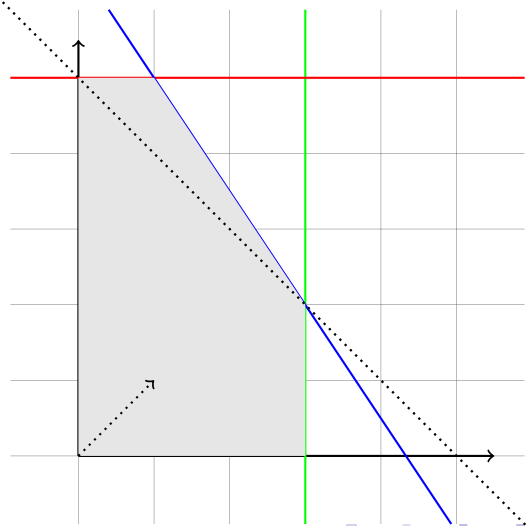

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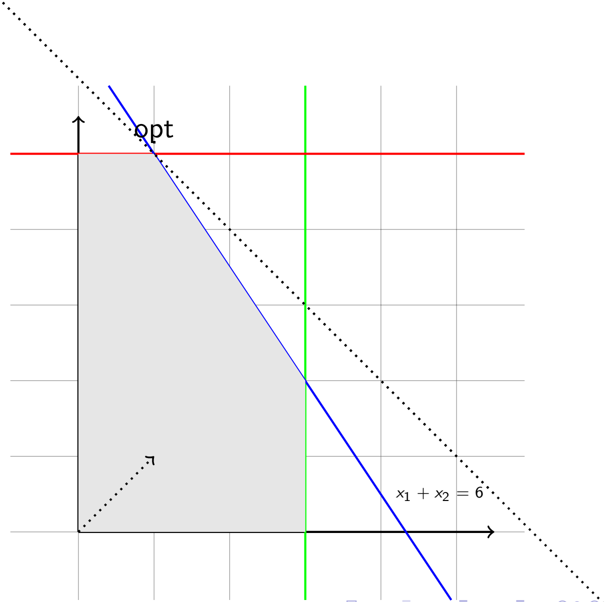
Optimal solution:
$x_1 = 1 \rightarrow$ 1k gallons of milk
$x_2 = 5 \rightarrow$ 5k gallons of juice
Profit $= 6$



Example 2: infeasible problem

$$\max x_1 + x_2$$
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$$0 \leq x_1 \leq 3$$
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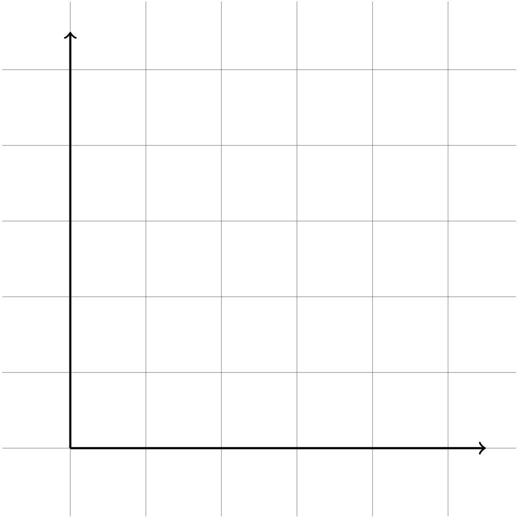
Impose to transport at least $7k$ gallons of liquid, in total.



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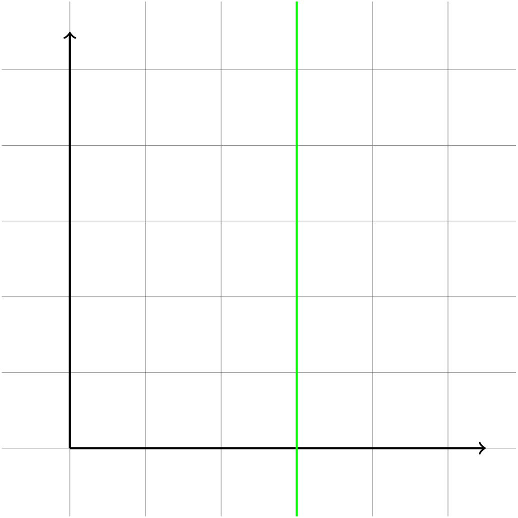

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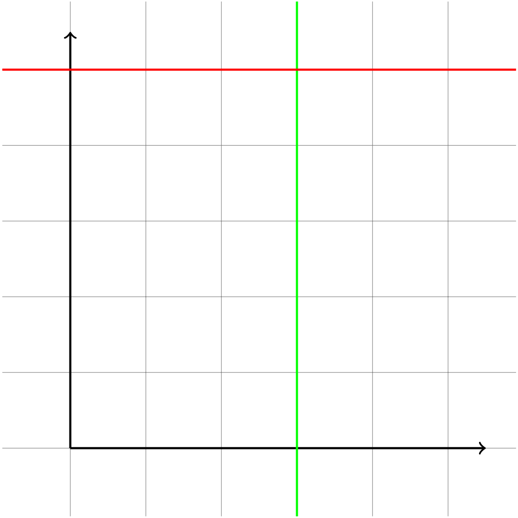

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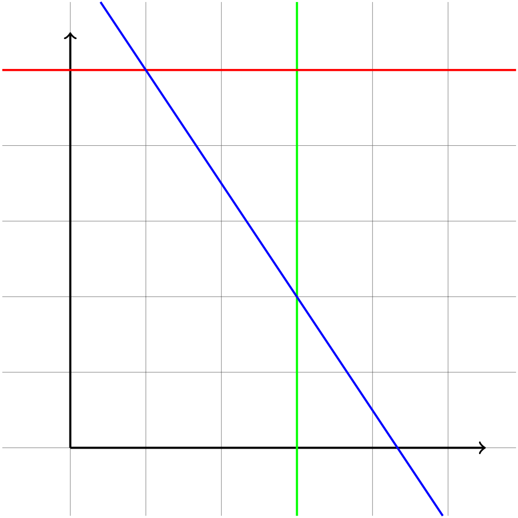

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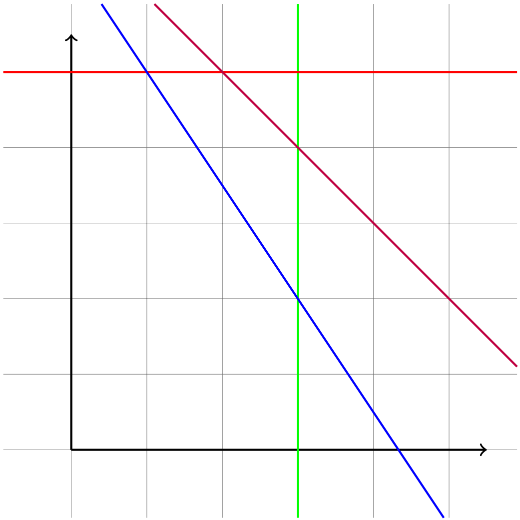

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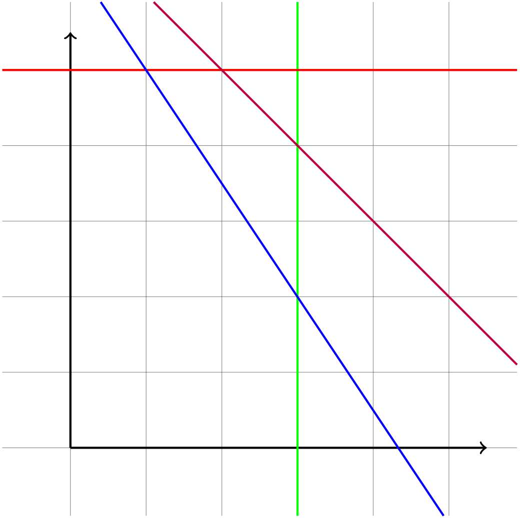
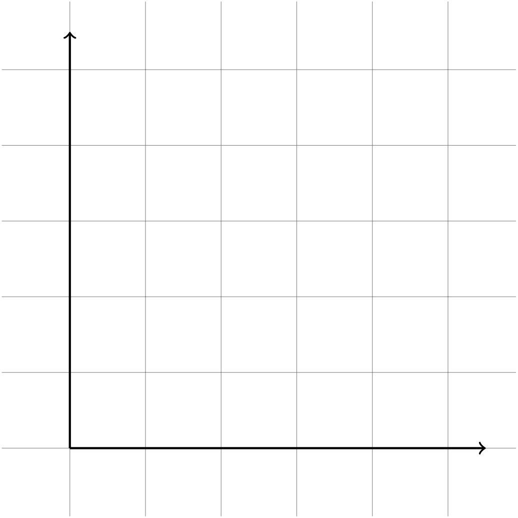
Solutions set: $\emptyset$



Example 3: unbounded problem

$$\max x_1 + 2x_2$$

$$\begin{aligned} x_1 - x_2 &\leq 1 \\ -x_1 + x_2 &\leq 3 \\ x_1 &\geq 0 \\ x_2 &\geq 0. \end{aligned}$$



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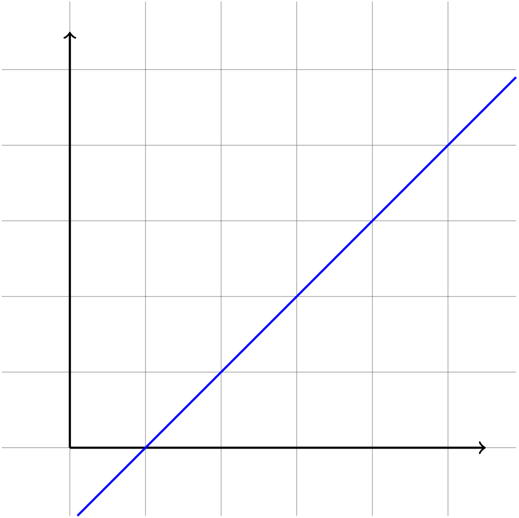

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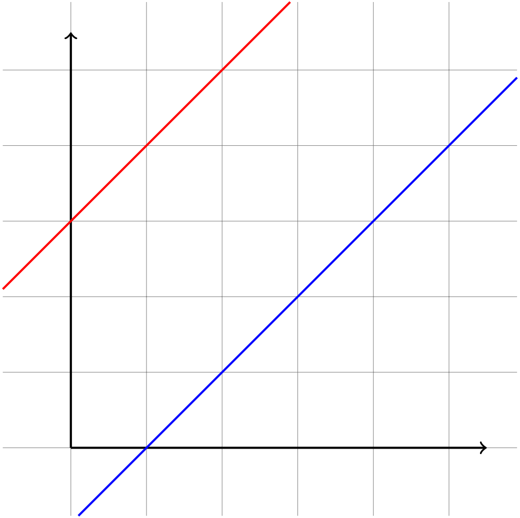

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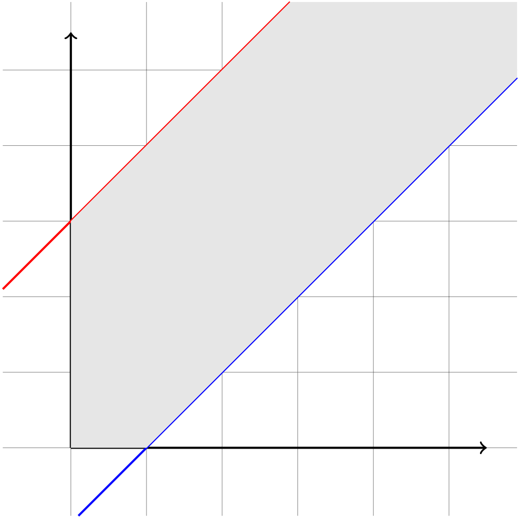

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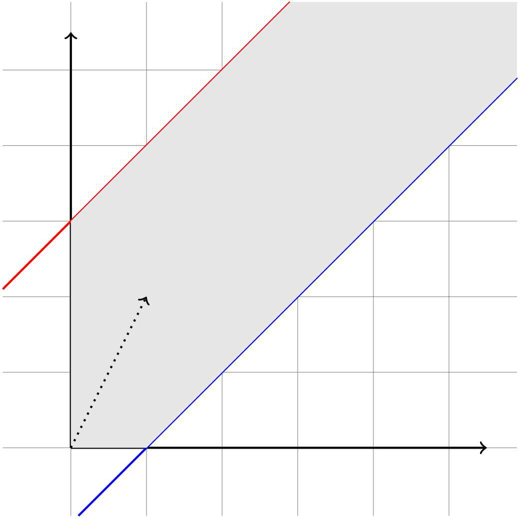
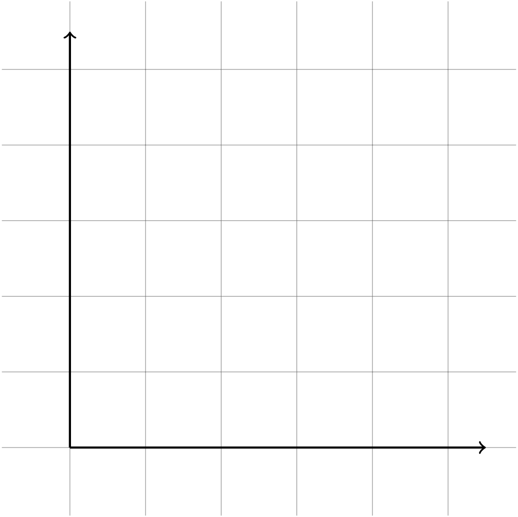


Example 4: degenerate case

$$\max 3x_1 + 2x_2$$

$$\begin{aligned} 3x_1 + 2x_2 &\leq 13 \\ 0 \leq x_1 &\leq 3 \\ 0 \leq x_2 &\leq 5 \end{aligned}$$



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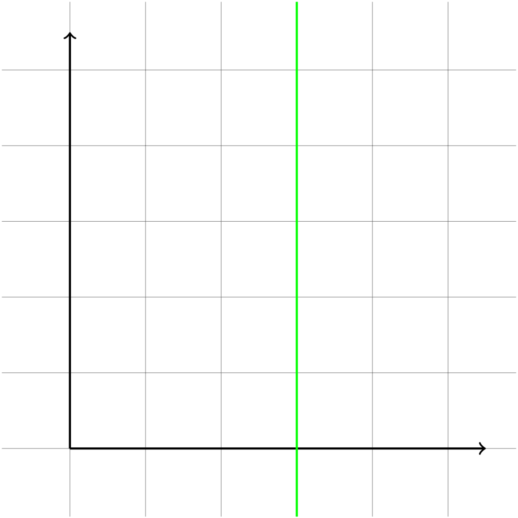

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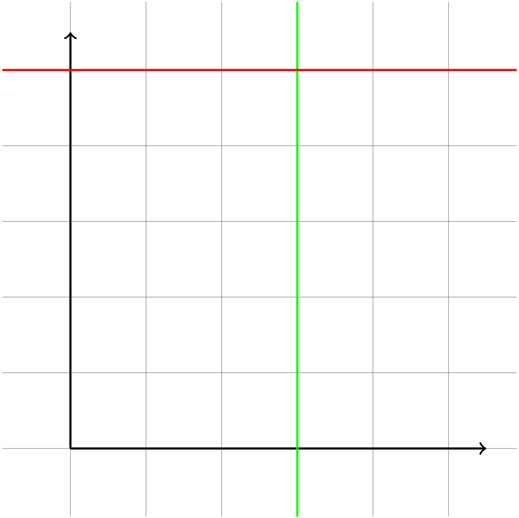

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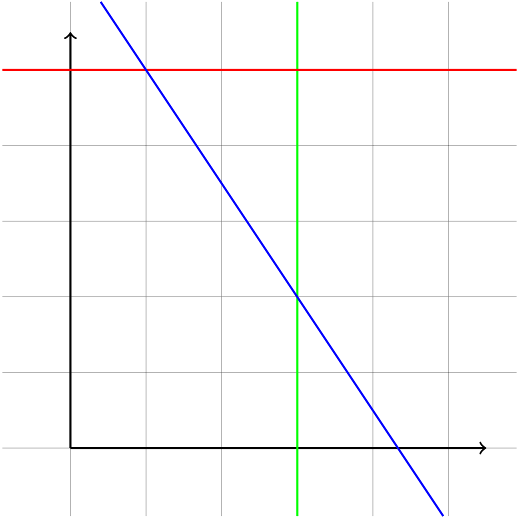

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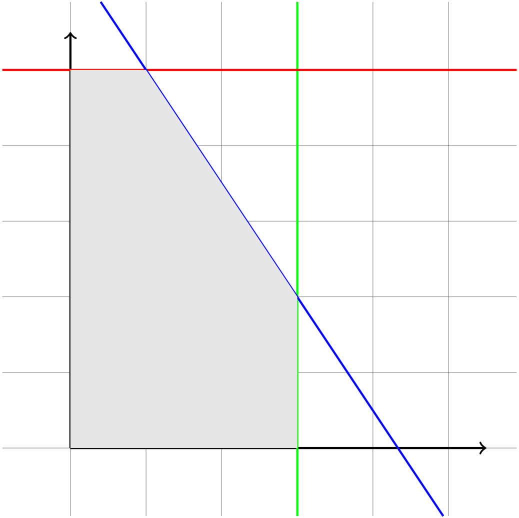

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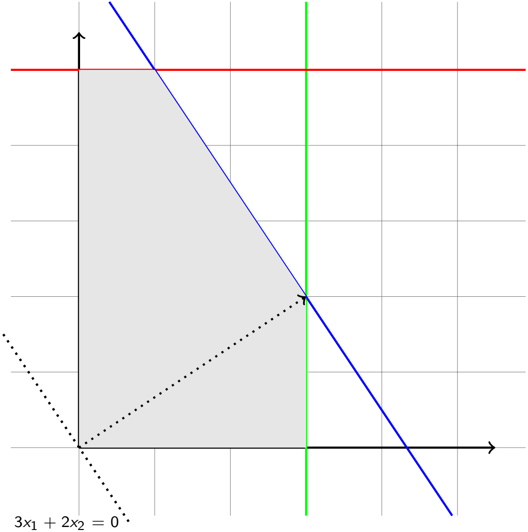

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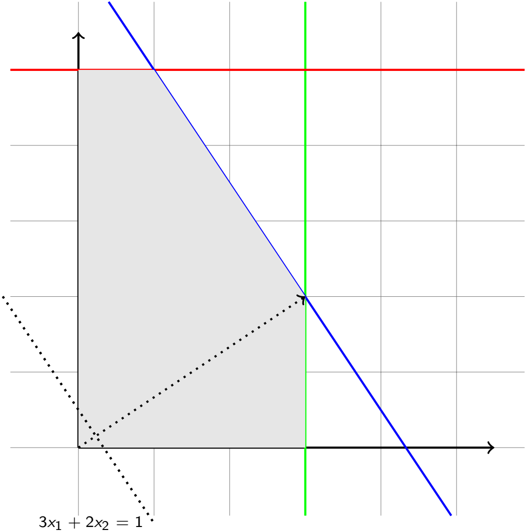

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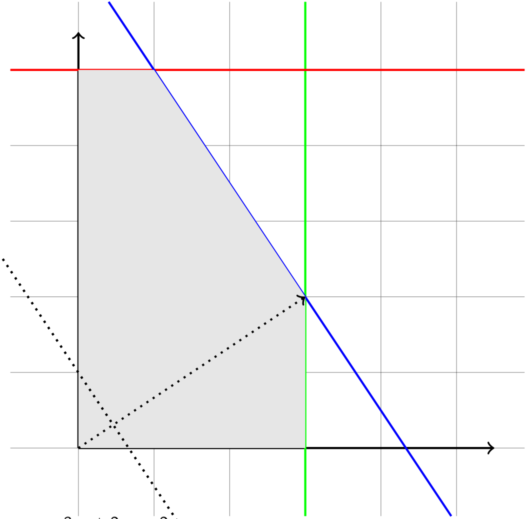

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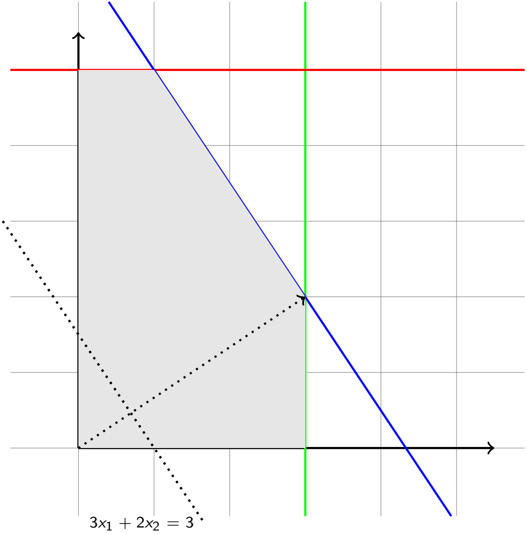

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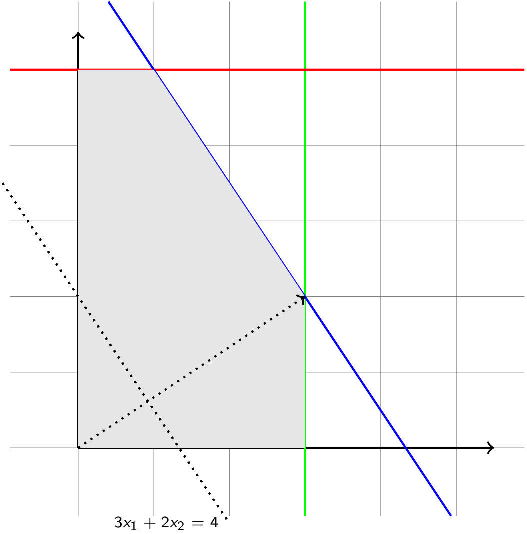

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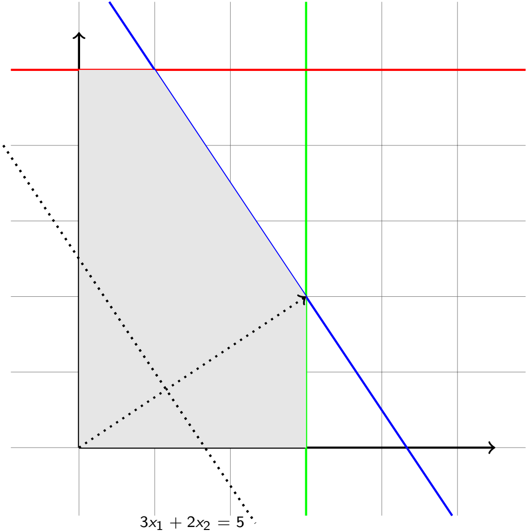

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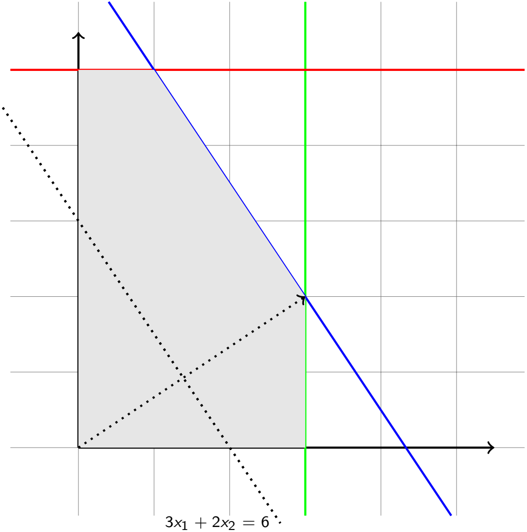

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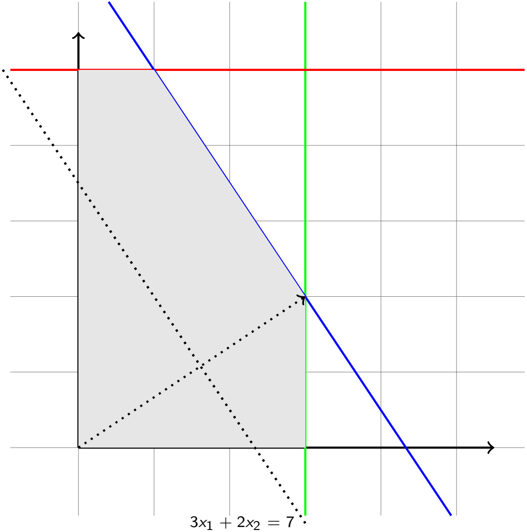

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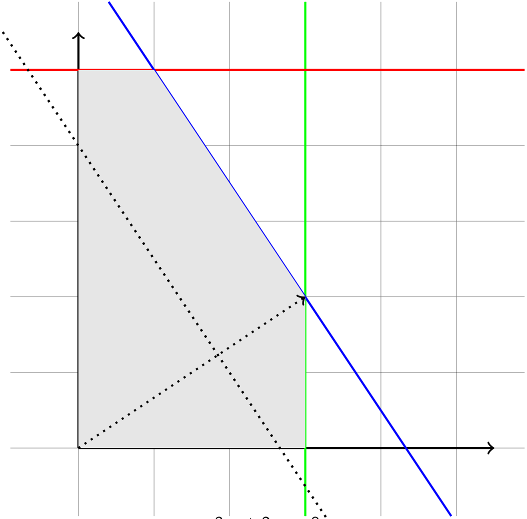

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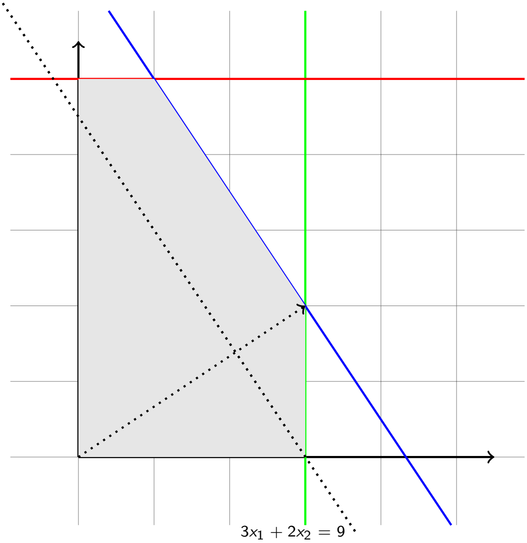

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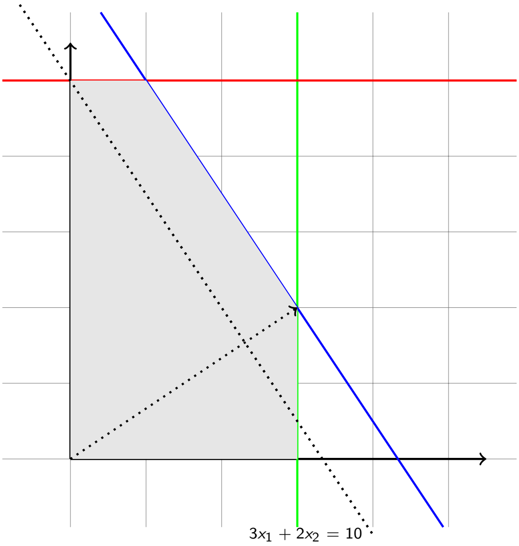

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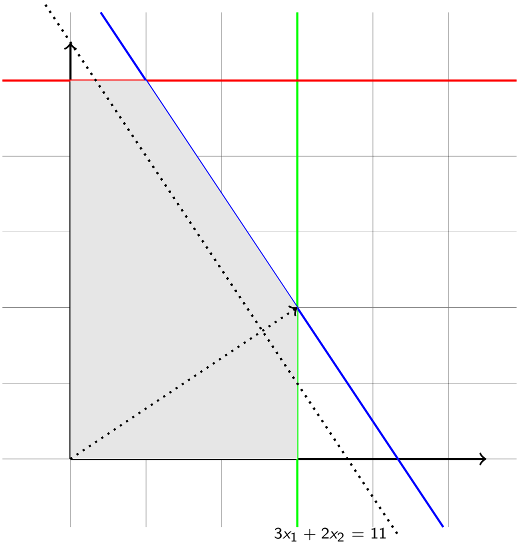

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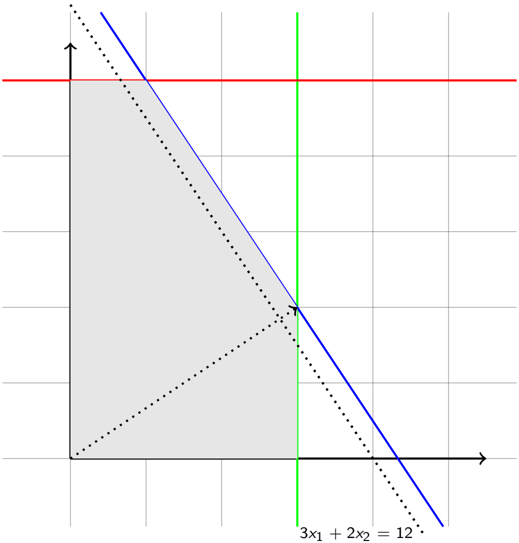

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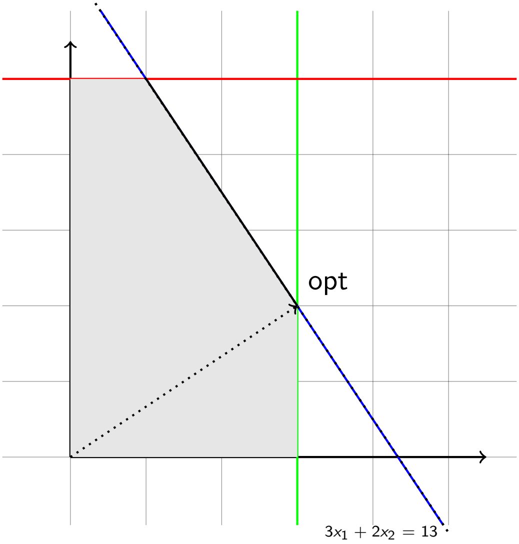

Some theoretical results

Theorem

Each convex combination of optimal vertices is optimal.

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Proof.

Let $v^1, \dots, v^p$ be the optimal vertices of the polyhedron corresponding to the feasible region of LP.
Let $x = \sum_{i=1}^{p} \alpha_i v^i$ with $\sum_{i=1}^{p} \alpha_i = 1, \alpha \geq 0$.
Then, its cost is

$$c^\top x = c^\top \sum_{i=1}^{p} \alpha_i v^i = c^\top v^1 \sum_{i=1}^{p} \alpha_i = c^\top v^1.$$

□

Outline

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Simplex Methods based on the property of LP that

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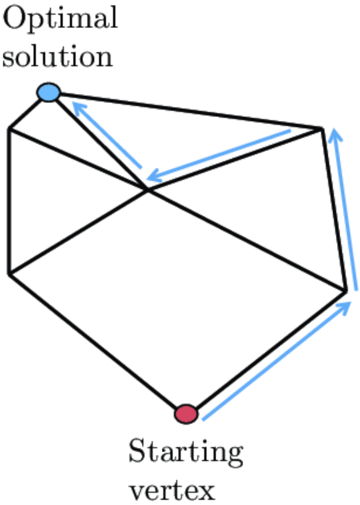
Phase 1 : find a feasible solution

Phase 2 : move from a vertex to an "improving" vertex

Intuition

Simplex Methods



From the Research Gate's page of by Laura Leal-Taixé

The Simplex Method

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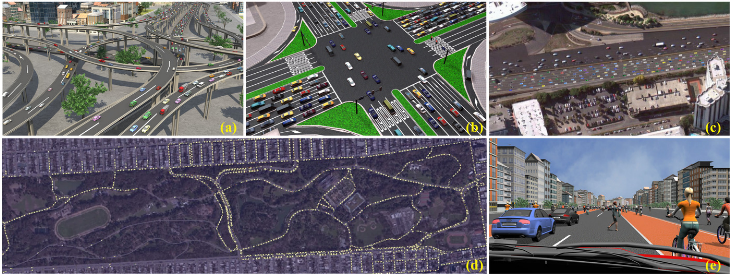
Simplex method → **exponential** worst-case running time

Ellipsoid Method → **polynomial** worst-case running time

Optimization vs. Simulation



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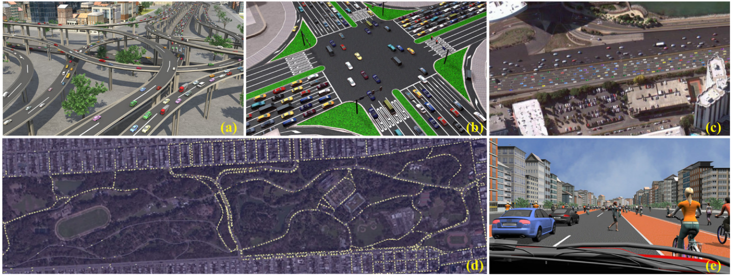

Optimization : decisions are made.

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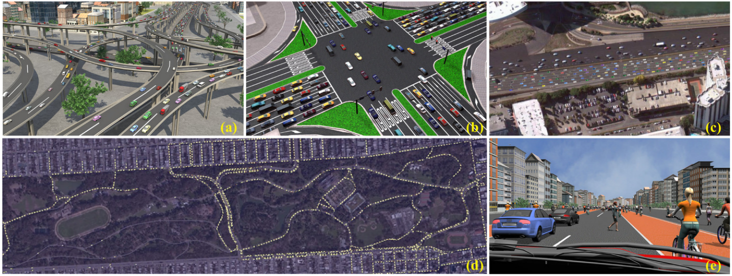

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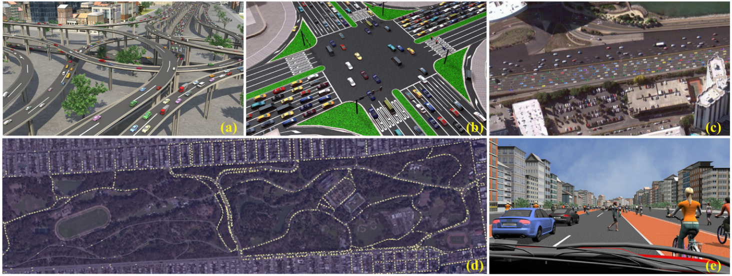

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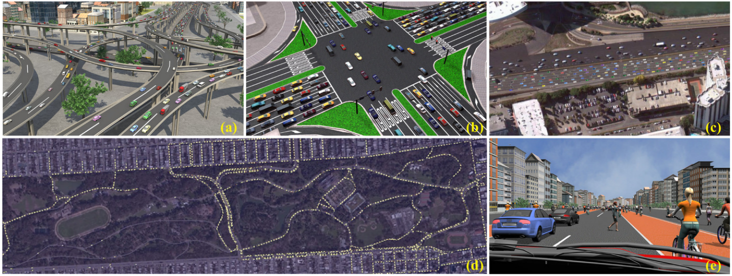

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Answer to the question "What would happen if I make this decision?"

Outline

A few references

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