A system of inference based on proof search

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Based on a paper in the LICS 2023 proceedings.
This is a work-in-progress.

Book Announcement:

Proof Theory and Logic Programming: Computation as proof search

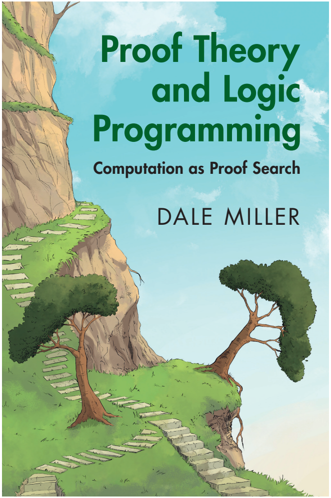
To be published by Cambridge University Press by December 2025.

Preprint available from my web page.
`https://www.lix.polytechnique.fr/Labo/Dale.Miller/ptlp/`
(317 pages, 90 exercises).

Organizes everything I learned about the intersection of proof theory and logic programming during four decades (1985-2025).

Uses classical, intuitionistic, and linear logic (first-order and higher-order) to design and reason about logic programs.

Art by Nadia Miller



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The plan for this talk:

First, formalize computing with symbols on sheets of paper

Second, deal with logical connectives.

Proof search using sheets of paper

Prove that $n(n+1)$ is even for natural number n.

Assume:

$\forall n.\ \text{even}\ n \vee \text{odd}\ n$

$\forall n.\ \text{odd}\ n \supset \text{even}\ (s\ n)$

$\forall n, m, p.\ (\text{even}\ n \vee \text{even}\ m) \supset$
$\qquad \text{times}\ n\ m\ p \supset \text{even}\ p$

$\vdots$

Hence:

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Hence:
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Observations:

1. Occurrences of formulas have two *senses*: as assumption and goal.
2. Some formula occurrences are *permanent*; others may get deleted and/or replaced.
3. One sheet can become 2, also 0 (if an assumption is the goal).

Abstraction of what happens on sheets of paper

$\mathbf{A_1}$
$\mathbf{A_2}$
$\mathbf{A_3}$

$\vdots$

A_0

Conventions:

1. The two senses: hypothesis are blue; goals are red. The vertical dots are no longer needed.

Abstraction of what happens on sheets of paper

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$\mathbf{A_2}$
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A_4

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1. The two senses: hypothesis are blue; goals are red. The vertical dots are no longer needed.
2. *Permanent* items are displayed in **bold**.
3. Sheets are encoded as multisets and not lists.

Our first step is to formalize an *enriched version of multiset rewriting*.

The distinction between hypothesis and goal is not part of the multiset rewriting system itself: it is added later.

The pre-logical framework

- Formulas will be tagged as "hypothesis" or "goal".
- We abstract away the internal structure of tagged formulas and replace them with *atomic expressions*.
- A sheet of paper is modeled by a multisets of atomic expressions.
- The current state is simply a set of sheets.
- Our first goal is to describe
  - how state can be encoded as *expressions* and
  - how state *evolves* by applying *rewriting rules*.

Multiplicative features: multiset rewriting

Multisets: $E ::= A \mid \mathbf{1} \mid E_1 \times E_2$, where A is an atomic expression.

E.g. $a \times a \times b$ denotes $\{a, a, b\}$

$$\frac{\vdash \Delta}{\vdash \mathbf{1}, \Delta} \qquad \frac{\vdash E_1, E_2, \Delta}{\vdash E_1 \times E_2, \Delta}$$

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Entailment between expressions and multisets provides an equality.

$$\frac{}{\mathbf{1} \vdash} \qquad \frac{}{E \vdash E} \qquad \frac{E_1 \vdash \Delta_1 \quad E_2 \vdash \Delta_2}{E_1 \times E_2 \vdash \Delta_1, \Delta_2}$$

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Rewriting multiset Δ to multiset Δ' using rule $E_1 \mapsto E_2$ is done in 3 steps.

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$$\frac{\dfrac{}{a \times b \mapsto c \vdash {\color{red}a}, {\color{blue}a}, {\color{blue}b}}}{\vdash a, a, b}\ \textit{decide}$$

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Additive feature: copying of multisets

We also need to be able to copy the content of a sheet. To this end, we add the following operators on expressions.

$$\frac{}{\vdash \mathbf{0}, \Delta} \qquad \frac{\vdash E_1, \Delta \quad \vdash E_2, \Delta}{\vdash E_1 + E_2, \Delta} \qquad \frac{E_i \vdash \Delta}{E_1 + E_2 \vdash \Delta}$$

Distributivity holds: the inference systems will not be able to distinguish $E_1 \times (E_2 + E_3)$ from $(E_1 \times E_2) + (E_1 \times E_3)$.

The $\times$ on the

- right builds contexts (by becoming a comma)
- left splits contexts

The $+$ on the

- right accumulates branches
- left selects a branch.

These two senses for $\times$ and $+$ allow us to prove results similar to the elimination of non-atomic initials and cuts.

Additional features: the linear and classical realms

As motivated before, some atomic expressions can remain in all evolutions of a multiset; others can be deleted and replaced.

Atomic expressions will belong to the *linear* or the *classical realms*.

Non-atomic expressions are not classified either way.

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Atomic expressions in the linear realm do not have this superpower.

Additional feature: debts

Because computation can be distributed, multiset rewriting needs some flexibility.

- Applying the rule $a \times b \mapsto c$ to Δ requires locating both a and b in Δ.
- Δ might be very large and distributed across a network.
- The expression a might be found quickly, but finding b could take time.
- Instead of *blocking* rewriting until b is found, we might allow a *debt* to be registered in our multiset and then resolve that debt later.

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All atomic expressions will have a positive or negative "credit rating."

- If b's rating is positive, then a debt can be constructed.

This debt mechanism will help account for the difference between

- *bottom-up* and *top-down* reasoning, and
- sequent calculus and natural deduction.

Expressions and Rules

The syntactic variable A ranges over some fixed set of atomic expressions.

Expressions and rules are defined inductively.

$$
\begin{aligned}
E &::= A \mid \mathbf{0} \mid E_1 + E_2 \mid \mathbf{1} \mid E_1 \times E_2 \\
R &::= A \mid \mathbf{0} \mid R_1 + R_2 \mid \mathbf{1} \mid R_1 \times R_2 \mid R \mapsto E \mid R \Mapsto E
\end{aligned}
$$

$\mapsto$ and $\Mapsto$ associate to the left; $+$ and $\times$ associate to the right.

A *debt* is an expression of the form $\overline{A}$.

Γ ranges over finite multisets containing R-expressions.

Δ ranges over multisets that can contain both E-expressions and debts.

$\mathcal{R}$ denotes some countable set of R-expressions.

Bias assignments

A *bias assignment* $\delta(\cdot)$ maps atomic expressions to $\{-2, -1, +1, +2\}$.

If $\delta(A) > 0$, then A can be converted into a debt.

A is in the *linear realm* if $\delta(A)$ is ± 1 and in the *classical realm* if $\delta(A)$ is ± 2.

$\boldsymbol{S}$ ranges over atomic expressions in the classical realm.

Υ ranges over finite multisets of atomic expressions in the classical realm.

The basic inference system: $\mathbf{B}$

RIGHT RULES

$$\frac{}{\Gamma \vdash \mathbf{0}, \Delta} \qquad \frac{\Gamma \vdash E_1, \Delta \quad \Gamma \vdash E_2, \Delta}{\Gamma \vdash E_1 + E_2, \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \mathbf{1}, \Delta} \qquad \frac{\Gamma \vdash E_1, E_2, \Delta}{\Gamma \vdash E_1 \times E_2, \Delta}$$

LEFT RULES

$$\frac{}{\mathbf{1} \vdash} \qquad \frac{R_1 \vdash \Delta_1 \quad R_2 \vdash \Delta_2}{R_1 \times R_2 \vdash \Delta_1, \Delta_2} \qquad \frac{R_i \vdash \Delta}{R_1 + R_2 \vdash \Delta}$$

$$\frac{R \vdash \Delta_1 \quad \vdash E, \Delta_2}{R \mapsto E \vdash \Delta_1, \Delta_2} \qquad \frac{R \vdash \Upsilon \quad \vdash E, \Delta}{R \Mapsto E \vdash \Upsilon, \Delta}$$

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$$\frac{R \vdash \Delta}{\vdash \Delta} \ \textit{decide}, \ \ R \in \mathcal{R}$$

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$$\frac{}{\Gamma \vdash \mathbf{0}, \Delta} \qquad \frac{\Gamma \vdash E_1, \Delta \quad \Gamma \vdash E_2, \Delta}{\Gamma \vdash E_1 + E_2, \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \mathbf{1}, \Delta} \qquad \frac{\Gamma \vdash E_1, E_2, \Delta}{\Gamma \vdash E_1 \times E_2, \Delta}$$

Left rules

$$\frac{}{\mathbf{1} \vdash} \qquad \frac{R_1 \vdash \Delta_1 \quad R_2 \vdash \Delta_2}{R_1 \times R_2 \vdash \Delta_1, \Delta_2} \qquad \frac{R_i \vdash \Delta}{R_1 + R_2 \vdash \Delta}$$

$$\frac{R \vdash \Delta_1 \quad \vdash E, \Delta_2}{R \mapsto E \vdash \Delta_1, \Delta_2} \qquad \frac{R \vdash \Upsilon \quad \vdash E, \Delta}{R \Mapsto E \vdash \Upsilon, \Delta}$$

$$\frac{R \vdash \Delta}{\vdash \Delta}\ \textit{decide},\ R \in \mathcal{R}$$

$$\frac{\vdash \bar{A}, \Delta}{A \vdash \Delta}\ \textit{debit}_1,\ \text{if } \delta(A) = +1 \qquad \frac{\vdash \bar{\boldsymbol{S}}, \Upsilon}{\boldsymbol{S} \vdash \Upsilon}\ \textit{debit}_2,\ \text{if } \delta(\boldsymbol{S}) = +2$$

The basic inference system: $\mathbf{B}$

RIGHT RULES

$$\frac{}{\Gamma \vdash \mathbf{0}, \Delta} \qquad \frac{\Gamma \vdash E_1, \Delta \quad \Gamma \vdash E_2, \Delta}{\Gamma \vdash E_1 + E_2, \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \mathbf{1}, \Delta} \qquad \frac{\Gamma \vdash E_1, E_2, \Delta}{\Gamma \vdash E_1 \times E_2, \Delta}$$

LEFT RULES

$$\frac{}{\mathbf{1} \vdash} \qquad \frac{R_1 \vdash \Delta_1 \quad R_2 \vdash \Delta_2}{R_1 \times R_2 \vdash \Delta_1, \Delta_2} \qquad \frac{R_i \vdash \Delta}{R_1 + R_2 \vdash \Delta}$$

$$\frac{R \vdash \Delta_1 \quad \vdash E, \Delta_2}{R \mapsto E \vdash \Delta_1, \Delta_2} \qquad \frac{R \vdash \Upsilon \quad \vdash E, \Delta}{R \Mapsto E \vdash \Upsilon, \Delta}$$

$$\frac{R \vdash \Delta}{\vdash \Delta}\ \mathit{decide},\ R \in \mathcal{R}$$

$$\frac{\vdash \bar{A}, \Delta}{A \vdash \Delta}\ \mathit{debit}_1,\ \text{if } \delta(A) = +1 \qquad \frac{\vdash \bar{\boldsymbol{S}}, \Upsilon}{\boldsymbol{S} \vdash \Upsilon}\ \mathit{debit}_2,\ \text{if } \delta(\boldsymbol{S}) = +2$$

IDENTITY RULES

$$\frac{}{E \vdash E}\ \mathit{init} \qquad \frac{}{\vdash \bar{A}, A}\ \mathit{iou}$$

STRUCTURAL RULES

$$\frac{\Gamma \vdash \Delta, \boldsymbol{S}, \boldsymbol{S}}{\Gamma \vdash \Delta, \boldsymbol{S}}\ \text{contract} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \boldsymbol{S}}\ \text{weaken}$$

Some meta-theory of $\mathbf{B}$

Proposition

Provability in $\mathbf{B}$ does not change if the init rule is restricted to atomic expressions, i.e., $A \vdash A$ instead of $E \vdash E$.

Proposition

The following inference rule is admissible in $\mathbf{B}$.

$$\frac{\Gamma \vdash \Delta_1, E \qquad E \vdash \Delta_2}{\Gamma \vdash \Delta_1, \Delta_2}\ \mathit{clip}$$

Proposition

If $\delta(A) = +1$, the following $dclip_1$ rule is admissible.

$$\frac{\vdash \Delta_1, A \qquad \vdash \Delta_2, \overline{A}}{\vdash \Delta_1, \Delta_2}\ \mathit{dclip}_1$$

If $\delta(\boldsymbol{S}) = +2$, the following $dclip_2$ rule is admissible.

$$\frac{\vdash \Delta, \boldsymbol{S} \qquad \vdash \Upsilon, \overline{\boldsymbol{S}}}{\vdash \Delta, \Upsilon}\ \mathit{dclip}_2$$

Some meta theory of $\mathbf{B}$ (cont)

Proposition

If $\vdash \Delta$ has a $\mathbf{B}$-proof, it has a $\mathbf{B}$-proof without the debit$_1$ and debit$_2$ rules.

- If all debts are eventually paid, we can reorganize the proof so that the payments precede the formation of a debt.
- Of course, these proofs might vary a great deal in structure.

Proposition

The right rules are invertible. In particular, if E is not atomic and the sequent $\vdash E, \Delta$ is provable, then there is a proof of this sequent in which the last inference rule is an introduction rule for E.

Removing some non-determinism from $\mathbf{B}$

1. The right rules are invertible: do them in any order and to exhaustion.
   - A $\mathbf{B}$-proof Ξ is *reduced* if every occurrence of the *decide* rule has a right-hand side containing only atomic expressions or debts.
   - Proposition: If the sequent $\vdash \Delta$ has a $\mathbf{B}$-proof, it has a reduced proof.
2. There are two ways to prove $A \vdash A$ when $\delta(A) = +1$: *init* or a combination of $debit_1$ and *iou*. This has a simple resolution.
3. Major issue:
   The structural rules seem all wrong from the proof search perspective.

Structural rules: a major revision is needed

$$\frac{\Gamma \vdash \Delta, \boldsymbol{S}, \boldsymbol{S}}{\Gamma \vdash \Delta, \boldsymbol{S}} \text{ contract} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \boldsymbol{S}} \text{ weaken}$$

These can be applied almost anytime! We need a better treatment.

Structural rules: a major revision is needed

$$\frac{\Gamma \vdash \Delta, \boldsymbol{S}, \boldsymbol{S}}{\Gamma \vdash \Delta, \boldsymbol{S}}\ \text{contract} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \boldsymbol{S}}\ \text{weaken}$$

These can be applied almost anytime! We need a better treatment.

Consider again a multiplicative and an additive rule.

$$\frac{R_1 \vdash \Delta_1 \quad R_2 \vdash \Delta_2}{R_1 \times R_2 \vdash \Delta_1, \Delta_2} \qquad \frac{\Gamma \vdash E_1, \Delta \quad \Gamma \vdash E_2, \Delta}{\Gamma \vdash E_1 + E_2, \Delta}$$

In the multiplicative rule, every side-expression occurrence in the conclusion (a member of $\Delta_1 \cup \Delta_2$) also occurs in a *unique* premise.

In an additive rule, every side-expression occurrence in the conclusion (a member of Δ) occurs in *every* premise.

Structural rules: a major revision is needed

$$\frac{\Gamma \vdash \Delta, \boldsymbol{S}, \boldsymbol{S}}{\Gamma \vdash \Delta, \boldsymbol{S}}\ \text{contract} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \boldsymbol{S}}\ \text{weaken}$$

These can be applied almost anytime! We need a better treatment.

Consider again a multiplicative and an additive rule.

$$\frac{R_1 \vdash \Delta_1 \quad R_2 \vdash \Delta_2}{R_1 \times R_2 \vdash \Delta_1, \Delta_2} \qquad \frac{\Gamma \vdash E_1, \Delta \quad \Gamma \vdash E_2, \Delta}{\Gamma \vdash E_1 + E_2, \Delta}$$

In the multiplicative rule, every side-expression occurrence in the conclusion (a member of $\Delta_1 \cup \Delta_2$) also occurs in a *unique* premise.

In an additive rule, every side-expression occurrence in the conclusion (a member of Δ) occurs in *every* premise.

New treatment: Classical realm atomic expressions are *treated additively*, even in multiplicative rules.

Structural rules (continued)

This new treatment of structural rules produces rules of the following form.

$$\frac{}{\mathbf{1} \vdash \Upsilon} \qquad \frac{R_1 \vdash \mathcal{A}_1, \Upsilon \quad R_2 \vdash \mathcal{A}_2, \Upsilon}{R_1 \times R_2 \vdash \mathcal{A}_1, \mathcal{A}_2, \Upsilon}$$

Here, $\mathcal{A}_1$ and $\mathcal{A}_2$ have only linear realm atomic expressions or debts.

The two-phase inference system $\mathbf{F}$

$$\frac{}{\vdash \mathbf{0}, \Delta} \qquad \frac{\vdash E_1, \Delta \quad \vdash E_2, \Delta}{\vdash E_1 + E_2, \Delta} \qquad \frac{\vdash \Delta}{\vdash \mathbf{1}, \Delta} \qquad \frac{\vdash E_1, E_2, \Delta}{\vdash E_1 \times E_2, \Delta}$$

The two-phase inference system $\mathbf{F}$

$$\frac{}{\vdash \mathbf{0}, \Delta} \qquad \frac{\vdash E_1, \Delta \quad \vdash E_2, \Delta}{\vdash E_1 + E_2, \Delta} \qquad \frac{\vdash \Delta}{\vdash \mathbf{1}, \Delta} \qquad \frac{\vdash E_1, E_2, \Delta}{\vdash E_1 \times E_2, \Delta}$$

$$\frac{\Downarrow R \vdash \mathcal{A}, \Upsilon}{\vdash \mathcal{A}, \Upsilon} \ \textit{decide}, \ \ R \in \mathcal{R}$$

Three kinds of sequents: $\vdash \Delta \qquad \Downarrow R \vdash \mathcal{A}, \Upsilon \qquad \vdash E \Downarrow \mathcal{A}, \Upsilon$

The two-phase inference system $\mathbf{F}$

$$\frac{}{\vdash \mathbf{0}, \Delta} \qquad \frac{\vdash E_1, \Delta \quad \vdash E_2, \Delta}{\vdash E_1 + E_2, \Delta} \qquad \frac{\vdash \Delta}{\vdash \mathbf{1}, \Delta} \qquad \frac{\vdash E_1, E_2, \Delta}{\vdash E_1 \times E_2, \Delta}$$

$$\frac{\Downarrow R \vdash \mathcal{A}, \Upsilon}{\vdash \mathcal{A}, \Upsilon}\ \textit{decide},\ \ R \in \mathcal{R}$$

$$\frac{}{\Downarrow \mathbf{1} \vdash \Upsilon} \qquad \frac{\Downarrow R_1 \vdash \mathcal{A}_1, \Upsilon \quad \Downarrow R_2 \vdash \mathcal{A}_2, \Upsilon}{\Downarrow R_1 \times R_2 \vdash \mathcal{A}_1, \mathcal{A}_2, \Upsilon} \qquad \frac{\Downarrow R_i \vdash \mathcal{A}, \Upsilon}{\Downarrow R_1 + R_2 \vdash \mathcal{A}, \Upsilon}$$

$$\frac{\Downarrow R \vdash \mathcal{A}_1, \Upsilon \quad \vdash E \Downarrow \mathcal{A}_2, \Upsilon}{\Downarrow R \mapsto E \vdash \mathcal{A}_1, \mathcal{A}_2, \Upsilon} \qquad \frac{\Downarrow R \vdash \Upsilon \quad \vdash E \Downarrow \mathcal{A}, \Upsilon}{\Downarrow R \Mapsto E \vdash \mathcal{A}, \Upsilon}$$

The two-phase inference system $\mathbf{F}$

$$\frac{}{\vdash \mathbf{0}, \Delta} \qquad \frac{\vdash E_1, \Delta \quad \vdash E_2, \Delta}{\vdash E_1 + E_2, \Delta} \qquad \frac{\vdash \Delta}{\vdash \mathbf{1}, \Delta} \qquad \frac{\vdash E_1, E_2, \Delta}{\vdash E_1 \times E_2, \Delta}$$

$$\frac{\Downarrow R \vdash \mathcal{A}, \Upsilon}{\vdash \mathcal{A}, \Upsilon}\ \textit{decide},\ \ R \in \mathcal{R}$$

$$\frac{}{\Downarrow \mathbf{1} \vdash \Upsilon} \qquad \frac{\Downarrow R_1 \vdash \mathcal{A}_1, \Upsilon \quad \Downarrow R_2 \vdash \mathcal{A}_2, \Upsilon}{\Downarrow R_1 \times R_2 \vdash \mathcal{A}_1, \mathcal{A}_2, \Upsilon} \qquad \frac{\Downarrow R_i \vdash \mathcal{A}, \Upsilon}{\Downarrow R_1 + R_2 \vdash \mathcal{A}, \Upsilon}$$

$$\frac{\Downarrow R \vdash \mathcal{A}_1, \Upsilon \quad \vdash E \Downarrow \mathcal{A}_2, \Upsilon}{\Downarrow R \mapsto E \vdash \mathcal{A}_1, \mathcal{A}_2, \Upsilon} \qquad \frac{\Downarrow R \vdash \Upsilon \quad \vdash E \Downarrow \mathcal{A}, \Upsilon}{\Downarrow R \Mapsto E \vdash \mathcal{A}, \Upsilon}$$

$$\frac{\vdash \overline{A}, \mathcal{A}, \Upsilon}{\Downarrow A \vdash \mathcal{A}, \Upsilon}\ \textit{debit}_1,\ \text{if } \delta(A) = +1 \qquad \frac{\vdash \bar{A}, \Upsilon}{\Downarrow A \vdash \Upsilon}\ \textit{debit}_2,\ \text{if } \delta(A) = +2$$

The two-phase inference system $\mathbf{F}$

$$\frac{}{\vdash \mathbf{0}, \Delta} \qquad \frac{\vdash E_1, \Delta \quad \vdash E_2, \Delta}{\vdash E_1 + E_2, \Delta} \qquad \frac{\vdash \Delta}{\vdash \mathbf{1}, \Delta} \qquad \frac{\vdash E_1, E_2, \Delta}{\vdash E_1 \times E_2, \Delta}$$

$$\frac{\Downarrow R \vdash \mathcal{A}, \Upsilon}{\vdash \mathcal{A}, \Upsilon}\ \mathit{decide},\ \ R \in \mathcal{R}$$

$$\frac{}{\Downarrow \mathbf{1} \vdash \Upsilon} \qquad \frac{\Downarrow R_1 \vdash \mathcal{A}_1, \Upsilon \quad \Downarrow R_2 \vdash \mathcal{A}_2, \Upsilon}{\Downarrow R_1 \times R_2 \vdash \mathcal{A}_1, \mathcal{A}_2, \Upsilon} \qquad \frac{\Downarrow R_i \vdash \mathcal{A}, \Upsilon}{\Downarrow R_1 + R_2 \vdash \mathcal{A}, \Upsilon}$$

$$\frac{\Downarrow R \vdash \mathcal{A}_1, \Upsilon \quad \vdash E \Downarrow \mathcal{A}_2, \Upsilon}{\Downarrow R \mapsto E \vdash \mathcal{A}_1, \mathcal{A}_2, \Upsilon} \qquad \frac{\Downarrow R \vdash \Upsilon \quad \vdash E \Downarrow \mathcal{A}, \Upsilon}{\Downarrow R \Mapsto E \vdash \mathcal{A}, \Upsilon}$$

$$\frac{\vdash \overline{A}, \mathcal{A}, \Upsilon}{\Downarrow A \vdash \mathcal{A}, \Upsilon}\ \mathit{debit}_1,\ \text{if } \delta(A) = +1 \qquad \frac{\vdash \bar{A}, \Upsilon}{\Downarrow A \vdash \Upsilon}\ \mathit{debit}_2,\ \text{if } \delta(A) = +2$$

$$\frac{\delta(A) < 0}{\Downarrow A \vdash A, \Upsilon}\ \mathit{initL} \qquad \frac{\delta(A) > 0}{\vdash A \Downarrow \overline{A}, \Upsilon}\ \mathit{initR} \qquad \frac{\vdash E, \mathcal{A}, \Upsilon}{\vdash E \Downarrow \mathcal{A}, \Upsilon}\ \mathit{release}\dagger \qquad \frac{\delta(A) > 0}{\vdash \overline{A}, A, \Upsilon}\ \mathit{iou}$$

The proviso † for *release*: E is either not atomic or it is atomic and $\delta(E) < 0$.

Synthetic rules in $\mathbf{F}$

$\vdash \mathcal{A}, \boldsymbol{\Upsilon}$ is a *border sequent*: only of atomic expressions and debts.

A *synthetic rule* is built from right phases above a left phase: their premises and conclusions are border sequents.

$$
\dfrac{\dfrac{\dots \vdots \dots \qquad \dfrac{\dfrac{\vdash \mathcal{A}', \boldsymbol{\Upsilon}'}{\vdots}}{\vdash \Delta, \boldsymbol{\Upsilon}}\,\dagger \qquad \dots \vdots \dots}{\Downarrow R \vdash \mathcal{A}, \boldsymbol{\Upsilon}}}{\vdash \mathcal{A}, \boldsymbol{\Upsilon}}\ \textit{decide}
\qquad
\begin{array}{l}\Leftarrow \text{right phase} \\[2em] \Leftarrow \text{left phase}\end{array}
$$

$\dagger$ is either *release*, *debit*$_1$, or *debit*$_2$.

The *right phase* is invertible and additive.

The *left phase* is not invertible and multiplicative.

Different levels of adequacy when encoding proof systems

$\mathbf{F}$ presents an *assembly language* for inference. We want to *compile* logical inference rules into $\mathbf{F}$ and preserve the proof search semantics.

Three levels of adequacy of encodings are natural to identify.

1. *Relative completeness*: a formula has a proof in one system if it has a proof in the other system.
2. *Full completeness of proofs*: the complete proofs in one system naturally correspond to proofs in the other system.
3. *Full completeness of inference rules*: every inference rule is in one-to-one correspondence with those in the other system.

All encodings in this talk are at this highest level of adequacy:

> *A set of rules $\mathcal{R}$ encodes a proof system* **P** *means that the synthetic rules in $\mathbf{F}$ for elements of $\mathcal{R}$ corresponds to inference rules in the* **P***, and vice versa.*

Encoding sequents of formulas

Two-sided sequents are of the form

$$B_1, \ldots, B_n \vdash C_1, \ldots, C_m$$

which we encode as the expression

$$\lfloor B_1 \rfloor \times \cdots \times \lfloor B_n \rfloor \times \lceil C_1 \rceil \times \cdots \times \lceil C_m \rceil$$

or, equivalently, by the multiset

$$\lfloor B_1 \rfloor, \ldots, \lfloor B_n \rfloor, \lceil C_1 \rceil, \ldots, \lceil C_m \rceil.$$

In *classical logic*, formulas on the left and right are subject to weakening and contraction: thus, $\delta(\lfloor \cdot \rfloor) = \pm 2$ and $\delta(\lceil \cdot \rceil) = \pm 2$.

In *intuitionistic logic*, only the formulas on the left are subject to weakening and contraction: thus, $\delta(\lfloor \cdot \rfloor) = \pm 2$ and $\delta(\lceil \cdot \rceil) = \pm 1$.

Rules for classical and intuitionistic logic $\mathcal{R}_1$

$$
\begin{array}{lrcl}
(\supset L) & \lfloor A \supset B \rfloor & \mapsto & \lceil A \rceil \Rrightarrow \lfloor B \rfloor \\
(\supset R) & \lceil A \supset B \rceil & \mapsto & \lfloor A \rfloor \times \lceil B \rceil \\
(\wedge L) & \lfloor A \wedge B \rfloor & \mapsto & \lfloor A \rfloor \\
(\wedge L) & \lfloor A \wedge B \rfloor & \mapsto & \lfloor B \rfloor \\
(\wedge R) & \lceil A \wedge B \rceil & \mapsto & \lceil A \rceil + \lceil B \rceil \\
(\vee L) & \lfloor A \vee B \rfloor & \mapsto & \lfloor A \rfloor + \lfloor B \rfloor \\
(\vee R) & \lceil A \vee B \rceil & \mapsto & \lceil A \rceil \\
(\vee R) & \lceil A \vee B \rceil & \mapsto & \lceil B \rceil \\
(\bot L) & \lfloor \bot \rfloor & \mapsto & \mathbf{0} \\
(\top R) & \lceil \top \rceil & \mapsto & \mathbf{0} \\
(Id_1) & \lfloor C \rfloor \times \lceil C \rceil & & \\
(Id_2) & \mathbf{1} & \mapsto & \lceil C \rceil \Rrightarrow \lfloor C \rfloor
\end{array}
$$

Choosing the correct bias assignment for sequent calculi

Using the bias assignment that returns only negative numbers, then we get *sequent calculi* similar to Gentzen's $\mathbf{LK}$ and $\mathbf{LJ}$.

- If $\delta(\lfloor\cdot\rfloor) = -2$ and $\delta(\lceil\cdot\rceil) = -1$, then deciding on $(\supset L)$ yields

$$\dfrac{\dfrac{\dfrac{}{\Downarrow \lfloor A \supset B \rfloor \vdash \lfloor A \supset B \rfloor, \Upsilon} \quad \vdash \lceil A \rceil, \lfloor A \supset B \rfloor, \Upsilon}{\Downarrow \lfloor A \supset B \rfloor \mapsto \lceil A \rceil \vdash \lceil A \rceil, \lfloor A \supset B \rfloor, \Upsilon} \qquad \vdash \lfloor B \rfloor, \lfloor A \supset B \rfloor, \mathcal{A}, \Upsilon}{\dfrac{\Downarrow \lfloor A \supset B \rfloor \mapsto \lceil A \rceil \Mapsto \lfloor B \rfloor \vdash \lfloor A \supset B \rfloor, \mathcal{A}, \Upsilon}{\vdash \lfloor A \supset B \rfloor, \mathcal{A}, \Upsilon}}$$

which encodes (assuming that $\mathcal{A}$ is $\{\lceil C \rceil\}$).

$$\dfrac{A \supset B, \Gamma \vdash A \qquad A \supset B, B, \Gamma \vdash C}{A \supset B, \Gamma \vdash C}$$

Choosing the correct bias assignment for sequent calculi

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$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Downarrow \lfloor A \supset B \rfloor \vdash \lfloor A \supset B \rfloor, \Upsilon} \quad \vdash \lceil A \rceil, \lfloor A \supset B \rfloor, \Upsilon}{\Downarrow \lfloor A \supset B \rfloor \mapsto \lceil A \rceil \vdash \lceil A \rceil, \lfloor A \supset B \rfloor, \Upsilon} \qquad \vdash \lfloor B \rfloor, \lfloor A \supset B \rfloor, \mathcal{A}, \Upsilon}{\Downarrow \lfloor A \supset B \rfloor \mapsto \lceil A \rceil \Mapsto \lfloor B \rfloor \vdash \lfloor A \supset B \rfloor, \mathcal{A}, \Upsilon}}{\vdash \lfloor A \supset B \rfloor, \mathcal{A}, \Upsilon}$$

  which encodes (assuming that $\mathcal{A}$ is $\{\lceil C \rceil\}$).

$$\dfrac{A \supset B, \Gamma \vdash A \qquad A \supset B, B, \Gamma \vdash C}{A \supset B, \Gamma \vdash C}$$

- If we set $\delta(\lfloor \cdot \rfloor) = -2$ and $\delta(\lceil \cdot \rceil) = -2$, then we have

$$\dfrac{A \supset B, \Gamma \vdash A, \Psi \qquad A \supset B, B, \Gamma \vdash \Psi}{A \supset B, \Gamma \vdash \Psi} \; .$$

The two identity rules: initial and cut

The (Id_1) and (Id_2) rules have special roles.

$$\dfrac{\dfrac{}{\Downarrow \lfloor C \rfloor \vdash \lfloor C \rfloor, \Upsilon}\ \mathit{initL} \quad \dfrac{}{\Downarrow \lceil C \rceil \vdash \lceil C \rceil, \Upsilon}\ \mathit{initL}}{\dfrac{\Downarrow \lfloor C \rfloor \times \lceil C \rceil \vdash \lfloor C \rfloor, \lceil C \rceil, \Upsilon}{\vdash \lfloor C \rfloor, \lceil C \rceil, \Upsilon}\ \mathit{decide}\ Id_1}$$

$$\dfrac{\dfrac{\dfrac{}{\Downarrow \mathbf{1} \vdash \Upsilon} \quad \vdash \lceil C \rceil, \Upsilon \quad \vdash \lfloor C \rfloor, \mathcal{A}, \Upsilon}{\Downarrow \mathbf{1} \Rrightarrow \lceil C \rceil \mapsto \lfloor C \rfloor \vdash \mathcal{A}, \Upsilon}}{\vdash \mathcal{A}, \Upsilon}\ \mathit{decide}\ Id_2$$

These justify the synthetic rules

$$\dfrac{}{\vdash \lfloor C \rfloor, \lceil C \rceil, \Upsilon} \qquad \dfrac{\vdash \lceil C \rceil, \Upsilon \quad \vdash \lfloor C \rfloor, \mathcal{A}, \Upsilon}{\vdash \mathcal{A}, \Upsilon}$$

In the intuitionistic setting, the variable Υ contains only $\lfloor \cdot \rfloor$ atomic expressions while $\mathcal{A}$ contains only a single expression, which is of the form $\lceil \cdot \rceil$.

(Id_1) and (Id_2) encode the *init* and *cut* rules of sequent calculus.

Negative bias encodes sequent calculi

Proposition

Let $\delta(\lfloor \cdot \rfloor) = -2$.

1. *If* $\delta(\lceil \cdot \rceil) = -1$ *then* $\mathcal{R}_1$ *encodes (essentially) Gentzen's* $\mathbf{LJ}$ *proof system.*
2. *If* $\delta(\lceil \cdot \rceil) = -2$*, then* $\mathcal{R}_1$ *encodes (essentially) Gentzen's* $\mathbf{LK}$ *proof system.*

Natural deduction for intuitionistic logic

$$\frac{\Gamma \vdash A \supset B \downarrow \quad \Gamma \vdash A \uparrow}{\Gamma \vdash B \downarrow}\ [\supset E] \qquad \frac{\Gamma, A \vdash B \uparrow}{\Gamma \vdash A \supset B \uparrow}\ [\supset I]$$

$$\frac{\Gamma \vdash A \wedge B \downarrow}{\Gamma \vdash A \downarrow}\ [\wedge E] \qquad \frac{\Gamma \vdash A \wedge B \downarrow}{\Gamma \vdash B \downarrow}\ [\wedge E] \qquad \frac{\Gamma \vdash A \uparrow \quad \Gamma \vdash B \uparrow}{\Gamma \vdash A \wedge B \uparrow}\ [\wedge I]$$

$$\frac{}{\Gamma \vdash \top \uparrow}\ [\top I] \qquad \frac{\Gamma \vdash \bot \downarrow}{\Gamma \vdash C \uparrow}\ [\bot E]$$

$$\frac{}{\Gamma, A \vdash A \downarrow}\ [\mathrm{I}] \qquad \frac{\Gamma \vdash A \downarrow}{\Gamma \vdash A \uparrow}\ [\mathrm{M}] \qquad \frac{\Gamma \vdash A \uparrow}{\Gamma \vdash A \downarrow}\ [\mathrm{S}]$$

Natural deduction in the style of Sieg and Byrnes (*Studia Logica*, 1998).

A proof is *normal* if it does not contain the switch rule $[S]$.

$\mathcal{R}_1$ can also capture natural deduction

Let $\delta(\lfloor \cdot \rfloor) = +2$ and $\delta(\lceil \cdot \rceil) = -1$.

The $\uparrow$ and $\downarrow$ judgments are encoded as follows.

- $\Gamma \vdash C \uparrow$ is encoded using $\vdash \lfloor \Gamma \rfloor, \lceil C \rceil$.
- $\Gamma \vdash C \downarrow$ is encode using $\vdash \lfloor \Gamma \rfloor, \overline{\lfloor C \rfloor}$.

$\mathcal{R}_1$ can also capture natural deduction

Let $\delta(\lfloor \cdot \rfloor) = +2$ and $\delta(\lceil \cdot \rceil) = -1$.

The $\uparrow$ and $\downarrow$ judgments are encoded as follows.

- $\Gamma \vdash C \uparrow$ is encoded using $\vdash \lfloor \Gamma \rfloor, \lceil C \rceil$.
- $\Gamma \vdash C \downarrow$ is encode using $\vdash \lfloor \Gamma \rfloor, \overline{\lfloor C \rfloor}$.

Using *decide* on the R-formula $(\supset L)$ yields

$$\dfrac{\dfrac{\dfrac{\vdash \overline{\lfloor A \supset B \rfloor}, \Upsilon}{\Downarrow \lfloor A \supset B \rfloor \vdash \Upsilon}\ \mathit{debit}_2 \quad \dfrac{\vdash \lceil A \rceil, \Upsilon}{\vdash \lceil A \rceil \Downarrow \Upsilon}\ \mathit{release}}{\Downarrow \lfloor A \supset B \rfloor \mapsto \lceil A \rceil \vdash \Upsilon} \qquad \dfrac{}{\vdash \lfloor B \rfloor \Downarrow \overline{\lfloor B \rfloor}, \Upsilon}\ \mathit{initR}}{\dfrac{\Downarrow(\lfloor A \supset B \rfloor \mapsto \lceil A \rceil) \Mapsto \lfloor B \rfloor \vdash \overline{\lfloor B \rfloor}, \Upsilon}{\vdash \overline{\lfloor B \rfloor}, \Upsilon}\ \mathit{decide}}$$

$\mathcal{R}_1$ can also capture natural deduction

Let $\delta(\lfloor\cdot\rfloor) = +2$ and $\delta(\lceil\cdot\rceil) = -1$.

The $\uparrow$ and $\downarrow$ judgments are encoded as follows.

- $\Gamma \vdash C \uparrow$ is encoded using $\vdash \lfloor\Gamma\rfloor, \lceil C\rceil$.
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Using *decide* on the R-formula ($\supset L$) yields

$$\cfrac{\cfrac{\cfrac{\vdash \overline{\lfloor A \supset B\rfloor}, \Upsilon}{\Downarrow \lfloor A \supset B\rfloor \vdash \Upsilon}\ \mathit{debit}_2 \quad \cfrac{\vdash \lceil A\rceil, \Upsilon}{\vdash \lceil A\rceil \Downarrow \Upsilon}\ \mathit{release}}{\Downarrow \lfloor A \supset B\rfloor \mapsto \lceil A\rceil \vdash \Upsilon} \qquad \cfrac{}{\vdash \lfloor B\rfloor \Downarrow \overline{\lfloor B\rfloor}, \Upsilon}\ \mathit{initR}}{\cfrac{\Downarrow(\lfloor A \supset B\rfloor \mapsto \lceil A\rceil) \Mapsto \lfloor B\rfloor \vdash \overline{\lfloor B\rfloor}, \Upsilon}{\vdash \overline{\lfloor B\rfloor}, \Upsilon}\ \mathit{decide}}$$

This yields the synthetic rule, which encodes the $[\supset E]$ inference rule.

$$\frac{\vdash \overline{\lfloor A \supset B\rfloor}, \Upsilon \quad \vdash \lceil A\rceil, \Upsilon}{\vdash \overline{\lfloor B\rfloor}, \Upsilon}$$

The $[M]$ and $[S]$ rules

Deciding on (Id_1) and (Id_2), respectively, yields

$$\dfrac{\dfrac{\vdash \overline{\lfloor B\rfloor}, \Upsilon}{\Downarrow \lfloor B\rfloor \vdash \Upsilon}\ \mathit{debit}_2 \qquad \dfrac{}{\Downarrow \lceil B\rceil \vdash \lceil B\rceil, \Upsilon}\ \mathit{initL}}{\dfrac{\Downarrow \lfloor B\rfloor \times \lceil B\rceil \vdash \lceil B\rceil, \Upsilon}{\vdash \lceil B\rceil, \Upsilon}\ \mathit{decide}}$$

$$\dfrac{\dfrac{\dfrac{}{\Downarrow \mathbf{1} \vdash \Upsilon} \qquad \dfrac{\vdash \lceil B\rceil, \Upsilon}{\vdash \lceil B\rceil \Downarrow \Upsilon}\ \mathit{release}}{\Downarrow \mathbf{1} \mapsto \lceil B\rceil \vdash \Upsilon} \qquad \dfrac{}{\vdash \lfloor B\rfloor \Downarrow \overline{\lfloor B\rfloor}, \Upsilon}\ \mathit{initL}}{\dfrac{\Downarrow \mathbf{1} \mapsto \lceil B\rceil \Mapsto \lfloor B\rfloor \vdash \overline{\lfloor B\rfloor}, \Upsilon}{\vdash \overline{\lfloor B\rfloor}, \Upsilon}\ \mathit{decide}}$$

The $[M]$ and $[S]$ rules

Deciding on (Id_1) and (Id_2), respectively, yields

$$\dfrac{\dfrac{\dfrac{\vdash \overline{\lfloor B\rfloor}, \boldsymbol{\Upsilon}}{\Downarrow \lfloor B\rfloor \vdash \boldsymbol{\Upsilon}}\;\textit{debit}_2 \qquad \dfrac{}{\Downarrow \lceil B\rceil \vdash \lceil B\rceil, \boldsymbol{\Upsilon}}\;\textit{initL}}{\Downarrow \lfloor B\rfloor \times \lceil B\rceil \vdash \lceil B\rceil, \boldsymbol{\Upsilon}}}{\vdash \lceil B\rceil, \boldsymbol{\Upsilon}}\;\textit{decide}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Downarrow \mathbf{1} \vdash \boldsymbol{\Upsilon}} \qquad \dfrac{\vdash \lceil B\rceil, \boldsymbol{\Upsilon}}{\vdash \lceil B\rceil \Downarrow \boldsymbol{\Upsilon}}\;\textit{release}}{\Downarrow \mathbf{1} \mapsto \lceil B\rceil \vdash \boldsymbol{\Upsilon}} \qquad \dfrac{}{\vdash \lfloor B\rfloor \Downarrow \overline{\lfloor B\rfloor}, \boldsymbol{\Upsilon}}\;\textit{initL}}{\Downarrow \mathbf{1} \mapsto \lceil B\rceil \Mapsto \lfloor B\rfloor \vdash \overline{\lfloor B\rfloor}, \boldsymbol{\Upsilon}}}{\vdash \overline{\lfloor B\rfloor}, \boldsymbol{\Upsilon}}\;\textit{decide}$$

and these yield the two synthetic rules (encoding $[M]$ and $[S]$)

$$\dfrac{\vdash \overline{\lfloor B\rfloor}, \boldsymbol{\Upsilon}}{\vdash \lceil B\rceil, \boldsymbol{\Upsilon}} \qquad \text{and} \qquad \dfrac{\vdash \lceil B\rceil, \boldsymbol{\Upsilon}}{\vdash \overline{\lfloor B\rfloor}, \boldsymbol{\Upsilon}}\;.$$

The R-expression (Id_2) corresponds to cut in sequent calculus and to the switch $[S]$ in natural deduction.

Encoding natural deduction

Proposition

Assume that $\delta(\lceil \cdot \rceil) = -1$ *and* $\delta(\lfloor \cdot \rfloor) = +2$. *Then*

- $\Gamma \vdash C \uparrow$ *if and only if* $\vdash \lfloor \Gamma \rfloor, \lceil C \rceil$ *is provable using* $\mathcal{R}_1$, *and*
- $\Gamma \vdash C \downarrow$ *if and only if* $\vdash \lfloor \Gamma \rfloor, \overline{\lfloor C \rfloor}$ *is provable using* $\mathcal{R}_1$.

Normal proofs are captured by removing (Id_2) *from consideration.*

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Normal proofs are captured by removing (Id_2) *from consideration.*

The following rules can also be captured.

$$\frac{\Gamma \vdash A \vee B \downarrow \quad \Gamma, A \vdash C \uparrow(\downarrow) \quad \Gamma, B \vdash C \uparrow(\downarrow)}{\Gamma \vdash C \uparrow(\downarrow)} \; [\vee E]$$

$$\frac{\Gamma \vdash A_i \uparrow}{\Gamma \vdash A_1 \vee A_2 \uparrow} \; [\vee I]$$

Other proof systems and logics

The LICS 2023 paper discusses additional proof systems:

- Generalized elimination rules [Schroeder-Heister, 1984], [von Plato, 2001]
- Free deduction for classical logic [Parigot 1992]
- Sequent calculus for linear logic: uses four tags, not just the two used here.
- Quantificational logic

Future work

- Use **PSF** to help prove object-level results: eg, cut elimination, etc
- Accommodate a feature similar to *sub-exponentials* should permit capturing more proof systems.
  - Multi-conclusion proof systems for intuitionistic logic [Maehara, 1954]
  - **G1m**, **LJQ***, etc [Nigam, Pimentel and Reis, 2011].
- Consider higher-level rules

First-order quantification

Early logical frameworks (λProlog, LF) were notable for their treatment of quantification via *binder mobility*: term-level bindings move to formula-level bindings (quantifier) to proof-level bindings (eigenvariables).

- Add quantified expressions and rules: $\boldsymbol{Q}\, x.(E\, x)$ and $\boldsymbol{Q}\, x.(R\, x)$.
- Sequents are enriched: Σ binds over sequents: $\Sigma : \Gamma \vdash \Delta$ and $\Sigma : \Downarrow \Gamma \vdash \Delta$.
- Add two rules to $\mathbf{B}$ (in the first rule, $x \notin \Sigma$).

$$\frac{\Sigma, x : \Gamma \vdash E\, x, \Delta}{\Sigma : \Gamma \vdash \boldsymbol{Q}\, x.E\, x, \Delta} \qquad \frac{\Sigma : \Gamma, R\, t \vdash \Delta \quad t \text{ is a } \Sigma\text{-term}}{\Sigma : \Gamma, \boldsymbol{Q}\, x.R\, x \vdash \Delta}$$

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The rule $(\supset L)$ in $\mathcal{R}_1$ can be written more explicitly as

$$\boldsymbol{Q}\, A.\boldsymbol{Q}\, B.\lfloor A \supset B \rfloor \mapsto \lceil A \rceil \Rrightarrow \lfloor B \rfloor$$

We can now add the following to $\mathcal{R}_1$.

$$\begin{array}{rlcl}
\boldsymbol{Q}\, B.\boldsymbol{Q}\, t. & \lfloor \forall x.B\, x \rfloor & \mapsto & \lfloor B\, t \rfloor \\
\boldsymbol{Q}\, B. & \lceil \forall x.B\, x \rceil & \mapsto & \boldsymbol{Q}\, x.\lceil B\, x \rceil \\
\boldsymbol{Q}\, B. & \lfloor \exists x.B\, x \rfloor & \mapsto & \boldsymbol{Q}\, x.\lfloor B\, x \rfloor \\
\boldsymbol{Q}\, B.\boldsymbol{Q}\, t. & \lceil \exists x.B\, x \rceil & \mapsto & \lceil B\, t \rceil
\end{array}$$

Related work

- Logical frameworks in the 1980s and 1990s: Framework based on intuitionistic logic and typed λ-terms.
  - E.g., λProlog, LF.
- Frameworks based on linear logic (with subexponentials)
  - M, Pimentel, Nigam, and Reis et al. [1996-2014] have considered many proof systems and logic.
  - Sufficient (and decidable) conditions that ensure that a sequent calculus for a first-order logic has the cut-elimination property.
  - Various implementations have been developed.
- This paper grew out of the desire to supplant linear logic with something more basic and pre-logical.
- There are related approaches using algebraic and model-theoretic semantics as frameworks: e.g., A. Avron and I. Lev [IJCAR 2001].

Conclusion

$\mathbf{PSF}$ is a framework for specifying proof systems.

- It separates the semantics of inference rules into two parts:
  - the rule, i.e., $\lfloor A \supset B \rfloor \mapsto \lceil A \rceil \Rrightarrow \lfloor B \rfloor$
  - the bias assignment, i.e., values for $\delta(\lceil \cdot \rceil)$ and $\delta(\lfloor \cdot \rfloor)$.
- Inference rules in, say $\mathbf{NJ}$ and $\mathbf{LK}$, are identified as synthetic rules containing two phases of $\mathbf{PSF}$ rewriting steps.
- Many features shared with linear logic appear naturally.
  - Inference rules are characterized as multiplicative and additive.
  - The tagged formulas are either deletable or permanent.
  - Importance of contraction and weakening.
- Centrality of don't-know-nondeterminism and don't-care-nondeterminism
- In $\mathbf{PSF}$, cut-elimination is used to reason about the framework instead of specifying computation *à la* Curry-Howard correspondence.

Questions?

Art by Nadia Miller