Linear logic using negative connectives

Dale Miller
Inria Saclay & LIX Institut Polytechnique de Paris, France

Abstract

In linear logic, a connective's right-introduction rule is invertible if and only if its left-introduction rule is not. This fact suggests the following notion of *polarity*: a connective is *negative* if its right-introduction rule is invertible and *positive* otherwise. Negation inverts polarity. A two-sided sequent calculus for first-order linear logic with only negative connectives has an appealing proof theory. Proof search proceeds by alternating phases of invertible (right-introduction) rules and non-invertible (left-introduction) rules, corresponding to *goal-reduction* and *backchaining*, respectively. These phases are formalized by *multifocused proofs*, which illuminate differences between proofs in intuitionistic and linear logic. We decompose linear logic into three sublogics: $\mathcal{L}_0$ (first-order intuitionistic logic with conjunction, implication, and universal quantification); $\mathcal{L}_1$ (extending $\mathcal{L}_0$ with linear implication, retaining its intuitionistic character); and $\mathcal{L}_2$ (containing multiplicative falsity $\perp$, encompassing classical linear logic). Notably, the single-conclusion restriction on sequents does not need to be imposed (as Gentzen did) to define intuitionistic logic proofs since it arises as a feature of multifocused proofs of $\mathcal{L}_0$ and $\mathcal{L}_1$ sequents. While multifocused proofs of $\mathcal{L}_2$ sequents can contain parallel applications of left-introduction rules, proofs of $\mathcal{L}_0$ and $\mathcal{L}_1$ sequents cannot exploit such parallel rule application. This notion of parallelism in proofs facilitates a novel treatment of disjunctions and existential quantifiers in natural deduction for intuitionistic logic. The cut-elimination theorem for the focused proof system is proved in the appendix.

2012 ACM Subject Classification Theory of computation → Proof theory; Theory of computation → Linear logic

Keywords and phrases Linear logic, multifocused proofs, sequent calculus

Digital Object Identifier 10.4230/LIPIcs...

1 Introduction

The fact that an inference rules is invertible or not is an important property of a rule. Although Gentzen apparently did not consider this property of his inference rules [27], Ketonen recognized its significance shortly after Gentzen's work. In fact, Ketonen restructured Gentzen's LK calculus around invertible rules, and that enabled him to establish some decidability and independence results for classical provability [17, 18]. Maximizing the presence of invertible inference rules in a proof system is a goal of the popular G3 two-sided sequent calculus proof system [31].

One intriguing aspect of linear logic is that the right introduction of a connective is not invertible if and only if the right introduction of the dual connective is invertible. (Linear negation will not be a primitive logical connective here.) This fact suggests introducing a notion of *polarity*. Following Girard [12] and Andreoli [1], we say that a connective is negative if its right-introduction rule is invertible and positive otherwise. Given the observation above, De Morgan duality flips polarities. We shall say that a non-atomic formula is negative (positive) if its top-level connective is negative (resp., positive). In order to extend the notion of polarity to all formulas, we must assign a polarity to atomic formulas as well: while this can be done in a arbitrary way (see Section 7), we follow Andreoli [1] and assign all atomic formulas the negative polarity.

In linear logic, the logical connectives $\top, \&, \perp, ⅋, \forall, ?$ are negative and $\mathbf{0}, \oplus, \mathbf{1}, \otimes, \exists, !$ are positive. Here, we follow [23] in making two different choices in the selection of connectives in presenting linear logic. First we shall take the two implications $\multimap$ (linear implication) and

Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

$\Rightarrow$ (intuitionistic implication) as primitive. When $\Rightarrow$ is not a primitive, it is usually defined so that $A \Rightarrow B$ is $!\,A \multimap B$. As a result of using $\Rightarrow$, we will not take ! as a primitive. Since both of these implications have invertible right-introduction rules, they are both negative connectives. We choose also to work only with negative connectives which means that we need a two-sided sequent and we do not need to work explicitly with De Morgan duals.

In this paper, we develop the proof theory for full linear logic using only negative connectives. This involves slicing full linear logic into the following three classes of connectives:

- $\mathcal{L}_0$ captures the core of intuitionistic logic using the linear logic connectives $\{\top, \&, \Rightarrow, \forall\}$.
- $\mathcal{L}_1$ is $\mathcal{L}_0$ with $\multimap$ added and corresponds to linear intuitionistic logic.
- $\mathcal{L}_2$ is $\mathcal{L}_1$ with $\bot$ and $\parr$ added and is a complete set of connectives for linear logic.

Thus, $\mathcal{L}_0$ is $\{\top, \&, \Rightarrow, \forall\}$, $\mathcal{L}_1$ is $\mathcal{L}_0 \cup \{\multimap\}$, and $\mathcal{L}_2$ is $\mathcal{L}_1 \cup \{\bot, \parr\}$. For $i \in \{0, 1, 2\}$, we say that a formula is an $\mathcal{L}_i$-formula if all connectives occurring in it are from the set $\mathcal{L}_i$. In this paper, $\forall$ denotes a first-order quantifier only.

Considering proof systems that focus solely on negative polarity connectives is a common approach in the literature on intuitionistic logic. For instance, the connectives in $\mathcal{L}_0$ are primarily the ones discussed in the first half of Girard's textbook [13]. The positive connectives, such as disjunction, falsehood, and existential quantification, are only briefly mentioned in Chapter 10. Similarly, those studying the normalization procedure for natural deduction in Prawitz's work [28] will observe how straightforward the treatment of negative connectives is compared to the complexity involved in handling positive connectives.

As the following equivalences reveal, the set $\mathcal{L}_2$ is a complete set of connectives. (Here, the equivalence $A \equiv B$ is defined as the formula $(A \multimap B) \,\&\, (B \multimap A)$.)

$$
\begin{array}{lll}
\mathbf{0} \equiv \top \Rightarrow \bot & !\,B \equiv (B \Rightarrow \bot) \multimap \bot & B \oplus C \equiv ((B \multimap \bot) \,\&\, (C \multimap \bot)) \multimap \bot \\
\mathbf{1} \equiv \bot \multimap \bot & ?\,B \equiv (B \multimap \bot) \Rightarrow \bot. & B \otimes C \equiv (B \multimap \bot) \multimap (C \multimap \bot) \multimap \bot \\
 & & \exists x.B \equiv (\forall x.B \multimap \bot) \multimap \bot
\end{array}
$$

The set $\mathcal{L}_2$ is redundant since $B \parr C$ is equivalent to both $(B \multimap \bot) \multimap (C \multimap \bot) \multimap \bot$ and to $(B \multimap \bot) \multimap C$. We shall find it convenient to keep $\parr$ in $\Downarrow\mathcal{L}_2$, particularly when we discuss multiset rewriting in Section 3.1.

In many cases, when a positive connective is convenient to write a specification, they appear on the left of an implication. In such cases, the curry/uncurry equivalences can be employed (hence, avoiding the double-negation expressions above).

$$
\begin{array}{ll}
\mathbf{1} \multimap H \equiv H & (B \otimes C) \multimap H \equiv B \multimap C \multimap H \\
\mathbf{0} \multimap H \equiv \top & (B \oplus C) \multimap H \equiv (B \multimap H) \,\&\, (C \multimap H) \\
 & (\exists x.Bx) \multimap H \equiv \forall x.(Bx \multimap H)
\end{array}
$$

The main theoretical tools used in this paper are the $\Downarrow\mathcal{L}_2$ focused proof system and its extension $\Downarrow^{+}\mathcal{L}_2$ that includes (versions of) the cut rule. Let $i \in \{0, 1, 2\}$. A sequent is an $\mathcal{L}_i$ sequent if all formulas occurring in it are $\mathcal{L}_i$ formulas.

While this paper presents different ways to present several known results in structural proof theory, it also contains the following novelties.

1. $\Downarrow\mathcal{L}_2$ proofs of $\mathcal{L}_0$ and $\mathcal{L}_1$ formulas has the usual intuitionistic structure: i.e., they are necessarily single conclusion. Classical proof structure only appears once the $\bot$ and $\parr$ connectives are admitted.
2. As we shall see, parallel rule application is captured using *multifocusing.* As it turns out, multifocused proofs based on $\mathcal{L}_0$ and $\mathcal{L}_1$ formulas are, in fact, single-focused. As a result, such proofs do not permit the parallel application of rules. Non-single-focused proofs are possible with $\mathcal{L}_2$-sequents.

3. Our proof of cut elimination in $\Downarrow^{+}\mathcal{L}_2$ (Theorem 10) contains some novelty.
4. The admissibility of cut in $\Downarrow\mathcal{L}_2$ provides a new proof of the completeness of $\Downarrow\mathcal{L}_2$: earlier proofs relied on permutation arguments within cut-free proofs [2, 23].
5. We provide an improved treatment of disjunction and existential quantification within the LJT intuitionistic proof system of [14] and use it to motivate a parallel elimination for $\vee$ and $\exists$ in natural deduction proofs for intuitionistic logic.

It is well known that while cut elimination can be challenging to prove given the many cases that need to be considered, once it is proved many important results following immediately (witness the fact that Gentzen was able to prove the consistency of classical and intuitionistic logic using a one-line proof that invoked his *Hauptsatz*). We have placed the cut-elimination theorem for $\Downarrow\mathcal{L}_2$ in the appendix so that the bulk of this paper can focus on the number of consequences that follow rather simply from that result.

2 The focused proof systems $\Downarrow\mathcal{L}_2$ and $\Downarrow^{+}\mathcal{L}_2$

The inference rules in Figure 1 involve two kinds of sequents, namely, $\Sigma : \Psi; \Gamma \vdash \Delta$ and $\Sigma : \Psi; \Gamma \Downarrow \Theta \vdash \Theta' \Downarrow \Delta$. The *signature* of these sequents Σ is a binder of eigenvariables within the scope of the sequent. Any variable free in any formula occurring in any zone of the sequent must be explicitly bound (and typed) in Σ. The other components of these sequents—the *left unbounded zone* Ψ, the *left bounded zone* Γ, the *right bounded zone* Δ, the *left focused zone* Θ, and the *right focused zone* Θ'—are all multisets of formulas.

The $decide_m$ rule contains the two schema variables Ψ_2 and $\hat{\Psi}_2$: we require these two variables to be instantiated with multisets of formulas in such a way that every formula with a non-zero multiplicity in one of them also has a non-zero multiplicity (not necessarily equal) in the other. The $decide_m$ rule is also constrained so that the multiset union $\hat{\Psi}_2, \Gamma_2$ is non-empty. If we make no further restrictions on the $decide_m$ inference rule, we call the proof system in Figure 1 the *near-focused* proof system for $\mathcal{L}_2$. The $\Downarrow\mathcal{L}_2$ proof system is the result of requiring that the schema variable Δ in the $decide_m$ is a multiset of atomic formulas. Given that restriction on $decide_m$, it is then the case that all instances of the left phase rules are such that the right bounded zone contains only atomic formulas. Thus, in $\Downarrow\mathcal{L}_2$ proofs, the *init* rule takes place between two occurrences of the same atomic formula.

Although the following observation is not important for this paper since we are only interested in first-order quantification, an important property of near-focused proofs is that they are stable under higher-order substitution: that is, if one substitutes a predicate with a λ-term possibly containing logical connectives in a near-focused proof, the resulting instantiation will also be a near-focused proof. A similar statement is not true for $\Downarrow\mathcal{L}_2$ proofs since such substitutions can change an atomic formula into a non-atomic formula.

The $\Downarrow^{+}\mathcal{L}_2$ proof system is the result of adding the following two cut rules to $\Downarrow\mathcal{L}_2$.

$$\frac{\Sigma : \Psi; \cdot \vdash B \qquad \Sigma : \Psi, B; \Gamma \vdash \Delta}{\Sigma : \Psi; \Gamma \vdash \Delta}\ cut! \qquad \frac{\Sigma : \Psi; \Gamma_1 \vdash B, \Delta_1 \qquad \Sigma : \Psi; \Gamma_2, B \vdash \Delta_2}{\Sigma : \Psi; \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ cut_l$$

The formula B is the *cut-formula* in both of these rules. We say that a Σ-formula B has a proof in a given sequent calculus proof system if the sequent $\Sigma : \cdot; \cdot \vdash B$ has a proof.

The *site* of an inference rule is a set of occurrences of formulas in the conclusion of that rule defined as follows: (i) the site for an introduction rule contains just the formula occurrence being introduced, (ii) the site of an *init* rule contains the two formula occurrences labeled by B in Figure 1, and (iii) the site of the rules *release*, $decide_m$, $cut!$, and cut_l are all empty. An occurrence of a formula in the conclusion of an inference rule is a *side-formula occurrence* if it is not in the site of that rule. For example, all formula occurrences in the conclusion of *release*, $decide_m$, $cut\,!$, and cut_l are side formula occurrences. Side-formula occurrences can appear in any zone in the two different styles of sequents.

An essential feature of linear logic is its classification of logical connectives into *multiplicative*, *additive*, and *exponential*. The $\Downarrow\mathcal{L}_2$ proof system applies that classification to inference rules by how the rule treats side-formula occurrences. All inference rules treat side formula occurrences in the unbounded zone the same: formulas occurring in the unbounded zone of the conclusion occur in the unbounded left zone of every premise (this accounts for the exponential ! of linear logic). An inference is *additive* if every side formula occurrence in a bounded zone in the rule's conclusion has an occurrence in the same bounded zone in *every* premise of the rule. An inference is *multiplicative* if every side formula occurrence in a bounded zone in the rule's conclusion has an occurrence in *exactly one* premise and that occurrence is within the same kind of bounded zone. (Here, the left and right-focused zones are also considered to be bounded zones.) Note that all right phase rules are additive, all left rules are multiplicative, and all phase switching rules are additive and multiplicative.

Proofs in the $\Downarrow\mathcal{L}_2$ proof system are *multifocused* proofs. In the case that every occurrences of the $decide_m$ rule in a proof selects exactly one formula, we say that that proof is *single-focused*. Similarly, proofs in the $\Downarrow\mathcal{L}_2$ proof system are *multiple conclusion* proofs. In the case that every sequent in a proof has exactly one formula on its right-hand side, we say that that proof is a *single-conclusion* proof.

One reason to use the focused proof system $\Downarrow\mathcal{L}_2$ is that it provides a powerful normal form to sequent calculus proofs. Another reason is that phases can be viewed as derived inference rules and, as such, they can abstract away from the specific details in how they are constructed. A *border sequent* is a sequent of the form $\Sigma : \Psi; \Gamma \vdash \Delta$ where Δ is a multiset of atomic formulas. Above a border sequent is a $decide_m$ rule and above that is a left phase. Any open premise of the left phase are the conclusion of a *release* rule and above that is a right phase. Any open premises of this right phase (and those which are the left-premise of $\Rightarrow$L) must again be border sequents. Such a collection of inference rules that have border sequents as (open) premises, a border sequent as the conclusion, and exactly one instance of the $decide_m$ rule is called a *bipole*. The *synthetic rule* justified by such a bipole is the result of deleting all the internal inference rules of the left and right phases and simply maintaining the border sequents as premises and conclusion.

▶ **Example 1.** Let a, b, c be propositional constants and assume that Ψ contains the formula $a \multimap b \multimap c$. We have the following bipole and the synthetic inference rule it justifies.

$$\dfrac{\dfrac{\dfrac{\Sigma : \Psi; \Gamma_1 \vdash a, \Delta_1}{\Sigma : \Psi; \Gamma_1 \Downarrow \cdot \vdash a \Downarrow \Delta_1}\,\textit{release} \qquad \dfrac{\Sigma : \Psi; \Gamma_2 \vdash b, \Delta_2}{\Sigma : \Psi; \Gamma_2 \Downarrow \cdot \vdash b \Downarrow \Delta_2}\,\textit{release} \qquad \dfrac{}{\Sigma : \Psi; \cdot \Downarrow c \vdash \cdot \Downarrow c}\,\textit{init}}{\Sigma : \Psi; \Gamma_1, \Gamma_2 \Downarrow a \multimap b \multimap c \vdash \cdot \Downarrow c, \Delta_1, \Delta_2}\,\multimap \times 2}{\Sigma : \Psi; \Gamma_1, \Gamma_2 \vdash c, \Delta_1, \Delta_2}\,\textit{decide}_m$$

$$\dfrac{\Sigma : \Psi; \Gamma_1 \vdash a, \Delta_1 \qquad \Sigma : \Psi; \Gamma_2 \vdash b, \Delta_2}{\Sigma : \Psi; \Gamma_1, \Gamma_2 \vdash c, \Delta_1, \Delta_2}$$

If instead we assume that Ψ contains the formula $a \Rightarrow b \Rightarrow c$ then we have the following bipole and the synthetic inference rule it justifies.

$$\dfrac{\dfrac{\Sigma : \Psi; \cdot \vdash a, \cdot \quad \Sigma : \Psi; \cdot \vdash b \quad \dfrac{}{\Sigma : \Psi; \cdot \Downarrow c \vdash \cdot \Downarrow c}\; init}{\Sigma : \Psi; \cdot \Downarrow a \Rightarrow b \Rightarrow c \vdash \cdot \Downarrow c}\; \Rightarrow \times 2}{\Sigma : \Psi; \cdot \vdash c}\; decide_m \qquad \dfrac{\Sigma : \Psi; \cdot \vdash a \quad \Sigma : \Psi; \cdot \vdash b}{\Sigma : \Psi; \cdot \vdash c}$$

The following soundness theorem is straightforward to prove since every inference rule in $\Downarrow\mathcal{L}_2$ is derivable in linear logic: when translating the zoned sequents used in $\Downarrow\mathcal{L}_2$ to linear logic, simply place the exponential ! on all formulas in the unbounded zone and then replace the semicolon and the two occurrences of $\Downarrow$ with commas.

▶ **Theorem 2** (Soundness of $\Downarrow\mathcal{L}_2$ proofs). *If $\Sigma : \cdot; \cdot \vdash B$ has a $\Downarrow\mathcal{L}_2$ proof then B is a theorem of linear logic.*

2.1 Deriving $\Downarrow\mathcal{L}_0$ and $\Downarrow\mathcal{L}_1$ from $\Downarrow\mathcal{L}_2$

One of the important features of the $\Downarrow\mathcal{L}_2$ proof system for linear logic is that if we are interested in proving an $\mathcal{L}_1$ or an $\mathcal{L}_0$ formula, then various features of $\Downarrow\mathcal{L}_2$ proofs are not actually used and that proof system can be greatly simplified for proving such formulas. The following propositions will allows justify such simplifications of $\Downarrow\mathcal{L}_2$.

▶ **Lemma 3.** *There is no $\Downarrow\mathcal{L}_2$ proof of an $\mathcal{L}_1$ sequent with an empty right side.*

Proof. Assume that there is a $\Downarrow\mathcal{L}_2$ proof of a sequent with an empty right side and with only $\mathcal{L}_1$ formulas on the left side. Let Ξ be such a proof of minimal height. Consider the last inference rule of Ξ. This last inference rule cannot be right-introduction rule since these require a non-empty right side. Similarly, the last rule is not *decide*$_m$ since that would yield a premise with an empty right side with a shorter proof. Thus, the endsequent of Ξ must be of the form $\Sigma : \Psi; \Gamma \Downarrow \Theta \vdash \cdot \Downarrow \cdot$ where Ψ and Γ are multisets of $\mathcal{L}_1$ formulas over Σ. However, a check of all possible left-introduction rules ($\bot L$ and $\parr L$ are not possible) and the release rule yields at least one premise with an empty right side and shorter proof. This contradicts the choice of Ξ. ◀

▶ **Proposition 4.** *If Ξ is a $\Downarrow\mathcal{L}_2$ proof of a single-conclusion $\mathcal{L}_1$-sequent then Ξ is a single-conclusion proof.*

Proof. We proceed by induction on the structure of the $\Downarrow\mathcal{L}_2$ proof Ξ. By considering all the possible last inference rules of Ξ, we need to show that a single conclusion sequent in the conclusion will guarantee that all premises are also single conclusion: the inductive hypothesis then completes the proof. The only case that is not immediate is the case for the $\multimap L$ rule, namely,

$$\dfrac{\Sigma : \Psi; \Gamma_1 \Downarrow \Theta_1 \vdash \Theta_3, B \Downarrow \Delta_1 \quad \Sigma : \Psi; \Gamma_2 \Downarrow C, \Theta_2 \vdash \Theta_4 \Downarrow \Delta_2}{\Sigma : \Psi; \Gamma_1, \Gamma_2 \Downarrow B \multimap C, \Theta_1, \Theta_2 \vdash \Theta_3, \Theta_4 \Downarrow \Delta_1, \Delta_2}\; \multimap\mathrm{L}$$

and where $\Theta_3 \cup \Theta_4 \cup \Delta_1 \cup \Delta_2$ is a singleton multiset. By Lemma 3, we know that $\Theta_4 \cup \Delta_2$ is not empty. As a result, $\Theta_3 \cup \Delta_1$ must be empty. Thus, both premises of this inference rule are single-conclusion sequents. ◀

▶ **Proposition 5.** *If Ξ is a $\Downarrow\mathcal{L}_2$ proof of a single-conclusion $\mathcal{L}_1$ sequent then Ξ is single focused.*

Proof. Assume that there is a $\Downarrow\mathcal{L}_2$ proof of a $\mathcal{L}_1$ sequent that is not single-focused, and let Ξ be chosen as such a proof of minimal height. The endsequent of Ξ must be a $\Downarrow$-sequent with the focused zones containing at least two formulas. Consider the last inference rule in Ξ. That rule is not *init*. By Proposition 4, it is not *release*. Because of the minimality assumption, that rule is not $\forall L$, $\& L$, or $\Rightarrow L$. The only remaining case is when that rule is $\multimap L$. Thus, the last inference figure in Ξ is of the form

$$\frac{\Sigma : \Psi; \Gamma_1 \Downarrow \Theta_1 \vdash \Theta_3, B \Downarrow \Delta_1 \quad \Sigma : \Psi; \Gamma_2 \Downarrow C, \Theta_2 \vdash \Theta_4 \Downarrow \Delta_2}{\Sigma : \Psi; \Gamma_1, \Gamma_2 \Downarrow B \multimap C, \Theta_1, \Theta_2 \vdash \Theta_3, \Theta_4 \Downarrow \Delta_1, \Delta_2} \multimap\mathrm{L},$$

where at least one of the multisets $\Theta_1, \ldots, \Theta_4$ must be none empty. Thus, one of the premises must have a focused zone with two or more members, which contradicts the minimal height assumption about Ξ. ◀

Consider a $\Downarrow\mathcal{L}_2$ proof Ξ of an $\mathcal{L}_1$ Σ-formula B, that is, Ξ is a proof of the sequent $\Sigma : \cdot; \cdot \vdash B$. By Proposition 4, all sequents in Ξ are single conclusion and by Proposition 5, every $\Downarrow$ sequent has a focus zone (combining the left and right part) containing exactly one formula. The proof system in Figure 2 can describe all such proofs: this proof system arises from $\Downarrow\mathcal{L}_2$ by taking the following steps.

- Delete the inference rules that introduce $\perp$ and $⅋$.
- Simplify all sequents to have only one formula on the right side.
- Modify the decide rule to select exactly one formula by splitting it into *decide$_l$* (to select a formula from the left bound zone) and *decide*! (to select a formula from the left unbound zone).
- Drop the *release* rule since it can be merged into the left-premise of $\multimap L$. As a result, all $\Downarrow$ sequents no longer need their right focused zone.

The resulting of removing such aspects of $\Downarrow\mathcal{L}_2$ proofs yields the $\Downarrow\mathcal{L}_1$ proof system in Figure 2. A consequence of these propositions is if B is a $\mathcal{L}_1$ formula then there is an a $\Downarrow\mathcal{L}_2$ proof of B if and only if there is a $\Downarrow\mathcal{L}_1$ proof of B. Thus, in a proof of an $\mathcal{L}_1$ proof, the multiple conclusion sequents and multiple foci do not arise. Note that in other presentations of proof systems for intuitionistic (linear) logic, the use of single conclusion sequents is a requirement [11, 29, 32], while in our setting, is a consequent of the choice of connectives we have selected.

If we now turn our attention to proofs of $\mathcal{L}_0$ formulas, we find that an additional feature of $\Downarrow\mathcal{L}_1$ and $\Downarrow\mathcal{L}_2$ proofs is not needed.

▶ **Proposition 6.** *If B is a $\mathcal{L}_0$ Σ-formula and Ξ is a $\Downarrow\mathcal{L}_2$-proof of $\Sigma : \cdot; \cdot \vdash B$, then Ξ is a single-focused and single-conclusion proof in which all left bounded zones are empty.*

Proof. Let B be a $\mathcal{L}_0$ Σ-formula, and let Ξ be a $\Downarrow\mathcal{L}_2$-proof of $\Sigma : \cdot; \cdot \vdash B$. By the two preceding propositions, Ξ is easily be seen as an $\Downarrow\mathcal{L}_1$ proof. An easy induction on the structure of such proofs reveals that if B does not contain $\multimap$, then the left-bounded zone for all sequents in Ξ is empty. ◀

This proposition justifies introducing the $\Downarrow\mathcal{L}_0$ proof system in Figure 2, where the inference rules introducing $\multimap$ are dropped, and the left bounded zone is removed (since it will always be empty). The $\Downarrow\mathcal{L}_0$ proof system is also known as LJT [14, 15] and as uniform proofs with backchaining [24]. We will return to $\Downarrow\mathcal{L}_0$ when we discuss the LJT$^-$ proof system in Section 5.

It is worth noting here that while $\Downarrow\mathcal{L}_2$ is a multiple-conclusion proof system, both $\Downarrow\mathcal{L}_0$ and $\Downarrow\mathcal{L}_1$ are single-conclusion proof systems. This characteristic of $\Downarrow\mathcal{L}_0$ and $\Downarrow\mathcal{L}_1$ is not an imposition on the more general multiple-conclusion proof system (as Gentzen needed to impose on LK to get the LJ proof system [11]) but rather, it is simply a consequence of using fewer logical connectives.

2.2 Paths in formulas

Let the relationship $\cdot \uparrow \cdot$ on $\mathcal{L}_2$-formulas be defined as follows (here, A ranges over atomic formulas).

$$\frac{}{A \uparrow A} \quad \frac{B_1 \uparrow P}{B_1 \,\&\, B_2 \uparrow P} \quad \frac{B_2 \uparrow P}{B_1 \,\&\, B_2 \uparrow P} \quad \frac{B \uparrow P}{\forall_\tau x.B \uparrow \forall_\tau x.P}$$

$$\frac{B \uparrow P}{C \Rightarrow B \uparrow C \Rightarrow P} \quad \frac{B \uparrow P}{C \multimap B \uparrow C \multimap P} \quad \frac{}{\bot \uparrow \bot} \quad \frac{B_1 \uparrow P_1 \quad B_2 \uparrow P_2}{B_1 \parr B_2 \uparrow P_1 \parr P_2}$$

It is easy to prove $B \equiv \&_{B \uparrow P} P$ by using the following distributivity properties and quantifier movement rules:

$$\begin{array}{cc} C \multimap (B_1 \,\&\, B_2) \equiv (C \multimap B_1) \,\&\, (C \multimap B_2) & C \parr (B_1 \,\&\, B_2) \equiv (C \parr B_1) \,\&\, (C \parr B_2) \\ C \Rightarrow (B_1 \,\&\, B_2) \equiv (C \Rightarrow C_2) \,\&\, (B \Rightarrow B_2) & \forall x.\ (B_1 \,\&\, B_2) \equiv (\forall x.\ B_1) \,\&\, (\forall x.\ B_2) \end{array}$$

Paths have a reasonably simple normal form. Using the equivalences

$$B \parr \forall x.C \equiv \forall x.(B \parr C),\ B \multimap \forall x.C \equiv \forall x.(B \multimap C),\ \text{and } B \Rightarrow \forall x.C \equiv \forall x.(B \Rightarrow C),$$

a path can be written in the form $\forall x_1 \ldots \forall x_n.P'$ where $n \geq 0$, and every occurrence of $\forall$ in P' occurs to the left of either $\multimap$ or $\Rightarrow$. Similarly, using the equivalences

$$\begin{array}{ll} (B \multimap C_1) \parr C_2 \equiv B \multimap (C_1 \parr C_2), & B \multimap C \Rightarrow D \equiv C \Rightarrow B \multimap D, \\ (B \Rightarrow C_1) \parr C_2 \equiv B \Rightarrow (C_1 \parr C_2), & \bot \parr B \equiv B \parr \bot \equiv B \end{array}$$

and the commutativity of $\parr$, paths can be put into the normal form

$$\forall \bar{x}[C_1 \Rightarrow \ldots \Rightarrow C_n \Rightarrow B_1 \multimap \ldots \multimap B_m \multimap A_1 \parr \ldots \parr A_p],$$

where $\forall \bar{x}$ is a list of universal quantifiers, n, m, p are non-negative integers, $A_1, \ldots, A_p$ are atomic formulas, and $B_1, \ldots, B_m, C_1, \ldots, C_n$ are $\mathcal{L}_2$ formulas. If a path P has the normal form above, then we say that the multiset $\{C_1, \ldots, C_n\}$ is its *intuitionistic arguments*, the multiset $\{B_1, \ldots, B_m\}$ is its *linear arguments*, and the multiset $\{A_1, \ldots, A_p\}$ is its *targets*. Finally, $\bar{x}$ is the list of *bound variables* of P (we assume that all these bound variables are distinct and subject to α-conversion). Since these various components of the normal form of a path are multisets, this decomposition of a path is unique. We shall also display this normal form as the *associated* sequent $\bar{x} : C_1, \ldots, C_n; B_1, \ldots, B_m \vdash A_1, \ldots, A_p$. Paths can be used to describe both left and right phases in a more abstract setting than by appealing to introduction rules.

▶ **Proposition 7.** *Consider a $\Downarrow\mathcal{L}_2$-proof Ξ of the sequent $\Sigma : \Psi; \Gamma \vdash G, \Delta$. There is a $\Downarrow\mathcal{L}_2$-proof Ξ' of this same sequent that differs only in permutations of right-introduction* *rules such that the formula G is decomposed first. More specially, that right-introduction phase can be written as*

$$\dfrac{\left(\begin{array}{c}\Xi_i\\ \Sigma,\Sigma_i:\Psi,\Psi_i;\Gamma,\Gamma_i\vdash\mathcal{A}_i,\Delta\end{array}\right)_{G\uparrow P_i}}{\Sigma:\Psi;\Gamma\vdash G,\Delta}\ ,$$

where the path P_i is associated with the sequent $\Sigma_i:\Psi_i;\Gamma_i\vdash\mathcal{A}_i$ and where Ξ_i is the right phase of the i^{th} premise.

Concerning left phases in single-focused proofs with endsequent $\Sigma:\Psi;\Gamma\Downarrow B\vdash\cdot\Downarrow\mathcal{A}$ we note that in every left rule application, the signature and the left unbounded zone in the conclusion is the same in in every premise.

▶ **Proposition 8.** *Let Ξ be a $\Downarrow\mathcal{L}_2$-proof of the sequent $\Sigma{:}\Psi;\Gamma\Downarrow B\vdash\cdot\Downarrow\mathcal{A}$. The left-introduction phase at the bottom of Ξ, which has a multiset of premises $\mathcal{M}$, can be described as followings. There is a path P in B with the associated sequent $\Sigma':C_1,\ldots,C_n;B_1,\ldots,B_m\vdash A_1,\ldots,A_p$ and there is a substitution θ that maps the variables in Σ' to Σ-terms such that*

1. *$\mathcal{A}$ is equal to the multiset union $\{A_1\theta,\ldots,A_p\theta\}\cup\mathcal{A}_1\cup\cdots\cup\mathcal{A}_m$;*
2. *Γ is the multiset union $\Gamma_1\cup\cdots\cup\Gamma_m$; and*
3. *$\mathcal{P}$ is the multiset union $\{\Sigma:\Psi;\cdot\vdash C_i\theta\}_{i=1}^n\cup\{\Sigma:\Psi;\Gamma_i\vdash B_i\theta,\mathcal{A}_i\}_{i=1}^m$.*

This use of paths to characterize the two focusing phases can be seen as a generalization of the use of *game moves* in [25] and *patterns* in [34].

2.3 Cut elimination and completeness for $\Downarrow\mathcal{L}_2$

The one method to proving the (relative) completeness of $\Downarrow\mathcal{L}_2$ is to first prove that the general form of the initial rule and the cut rule are admissible. These two admissibility results are more formally stated as the following two theorems. Their proofs can be found in Appendices A.1 and A.2, respectively.

▶ **Theorem 9** (Admissibility of the generalized initial rule). *Let B be an $\mathcal{L}_2$ Σ-formula. The sequent $\Sigma:\cdot;B\vdash B$ has a $\Downarrow\mathcal{L}_2$ proof.*

▶ **Theorem 10** (Cut elimination for $\Downarrow^+\mathcal{L}_2$). *Let B be an $\mathcal{L}_2$ Σ-formula. If the sequent $\Sigma{:}\cdot;\cdot\vdash B$ has an $\Downarrow^+\mathcal{L}_2$ proof then it has a $\Downarrow\mathcal{L}_2$ proof.*

Below, we highlight the main novelty of our cut-elimination proof. We first introduce the following *key cut* inference rule.

$$\frac{\Sigma:\Psi;\Gamma_1\vdash B,\Delta\qquad\Sigma:\Psi;\Gamma_2\Downarrow B\vdash\cdot\Downarrow\mathcal{A}}{\Sigma:\Psi;\Gamma_1,\Gamma_2\vdash\Delta,\mathcal{A}}\ cut_k$$

When we allow this inference rule within a focused proof, we know that the right premise is proved by using a left-phase rule on B, while the left premise is proved by a right-introduction rule (and via permutation of right-introduction rules) on B.

Consider the following instance of $cut!$ in a single-focused proof Ξ.

$$\frac{\begin{array}{c}\Xi_l\\ \Sigma:\Psi;\cdot\vdash B\end{array}\qquad\begin{array}{c}\Xi_r\\ \Sigma:\Psi,B;\Gamma\vdash\Delta\end{array}}{\Sigma:\Psi;\Gamma\vdash\Delta}\ cut!$$

Consider also a subderivation of Ξ_r that ends in $decide_m$, such as

$$\dfrac{\begin{array}{c}\Xi_0\\ \Sigma, \Sigma' : \Psi, \Psi', B; \Gamma' \Downarrow B \vdash \cdot \Downarrow \mathcal{A}\end{array}}{\Sigma, \Sigma' : \Psi, \Psi', B; \Gamma' \vdash \mathcal{A}}\ decide_m,$$

where the variables bound in Σ' are not bound in Σ and where Ψ' and Γ' are multisets. This inference rule can be converted to the derivation

$$\dfrac{\begin{array}{c}\hat{\Xi}_l\\ \Sigma, \Sigma' : \Psi, \Psi'; \cdot \vdash B\end{array} \qquad \begin{array}{c}\Xi_0\\ \Sigma, \Sigma' : \Psi, \Psi', B; \Gamma' \Downarrow B \vdash \cdot \Downarrow \mathcal{A}\end{array}}{\Sigma, \Sigma' : \Psi, \Psi', B; \Gamma' \vdash \mathcal{A}}\ cut_k.$$

Here, $\hat{\Xi}_l$ is the result of weakening Ξ_l (using Proposition 20 in Appendix A.2). We can thus remove all occurrences of $decide_m$ on B in Ξ_r to obtain the proof Ξ'_r of $\Sigma : \Psi, B; \Gamma \vdash \Delta$. Since B is no longer used in this subproof, it can be strengthened (using Proposition 22 in Appendix A.2) to get a proof of $\Sigma : \Psi; \Gamma \vdash \Delta$. This proof can now replace our original redex. Similarly, an occurrence of cut_l can be used to rewrite instances of $decide_m$ into a key cut. The argument for eliminating key cuts follows the usual pattern of matching a left-introduction rule with a right-introduction.

One can draw some analogies between the proof theory of $\Downarrow\mathcal{L}_2$ and the meta-theory of typed λ-calculi. This connection is well developed for the $\Downarrow\mathcal{L}_0$ calculus (see Section 5). More generally, Theorems 9 and 10 are closely related to η-expansion and β-reduction in type λ-calculi, and Theorem 10 corresponds to a *weak normalization* theorem.

The completeness of $\Downarrow\mathcal{L}_2$ proofs for linear logic is now a simple consequent of this cut-elimination theorem since it is possible to prove all the rules in (an unfocused) proof system for linear logic are admissible in $\Downarrow^+\mathcal{L}_2$.

▶ **Theorem 11** (Completeness of $\Downarrow\mathcal{L}_2$). *Let B be an $\mathcal{L}_2$ Σ-formula provable in linear logic. The sequent $\Sigma : \cdot; \cdot \vdash B$ has a $\Downarrow\mathcal{L}_2$-proof.*

Several completeness theorems exist for focused proof systems. The first such theorem, stated by Andreoli [1], transformed cut-free proofs into focused proofs via permutation of inference rules. The completeness of $\Downarrow\mathcal{L}_2$ was proved in [23] by mapping the logical formulas and focused proofs used by Andreoli to those in $\Downarrow\mathcal{L}_2$. An alternative proof, based directly on phases rather than introduction rules, was given by Bruscoli and Guglielmi [2]. Other completeness proofs leveraged the cut rule and cut elimination, rather than the direct manipulation of cut-free proofs. Examples of this approach can be found in [5, 19, 30, 34] for various fragments of linear and intuitionistic logic, and in [21] for classical logic. Theorem 11, which relies on Theorems 9 and 10, falls into this latter category.

3 Parallel rule application within proofs

We now provide a few examples of $\Downarrow\mathcal{L}_2$ proofs in order to illustrate the ability of multifocused proofs to capture a notion of parallel rule application.

3.1 Multiset rewriting

An important class of examples supported by linear logic are those involved with multiset rewriting. Let H be the *multiset rewriting system* $\{\langle L_i, R_i\rangle \mid i \in I\}$ where for each $i \in I$ (a finite index set), L_i and R_i are finite multisets of atomic formulas. Define the relation

$M \Longrightarrow_H N$ on finite multisets to hold if there is some $i \in I$ and some multiset C such that M is $C \uplus L_i$ and N is $C \uplus R_i$. (Here, $\uplus$ is multiset union.) Let $\Longrightarrow_H^*$ be the reflexive and transitive closure of $\Longrightarrow_H$.

Given a multiset rewriting system H, we can encode the relation $\Longrightarrow_H$ easily into linear logic using one of two schemes. The first scheme employs the left-bounded multiset context of $\Downarrow\mathcal{L}_2$ sequents. In this setting, select a new propositional constant, say q, and encode the pair $\langle\{a_1,\dots,a_m\},\{b_1,\dots,b_n\}\rangle \in H$ as the formula $(b_1 \multimap \cdots \multimap b_n \multimap q) \multimap a_1 \multimap \cdots \multimap a_m \multimap q$.

▶ **Example 12.** Consider the multiset rewriting system $\{\langle\{a,b\},\{c\}\rangle,\langle\{d\},\{e\}\rangle\}$. Let Σ be the signature that declares that a, b, c, d, q are atomic formulas and let Ψ be the formulas $\{(c \multimap q) \multimap a \multimap b \multimap q, (e \multimap q) \multimap d \multimap q\}$. The following partial proof illustrates how these formulas can encode multiset rewriting of the left-bound context.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Sigma:\Psi;c,d,\Gamma\vdash q}{\Sigma:\Psi;d,\Gamma\vdash c\multimap q}}{\Sigma:\Psi;d,\Gamma\Downarrow\cdot\vdash c\multimap q\Downarrow\cdot}\quad \dfrac{\dfrac{\dfrac{}{\Sigma:\Psi;\cdot\Downarrow a\vdash\cdot\Downarrow a}}{\Sigma:\Psi;a\vdash a}}{\Sigma:\Psi;a\Downarrow\cdot\vdash a\Downarrow\cdot}\quad \dfrac{\dfrac{\dfrac{}{\Sigma:\Psi;\cdot\Downarrow b\vdash\cdot\Downarrow b}}{\Sigma:\Psi;b\vdash b}}{\Sigma:\Psi;b\Downarrow\cdot\vdash b\Downarrow\cdot}\quad \dfrac{}{\Sigma:\Psi;\cdot\Downarrow q\vdash\cdot\Downarrow q}}{\Sigma:\Psi;a,b,d,\Gamma\Downarrow(c\multimap q)\multimap a\multimap b\multimap q\vdash\cdot\Downarrow q}}{\Sigma:\Psi;a,b,d,\Gamma\vdash q}\ \mathit{decide}_m$$

Thus, there is a path in this proof that moves from the sequent $\Sigma:\Psi;a,b,d,\Gamma\vdash q$ to $\Sigma:\Psi;c,d,\Gamma\vdash q$ by the application of the rule given by the pair $\langle\{a,b\},\{c\}\rangle$. However, this partial proof is not a bipole; it is comprised of three bipoles.

From the proof-theoretic setting, there are at least three problems with this way of encoding multiset rewriting. First, an extraneous propositional constant q is required (in order to fill in the right-hand side of the context. Second, the core action in multiset rewriting (the rewrite step) does not correspond precisely to the core action in a focused proof system, namely, a bipole. Third, the parallel application of rewriting steps in multisets—a desirable feature of multiset rewriting—is also not captured in this setting: in principle, the extraneous constant q serves as a lock on the rewriting step (compare with Example 14).

A second approach to encoding multiset rewriting performs the rewriting within the right-bounded multiset of sequents. In particular, we can encode $\langle\{a_1,\dots,a_m\},\{b_1,\dots,b_n\}\rangle \in H$ as the formula $(b_1 \parr \cdots \parr b_n) \multimap a_1 \parr \cdots \parr a_m$.

▶ **Example 13.** Assume that a,b,c,d,e are atomic formulas and that the two formulas $c \multimap a \parr b$ and $e \multimap d$ are members of Ψ. Consider the following derivation.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Sigma:\Psi;\Delta\vdash c,e,\Gamma}{\Sigma:\Psi;\Delta\Downarrow\cdot\vdash c,e\Downarrow\Gamma}\ \mathit{release}\quad \dfrac{}{\Sigma:\Psi;\cdot\Downarrow d\vdash\cdot\Downarrow d,\Gamma}\ \mathit{init}}{\Sigma:\Psi;\Delta\Downarrow e\multimap d\vdash c\Downarrow d,\Gamma}\ {\multimap}L\quad \dfrac{}{\Sigma:\Psi;\cdot\Downarrow a\parr b\vdash\cdot\Downarrow a,b}\ {\parr}L,\mathit{init}}{\Sigma:\Psi;\Delta\Downarrow c\multimap a\parr b, e\multimap d\vdash\cdot\Downarrow a,b,d,\Gamma}\ {\multimap}L}{\Sigma:\Psi;\Delta\vdash a,b,d,\Gamma}\ \mathit{decide}_m$$

This bipole, which encodes the *parallel composition* of two rewriting steps, corresponds to the following synthetic inference rule. $\dfrac{\Sigma:\Psi;\Gamma\vdash c,e,\Delta}{\Sigma:\Psi;\Gamma\vdash a,b,d,\Delta}$

▶ **Example 14.** Assume that a,b,c,d,l are atomic formulas and that the two formulas $b \parr l \multimap a \parr l$ and $d \parr l \multimap c \parr l$ are members of Ψ. The atomic formula l serves as a kind of lock, and this lock makes it impossible for there to be a parallel application of these two rules unless there are two occurrences of the lock. The following is a synthetic inference rule

$$\frac{\Sigma : \Psi; \Gamma \vdash b, d, l, l, \Delta}{\Sigma : \Psi; \Gamma \vdash a, c, l, l, \Delta}$$

while the following is not a synthetic rule (assuming that Δ does not contain l).

$$\frac{\Sigma : \Psi; \Gamma \vdash b, d, l, \Delta}{\Sigma : \Psi; \Gamma \vdash a, c, l, \Delta}$$

The Lolli logic programming language [16] is based on the logic $\mathcal{L}_1$ and the only form of multiset rewriting it provided followed the indirect style described in Example 12. The Forum [23] extension to Lolli is based on $\mathcal{L}_2$ and it can encode multiset rewriting in the more direct style of Example 13, although it did not provide for parallel rewriting steps since it was described using a single-focused proof system. The LolliMon logic programming language [22] and the Concurrent LF [33] extended $\mathcal{L}_1$ by allowing some occurrences of the positive linear logic connectives $\mathbf{1}$, $\otimes$, $!$, and $\exists$ and positively polarized atomic formulas. In that system, a direct form of multiset rewriting was also possible using the multiset encoded in the left bounded zone. The Concurrent LF did not permit multifocusing, but it did provide an equality theory within its dependently-type setting that could equate different non-overlapping rewrites occurring in different order.

3.2 Multifocusing as parallel rule application

Two notable aspects of the $\Downarrow\mathcal{L}_2$ proof system makes it possible to deal with parallel rule application within a sequent calculus setting.

First, focusing makes it possible to hide the sequential nature of the construction of synthetic inference rules. The order in which left introduction rules are applied within a multifocused proof is irrelevant since every order leads to the same result. The same applies to the order in which right-introduction rules are applied in multiple-conclusion proofs. Thus, the reliance on phases and synthetic rules means that the particular details of how a phase is constructed are hidden away.

Second, the $\Downarrow\mathcal{L}_2$ proof system contains a subtle feature: namely, the interaction between the right-hand zone between the $\vdash$ and the $\Downarrow$ and the use of the *release* rule to merged that zone with the rest of the right-hand context. Consider modifying sequents so that the right-hand zone between $\vdash$ and $\Downarrow$ is removed and rewriting the $\multimap$L inference rule as

$$\frac{\Sigma : \Psi; \Gamma_1 \Downarrow \Theta_1 \vdash B, \Delta_1 \quad \Sigma : \Psi; \Gamma_2 \Downarrow C, \Theta_2 \vdash \Delta_2}{\Sigma : \Psi; \Gamma_1, \Gamma_2 \Downarrow B \multimap C, \Theta_1, \Theta_2 \vdash \Delta_1, \Delta_2} \multimap\mathrm{L}^*.$$

The synthetic inference rules that we argued in Example 14 that should not exist can now be constructed with the rule $\multimap\mathrm{L}^*$.

Multifocusing sequent calculus proofs have been used to capture parallelism within well-known proof structures: in particular, *maximally multifocused* proofs have been used to capture the parallel rule applications that occur within expansion proofs [3] and proof nets [4]. To the extent that we are using multifocusing to capture parallel rule application, a $\Downarrow\mathcal{L}_2$ proof of a sequent that does not mention $\perp$ and $\parr$ will not exhibit this kind of parallelism.

4 Linear negation in proofs

The multiplicative false $\perp$ separates the intuitionistic frameworks $\Downarrow\mathcal{L}_0$ and $\Downarrow\mathcal{L}_1$, where proofs are single-conclusion and single-focused, from the full linear logic framework $\Downarrow\mathcal{L}_2$. (As we mentioned earlier, $\parr$ can be defined using $\mathcal{L}_1$ and $\perp$.) Once $\perp$ is present, it is natural to deal with the notion of linear negation, which can be encoded in $\Downarrow\mathcal{L}_2$ using the "implies false” construction. In the following pairs of sequents, the first sequent has a $\Downarrow\mathcal{L}_2$ proof if and only if the second also has a $\Downarrow\mathcal{L}_2$ proof.

$$\begin{array}{ccc} \Sigma : \Psi; \Gamma, B \multimap \bot \vdash \Delta & \dashv\vdash & \Sigma : \Psi; \Gamma \vdash B, \Delta \\ \Sigma : \Psi, (B \Rightarrow \bot) \multimap \bot; \Gamma \vdash \Delta & \dashv\vdash & \Sigma : \Psi; \Gamma, (B \Rightarrow \bot) \multimap \bot \vdash \Delta \\ \Sigma : \Psi, B; \Gamma \vdash \Delta & \dashv\vdash & \Sigma : \Psi; \Gamma, (B \Rightarrow \bot) \multimap \bot \vdash \Delta \end{array}$$

We define the *delay* operator $\partial(B)$ to be $(B \multimap \bot) \multimap \bot$. While B and $\partial(B)$ are provably equivalent, their roles within $\Downarrow\mathcal{L}_2$ proofs can differ. Consider the following derivation.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\Sigma : \Psi, \partial(B); \Gamma, B \vdash \Delta}{\Sigma : \Psi, \partial(B); \Gamma, B \vdash \bot, \Delta}}{\Sigma : \Psi, \partial(B); \Gamma \vdash B \multimap \bot, \Delta}}{\Sigma : \Psi, \partial(B); \Gamma \Downarrow \cdot \vdash B \multimap \bot \Downarrow \Delta} \quad \cfrac{}{\Sigma : \Psi, \partial(B); \cdot \Downarrow \bot \vdash \cdot \Downarrow \cdot}}{\Sigma : \Psi, \partial(B); \Gamma \Downarrow \partial(B) \vdash \cdot \Downarrow \Delta}}{\Sigma : \Psi, \partial(B); \Gamma \vdash \Delta}$$

Thus, the following is an admissible rule

$$\frac{\Sigma : \Psi, \partial(B); \Gamma, B \vdash \Delta}{\Sigma : \Psi, \partial(B); \Gamma \vdash \Delta}, \qquad \text{although we do not generally have the rule} \qquad \frac{\Sigma : \Psi, B; \Gamma, B \vdash \Delta}{\Sigma : \Psi, B; \Gamma \vdash \Delta}.$$

This latter rule is a form of contraction that is not immediately associated with focusing (as is the case with the *decide*$_m$ rule).

5 The LJT$^{\pm}$ proof system for intuitionistic logic

Let Neg be the negative intuitionistic connectives $\{\mathsf{t}, \wedge, \supset, \forall\}$ and let Pos be the positive intuitionistic connectives $\{\mathsf{f}, \vee, \exists\}$.

We map intuitionistic logic formulas over the connectives in Neg to formulas in linear logic connectives using the following obvious translation: $A^\circ = A$ for atomic formulas and

$$\mathsf{t}^\circ = \top, \quad (B \wedge C)^\circ = B^\circ \mathbin{\&} C^\circ, \quad (B \supset C)^\circ = B^\circ \Rightarrow C^\circ, \quad (\forall x.B)^\circ = \forall x.B^\circ$$

Let LJT$^-$ be the proof system in Figure 4 for intuitionistic logic over the connectives in Neg that results from renaming the $\mathcal{L}_0$ connectives in Figure 3 with the corresponding connectives in Neg. The implication-only fragment of this proof system is exactly the LJT proof system of [14]. The following proposition has an immediate proof, given the structural properties we have seen of $\Downarrow\mathcal{L}_2$ proofs of $\mathcal{L}_0$ sequents.

▶ **Proposition 15.** *Let B be an intuitionistic formula over the connectives in Neg. The sequent $\Sigma : \cdot; \cdot \vdash B^\circ$ is provable in $\Downarrow\mathcal{L}_2$ if and only if the sequent $\Sigma : \cdot \vdash B$ has an LJT$^-$ proof.*

To the extent that maximal multifocused proofs are candidates for canonical proofs, we can conclude that LJT$^-$ proofs are canonical for the negative connectives since all multifocused proofs are single focused, and, hence, are maximal multifocused.

We now extend the mapping of intuitionistic logic formulas into $\mathcal{L}_2$ formulas in a rather natural fashion in order to treat also the positive connectives: $\mathsf{f}^\circ = \top \multimap \bot$, $(B \vee C)^\circ = ((B^\circ \Rightarrow \bot) \mathbin{\&} (C^\circ \Rightarrow \bot)) \multimap \bot$, and $(\exists x.B)^\circ$ as $(\forall x.(B^\circ \Rightarrow \bot)) \multimap \bot$. To make for a stronger result, we add a positive truth t^+ and a positive conjunction $\wedge^+$ to our intuitionistic logic.

These connectives are superfluous since we will be able to prove the formulas t and t^+ and the formulas $B \wedge C$ and $B \wedge^+ C$ are equivalent (in intuitionistic logic). None-the-less, we shall map them into linear logic differently: $(\mathsf{t}^+)^\circ$ as $\perp \multimap \perp$ and $(B \wedge^+ C)^\circ$ as $(B^\circ \Rightarrow C^\circ \Rightarrow \perp) \multimap \perp$. We shall also derive different inference rules for them. Note two things about this extension. First, the results of such translations are much richer than for the negative connectives: for example, one occurrence of $\vee$ yields seven occurrences of linear logic connectives. Second, this translation uses $\perp$, which leaves open the possibility to have multifocused proofs that are not single-focused.

The soundness of this translation (Proposition 16) is proved by a simple induction on the structure of $\mathsf{LJT}^\pm$ proofs; the proof of completeness (Proposition 17) is in Appendix A.3.

▶ **Proposition 16** (Soundness of $(\cdot)^\circ$). *Let B be a formula over the connectives in $\mathit{Neg} \cup \mathit{Pos}$. If B is provable in $\mathsf{LJT}^\pm$, then B° is provable in linear logic.*

▶ **Proposition 17** (Completeness of $(\cdot)^\circ$). *Let B be a formula over the connectives in $\mathit{Neg} \cup \mathit{Pos}$. If B° is provable in linear logic, then B is provable in the $\mathsf{LJT}^\pm$ proof system.*

▶ **Example 18.** The formula $(a \vee b) \supset p \supset p$ is clearly provable in intuitionistic logic. The LJF proof system [20] treats disjunctions and existentials on the left in a linear fashion: when such formulas appear on the left, they are introduced exactly once. Thus, the formula above has exactly one LJF proof, and that proof includes a (needless but harmless) case analysis. In $\mathsf{LJT}^\pm$, there are possibly many proofs of this formula, one for every invocation of the *invert* rule on a disjunctive assumption. This is similar to the proof system by Espírito Santo et al. [9] based on a polarized intuitionistic proof system that uses a “negation translation” for the disjunction

6 Revisiting natural deduction

Given the work of Herbelin [14], Espírito Santo [8], and others, the connection between focused proofs and natural deduction using only negative connectives is well established. It is also well known that the natural deduction treatment of the positive connectives is challenged by some of the same challenges experienced by the sequent calculus: elimination rules for the positive connectives can permute over each other without changing the essential nature of the proof. As a result, the definition and computation of normal-form proofs are complicated. In [13], Girard says, “one tends to think that natural deduction should be modified to correct such atrocities.” We illustrate one approach to making such a correction, but the cost will be an inference rule that can have a large number of premises. This approach is motivated by the treatment of left-introduction rules for the positive connectives in $\mathsf{LJT}^\pm$ proofs.

A formula is in *disjunctive normal form* if it is of the form

$$\exists x_1 \ldots \exists x_p \left(\bigvee\nolimits_{i=1}^{n} \bigwedge\nolimits^{+}{}_{j_i=1}^{m_i} N_{i,j_i} \right). \qquad (*)$$

The formula N_{i,j_i} must be either atomic or have a negative connective as its top level connective. It is easy to show the following facts about disjunctive normal forms.

1. These normal forms are unique up to renaming the existentially bound variables and the ordering of conjuncts and disjuncts (i.e., modulo commutativity for $\vee$ and $\wedge^+$ and identity for f and t^+.
2. The disjunctive normal form of a formula can be exponentially larger than the formula.
3. The following invariant holds for rules in $\mathsf{LJT}^\pm$: if a rule has $\Uparrow \Gamma \vdash$ in the conclusion and the premises contain $\Uparrow \Gamma_1 \vdash, \ldots, \Uparrow \Gamma_n \vdash$, then both $\bigwedge^+ \Gamma$ and $(\bigwedge^+ \Gamma_1) \vee \cdots \vee (\bigwedge^+ \Gamma_n)$ have the same disjunctive normal form.

The disjunctive normal form can be used to describe the following *parallel elimination* for the positive connectives, which can be given as the figure on the left.

$$\dfrac{P_1 \quad \cdots \quad P_p \qquad \left(\begin{array}{c} N_{i,1} \ \cdots \ N_{i,m_i} \\ \vdots \\ D \end{array}\right)_{i=1}^{n}}{D}$$

Here, $P_1, \ldots, P_p$ ($p \geq 1$) are positive formulas and the disjunctive normal form of $P_1 \wedge^+ \cdots \wedge^+ P_p$ is given by ($*$) above. The formula D can be restricted to being either a positive formula or an atomic formula. In this inference rule, $x_1, \ldots, x_p$ are treated as (new) eigenvariables.

As we mentioned before, the drawback of this rule for elimination of positive connectives is that the number of hypothetical premises can be an exponential in the number of occurrences of logical connectives in the formulas $P_1, \ldots, P_p$.

▶ **Example 19.** Assume that $p \geq 1$ and that $a_1, \ldots, a_p, b_1, \ldots, b_p$ are atomic formulas. A special case of the parallel elimination rule for positive formulas is the following.

$$\dfrac{a_1 \vee b_1 \quad \cdots \quad a_p \vee b_p \qquad \left(\begin{array}{c} \{a_i \,|\, i \in I\} \cup \{b_j \,|\, i \notin I\} \\ \vdots \\ D \end{array}\right)_{I \subseteq \{1,\ldots,p\}}}{D}$$

This rule has $p + 2^p$ premises.

7 Future work

As we mentioned at the start of Section 2, the $\Downarrow\mathcal{L}_2$ proof system assumes that atomic formulas are negative. Since this assumption is baked into the design of $\Downarrow\mathcal{L}_2$, it is unclear how one might accommodate atoms given a positive polarity. In the setting of intuitionistic and classical logics, various focused proof systems have considered settings where all atomic formulas are positive: see the classical proof system LKQ [6] and LJQ proof [7]. The LKF and LJF proof system [20] go further and allow positive and negative atomic formulas within the same proof. It would be interesting to consider modifying $\Downarrow\mathcal{L}_2$ to allow mixing both positive and negative polarized atomic formulas.

Extending this work to include higher-order quantification is a natural next step to consider given the successful higher-order extensions of $\Downarrow\mathcal{L}_0$ in [10] and LKQ and LKT in [6].

This paper suggests an interesting pedagogic presentation of linear logic and its proof theory: start with $\mathcal{L}_0$ and moves upward to the more expressive $\mathcal{L}_1$ and $\mathcal{L}_2$, thereby introducing into proofs notions of resources and parallel rule application.

8 Conclusion

This paper offers a new perspective on the proof theory of full linear logic, shifting the traditional classical foundation to an intuitionistic one. Instead of the standard progression through MLL, MALL, and full linear logic, which rely heavily on De Morgan duality, proof nets, and/or one-sided sequent calculi, our approach treats both intuitionistic and linear implications as primitives within the familiar framework of two-sided sequents. We propose an alternative development: from $\mathcal{L}_0$, a core intuitionistic logic, to $\mathcal{L}_1$, which incorporates linear implication, and finally to $\mathcal{L}_2$, which extends $\mathcal{L}_1$ with multiplicative falsity and disjunction.

Central to our contribution is the multifocused, multiple-conclusion proof system $\Downarrow\mathcal{L}_2$ for full linear logic. We demonstrate how $\Downarrow\mathcal{L}_2$ subsumes existing sequent systems for $\mathcal{L}_0$ and $\mathcal{L}_1$, while also introducing a formal definition of parallel rule application via multifocusing.

Crucially, we show that this form of parallelism, which is non-trivial in $\Downarrow\mathcal{L}_2$, is absent in proofs involving only $\mathcal{L}_0$ or $\mathcal{L}_1$ formulas. Furthermore, our work revisits and refines existing results, offering novel treatments of disjunction and existential quantification within intuitionistic sequent calculus and natural deduction. These innovations lead to more intuitive and modular proof systems. The cut elimination theorem, detailed in the appendix, provides the essential foundation for the results presented in this paper.

Acknowledgments I thank the reviewers of a previous draft of this paper for their comments.

References

1. Jean-Marc Andreoli. Logic programming with focusing proofs in linear logic. *J. of Logic and Computation*, 2(3):297–347, 1992. doi:10.1093/logcom/2.3.297.
2. Paola Bruscoli and Alessio Guglielmi. On structuring proof search for first order linear logic. *Theoretical Computer Science*, 360(1-3):42–76, 2006. doi:10.1016/j.tcs.2005.11.047.
3. Kaustuv Chaudhuri, Stefan Hetzl, and Dale Miller. A multi-focused proof system isomorphic to expansion proofs. *J. of Logic and Computation*, 26(2):577–603, 2016. doi:10.1093/logcom/exu030.
4. Kaustuv Chaudhuri, Dale Miller, and Alexis Saurin. Canonical sequent proofs via multi-focusing. In G. Ausiello, J. Karhumäki, G. Mauri, and L. Ong, editors, *Fifth International Conference on Theoretical Computer Science*, volume 273 of *IFIP*, pages 383–396. Springer, September 2008. doi:10.1007/978-0-387-09680-3_26.
5. Kaustuv Chaudhuri, Frank Pfenning, and Greg Price. A logical characterization of forward and backward chaining in the inverse method. *J. of Automated Reasoning*, 40(2-3):133–177, 2008. doi:10.1007/s10817-007-9091-0.
6. V. Danos, J.-B. Joinet, and H. Schellinx. LKT and LKQ: sequent calculi for second order logic based upon dual linear decompositions of classical implication. In J.-Y. Girard, Y. Lafont, and L. Regnier, editors, *Advances in Linear Logic*, number 222 in London Mathematical Society Lecture Note Series, pages 211–224. Cambridge University Press, 1995. doi:10.1017/CBO9780511629150.
7. R. Dyckhoff and S. Lengrand. LJQ: a strongly focused calculus for intuitionistic logic. In A. Beckmann and *et al.*, editors, *Computability in Europe 2006*, volume 3988 of *LNCS*, pages 173–185. Springer, 2006.
8. José Espírito Santo. Completing Herbelin's programme. In Simona Ronchi Della Rocca, editor, *Typed Lambda Calculi and Applications, 8th International Conference, TLCA 2007, Paris, France, June 26-28, 2007, Proceedings*, volume 4583 of *LNCS*, pages 118–132. Springer, 2007. doi:10.1007/978-3-540-73228-0_10.
9. José Espírito Santo, Ralph Matthes, and Luís Pinto. Coinductive proof search for polarized logic with applications to full intuitionistic propositional logic. In Ugo de'Liguoro, Stefano Berardi, and Thorsten Altenkirch;, editors, *26th International Conference on Types for Proofs and Programs (TYPES 2020)*, LIPIcs, pages 4:1–4:24, 2020. doi:10.4230/LIPIcs.TYPES.2020.4.
10. Amy Felty. Encoding the calculus of constructions in a higher-order logic. In M. Vardi, editor, *8th Symp. on Logic in Computer Science*, pages 233–244. IEEE, June 1993. doi:10.1109/LICS.1993.287584.
11. Gerhard Gentzen. Investigations into logical deduction. In M. E. Szabo, editor, *The Collected Papers of Gerhard Gentzen*, pages 68–131. North-Holland, Amsterdam, 1935. Translation of articles that appeared in 1934-35. Collected papers appeared in 1969. doi:10.1007/BF01201353.
12. Jean-Yves Girard. A new constructive logic: classical logic. *Math. Structures in Comp. Science*, 1:255–296, 1991. doi:10.1017/S0960129500001328.
13. Jean-Yves Girard, Paul Taylor, and Yves Lafont. *Proofs and Types*. Cambridge University Press, 1989.

14 Hugo Herbelin. A λ-calculus structure isomorphic to Gentzen-style sequent calculus structure. In *Computer Science Logic, 8th International Workshop, CSL '94*, volume 933 of *Lecture Notes in Computer Science*, pages 61–75. Springer, 1995. doi:10.1007/BFb0022247.

15 Hugo Herbelin and Gyesik Lee. Forcing-based cut-elimination for gentzen-style intuitionistic sequent calculus. In Hiroakira Ono, Makoto Kanazawa, and Ruy J. G. B. de Queiroz, editors, *Logic, Language, Information and Computation, 16th International Workshop, WoLLIC 2009, Tokyo, Japan, June 21-24, 2009. Proceedings*, volume 5514 of *LNCS*, pages 209–217. Springer, 2009. doi:10.1007/978-3-642-02261-6.

16 Joshua Hodas and Dale Miller. Logic programming in a fragment of intuitionistic linear logic. *Information and Computation*, 110(2):327–365, 1994. doi:10.1006/inco.1994.1036.

17 Oiva Ketonen. Untersuchungen zum Prädikatenkalkul. *Annales Academiae Scientiarum Fennicae, series A, I. Mathematica-physica 23*, 1944.

18 Oiva Ketonen. *Investigations into the Predicate Calculus*. College Publications, 2022. Ed. by S. Negri and J. von Plato.

19 Olivier Laurent. A proof of the focalization property of linear logic. Unpublished note, May 2004.

20 Chuck Liang and Dale Miller. Focusing and polarization in linear, intuitionistic, and classical logics. *Theoretical Computer Science*, 410(46):4747–4768, 2009. Abstract Interpretation and Logic Programming: A Special Issue in Honor of Professor Giorgio Levi. doi:10.1016/j.tcs.2009.07.041.

21 Chuck Liang and Dale Miller. Focusing Gentzen's LK proof system. In Thomas Piecha and Kai Wehmeier, editors, *Peter Schroeder-Heister on Proof-Theoretic Semantics*, Outstanding Contributions to Logic, pages 275–313. Springer, February 2024. doi:10.1007/978-3-031-50981-0_9.

22 Pablo López, Frank Pfenning, Jeff Polakow, and Kevin Watkins. Monadic concurrent linear logic programming. In Pedro Barahona and Amy P. Felty, editors, *Proceedings of the 7th International ACM SIGPLAN Conference on Principles and Practice of Declarative Programming (PPDP)*, pages 35–46. ACM, 2005. doi:10.1145/1069778.

23 Dale Miller. Forum: A multiple-conclusion specification logic. *Theoretical Computer Science*, 165(1):201–232, 1996. doi:10.1016/0304-3975(96)00045-X.

24 Dale Miller, Gopalan Nadathur, Frank Pfenning, and Andre Scedrov. Uniform proofs as a foundation for logic programming. *Annals of Pure and Applied Logic*, 51(1–2):125–157, 1991. doi:10.1016/0168-0072(91)90068-W.

25 Dale Miller and Alexis Saurin. A game semantics for proof search: Preliminary results. In M. Escardó, A. Jung, and M. Mislove, editors, *Proceedings of the 21st Annual Conference on Mathematical Foundations of Programming Semantics (MFPS XXI)*, volume 155 of *ENTCS*, pages 543–563. Elsevier, 2005. doi:10.1016/j.entcs.2005.11.072.

26 Grigorii E. Mints. Finite investigations of transfinite derivations. In *Selected papers in proof theory*, pages 17–71. Bibliopolis, 1992.

27 Jan von Plato. Proof theory of classical and intuitionistic logic. In Leila Haaparanta, editor, *The Development of Modern Logic*. Oxford University Press, 2009.

28 Dag Prawitz. *Natural Deduction*. Almqvist & Wiksell, Uppsala, 1965.

29 Harold Schellinx. Some syntactical observations on linear logic. *J. of Logic and Computation*, 1(4):537–559, September 1991. doi:10.1093/logcom/1.4.537.

30 Robert J. Simmons. Structural focalization. *ACM Trans. on Computational Logic*, 15(3):21, 2014. doi:10.1145/2629678.

31 A. S. Troelstra and H. Schwichtenberg. *Basic Proof Theory*. Cambridge University Press, 2nd edition, 2000.

32 Anne S. Troelstra. *Lectures on Linear Logic*. CSLI Lecture Notes 29, Center for the Study of Language and Information, Stanford, California, 1992.

33 Keven Watkins, Iliano Cervesato, Frank Pfenning, and David Walker. A concurrent logical framework I: The propositional fragment. In *Post-proceedings of TYPES 2003 Workshop*, number 3085 in LNCS, pages 355–377. Springer, 2003. doi:10.1007/978-3-540-24849-1_23.
34 Noam Zeilberger. Focusing and higher-order abstract syntax. In George C. Necula and Philip Wadler, editors, *Proceedings of the 35th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL 2008, San Francisco, California, USA, January 7-12, 2008*, pages 359–369. ACM, 2008. doi:10.1145/1328897.1328482.

A Some omitted proofs

We limit the $\Downarrow\mathcal{L}_2$ proofs we reason about in this appendix to single-focused proofs. This restriction does not limit the main results, which are essentially about *provability*. Dealing with the nature of, say, cut-elimination with multifocused proofs is an interesting project, but one that would only complicate the results we prove here.

A.1 The generalized initial rule

◀ **Theorem 9** (Admissibility of the generalized initial rule) *Let B be an $\mathcal{L}_2$ Σ-formula. The sequent $\Sigma : \cdot; B \vdash B$ has a $\Downarrow\mathcal{L}_2$ proof.*

Proof. Let $\Psi \cup \{B\}$ be a multiset of $\mathcal{L}_2$ Σ-formulas. We describe how to build an $\Downarrow\mathcal{L}_2$ proof of $\Sigma : \Psi; B \vdash B$ by induction on the structure of the formula B. By Proposition 7, there is a right phase with endsequent $\Sigma : \Psi; B \vdash B$ and with one premise for every path P in B. In particular, if the associated sequent for P is $\Sigma' : C_1, \ldots, C_n; B_1, \ldots, B_m \vdash A_1, \ldots, A_p$, then the premise of the right-introduction phase that corresponds to this path is $\Sigma, \Sigma' : C_1, \ldots, C_n; B, B_1, \ldots, B_m \vdash A_1, \ldots, A_p$. We can now use the *decide*$_m$ rule to select the occurrence of B in the left-bounded context. By Proposition 8, there is a left-introduction phase corresponding to P such that the sequents

$$\{\Sigma, \Sigma' : \Psi, C_1, \ldots, C_n;\ \cdot\ \vdash C_i\}_{i=1}^n \cup \{\Sigma, \Sigma' : \Psi, C_1, \ldots, C_n; B_i \vdash B_i\}_{i=1}^m$$

must all be provable (the θ in Proposition 8 is set to the identity substitution on the variables in Σ'). The inductive assumption proves the second group of sequents, and the first group is proved using the *decide*$_m$ rule to C_i. The inductive assumption completes this proof. ◀

A.2 The cut-elimination theorem for $\Downarrow\mathcal{L}_2$

Section 2 introduced two cut rules involving $\Downarrow\mathcal{L}_2$ sequents. We call those two cut rules the *regular cut rules* since we now introduce a new cut rule called the *key cut*.

$$\frac{\Sigma : \Psi; \Gamma_1 \vdash B, \Delta \quad \Sigma : \Psi; \Gamma_2 \Downarrow B \vdash \cdot \Downarrow \mathcal{A}}{\Sigma : \Psi; \Gamma_1, \Gamma_2 \vdash \Delta, \mathcal{A}}\ cut_k$$

Here, $\mathcal{A}$ is a multiset (possibly empty) of atomic formulas. The key cut is the only cut rule containing a $\Downarrow$-sequent. The formula B is the *cut-formula* in this rule. To help prove the cut-elimination theorem, we extend the $\Downarrow^+\mathcal{L}_2$ proof system to include the key cut. A proof is *cut-free* if it has no occurrences of these three cut rules.

The cut-elimination argument uses various measurements attached to occurrences of both regular and key-cut rules. A *thread* in the $\Downarrow^+\mathcal{L}_2$ proof Ξ is a list of sequent occurrences $S_1, \ldots, S_n$ in Ξ such that $n \geq 1$, S_1 is an occurrence of the *init* rule, S_n is the endsequent of Ξ, and, for $i = 1, \ldots, n-1$, there is an inference rule occurrence of Ξ that has S_i as a premise and S_{i+1} as its conclusion. Such a thread is said to have length n.

The *rank* of Ξ is the maximal number of occurrences of decide and cut rules in threads in Ξ that do not contain a sequent occurrence that is the left premise of a cut_l, $cut\,!$, or cut_k. The *degree* of a formula is the number of occurrences of logical connectives in that formula.

Every occurrence of a cut rule in a given proof is given a *measure* as follows. Let Ξ be the subproof determined by having that occurrence of cut as its last inference rule. We define $|\Xi|$ to be the pair of natural numbers $\langle d, w\rangle$, where d is the degree of its cut formula, and w is the rank of Ξ. Such pairs are well-ordered using the lexicographic ordering on pairs. The following two propositions are proved by simple inductions on the structure of $\Downarrow^{+}\mathcal{L}_2$ proofs.

▶ **Proposition 20** (Weakening $\Downarrow^{+}\mathcal{L}_2$ proofs). *If $\Sigma : \Psi; \Gamma \vdash \mathcal{A}$ has a $\Downarrow^{+}\mathcal{L}_2$ proof Ξ then $\Sigma, \Sigma' : \Psi, \Psi'; \Gamma \vdash \mathcal{A}$ has a $\Downarrow^{+}\mathcal{L}_2$ proof Ξ'. Furthermore, every instance of a cut rule in Ξ corresponds to an instance of cut in Ξ' and they have the same measure.*

▶ **Proposition 21** (Substitution into $\Downarrow^{+}\mathcal{L}_2$ proofs). *Let Σ be a signature, x be a variable not declared in Σ, τ be a primitive type, and t be a Σ-term of type τ. If $\Sigma, x : \tau : \Psi; \Gamma \vdash \mathcal{A}$ has a $\Downarrow^{+}\mathcal{L}_2$ proof Ξ then $\Sigma : \Psi[t/x]; \Gamma[t/x] \vdash \mathcal{A}[t/x]$ has a $\Downarrow^{+}\mathcal{L}_2$ proof Ξ'. Furthermore, every instance of a cut rule in Ξ corresponds to an instance of cut in Ξ' and they have the same measure.*

The following proposition states that if a formula occurrence in the unbounded zone of a sequent is never decided on within the proof of that sequent, then that occurrence can be removed from its zone. This proposition is proved by a simple induction on the structure of $\Downarrow^{+}\mathcal{L}_2$ proofs.

▶ **Proposition 22** (Strengthening $\Downarrow^{+}\mathcal{L}_2$ proofs). *Assume that we have a $\Downarrow^{+}\mathcal{L}_2$ proof Ξ of $\Sigma : \Psi, B; \Gamma \vdash \Delta$ (resp. $\Sigma : \Psi, B; \Gamma \Downarrow D \vdash \cdot \Downarrow \Delta$) in which there is no occurrence of $decide_m$ used with the formula B. Then there is a $\Downarrow^{+}\mathcal{L}_2$ proof Ξ' of $\Sigma : \Psi; \Gamma \vdash \Delta$ (respectively, $\Sigma : \Psi; \Gamma \Downarrow D \vdash \cdot \Downarrow \Delta$). Furthermore, every instance of a cut rule in Ξ corresponds to an instance of cut in Ξ', and they have the same measure.*

We single out instances of atomic cut_k rules for special treatment. Note that the right premise of an atomic cut_k rule can only be proved using *init*.

$$\dfrac{\Sigma : \Psi; \Gamma \vdash \Delta, A \qquad \dfrac{}{\Sigma : \Psi; \cdot \Downarrow A \vdash \cdot \Downarrow A}\ init}{\Sigma : \Psi; \Gamma \vdash \Delta, A}\ cut_k$$

This derivation can be written more simply as (assuming A is atomic).

$$\dfrac{\Sigma : \Psi; \Gamma \vdash \Delta, A}{\Sigma : \Psi; \Gamma \vdash \Delta, A}\ Rep$$

This rule resembles the *repetition rule* used by Mints [26] to prove a cut-elimination theorem for a different logic. An important feature of atomic key cut rules is that their measure is always $\langle 0, 1\rangle$ since the proof structure in their left premise is not part of the measure. Ultimately, our cut-elimination procedure will eliminate all cuts except for atomic key cuts. After those eliminations are made, a second procedure will eliminate all atomic key cuts.

A $\Downarrow^{+}\mathcal{L}_2$ proof is called a $\Downarrow^{a}\mathcal{L}_2$-proof if the only occurrences of cut rules in it are atomic key cuts. A *redex* is a $\Downarrow^{+}\mathcal{L}_2$ proof where the last inference rule is a regular or key cut and where that rule's two premises are $\Downarrow^{a}\mathcal{L}_2$-proofs. A redex is classified as atomic or non-atomic depending on whether the cut formula of its final cut rule is atomic or non-atomic. A redex is also classified by the kind of cut rule it has as its final rule.

It is easy to prove that the side-formulas on the right-bounded zone for the *Rep* rule (the schematic variable Δ above) can be restricted to contain only atomic formulas: that is, the conclusion of such rules can be assumed to be border sequents. As a result, Proposition 7 can be used to characterize additionally the right-introduction phase of $\Downarrow^a\mathcal{L}_2$-proofs.

We now provide several lemmas that show how various redexes can be replaced with proofs involving strictly smaller redexes.

▶ **Lemma 23** (Replace *cut*! with cut_k). *Let Ξ be a cut! redex. Then there exists a proof of the same endsequent in which the only instances of cut rules are either cut_l or atomic cut_k, and all such instances of cuts have a measure strictly less than $|\Xi|$.*

Proof. Consider the following *cut*!-redex Ξ.

$$\dfrac{\begin{matrix}\Xi_l\\ \Sigma : \Psi; \cdot \vdash B\end{matrix} \qquad \begin{matrix}\Xi_r\\ \Sigma : \Psi, B; \Gamma \vdash \Delta\end{matrix}}{\Sigma : \Psi; \Gamma \vdash \Delta}\ cut!$$

Here, the only occurrences of cut rules in the subproofs Ξ_l and Ξ_r are atomic key cuts. Consider a subderivation of Ξ_r that ends in $decide_m$, such as

$$\dfrac{\begin{matrix}\Xi_0\\ \Sigma, \Sigma' : \Psi, \Psi', B; \Gamma \Downarrow B \vdash \cdot \Downarrow \mathcal{A}\end{matrix}}{\Sigma, \Sigma' : \Psi, \Psi', B; \Gamma \vdash \mathcal{A}}\ decide_m,$$

where the variables bound in Σ' are not bound in Σ and where Ψ' is a multiset. This inference rule can be converted to the derivation

$$\dfrac{\begin{matrix}\hat{\Xi}_l\\ \Sigma, \Sigma' : \Psi, \Psi'; \cdot \vdash B\end{matrix} \qquad \begin{matrix}\Xi_0\\ \Sigma, \Sigma' : \Psi, \Psi', B; \Gamma \Downarrow B \vdash \cdot \Downarrow \mathcal{A}\end{matrix}}{\Sigma, \Sigma' : \Psi, \Psi', B; \Gamma \vdash \mathcal{A}}\ cut_k,$$

where $\hat{\Xi}_l$ is the result of weakening Ξ_l using Proposition 20. In this way, we can remove all occurrences of $decide_m$ on B in Ξ_r to obtain the proof Ξ_r' of $\Sigma : \Psi, B; \Gamma \vdash \Delta$. By Proposition 22, we can strengthen Ξ_r' to get a proof Ξ_r'' of $\Sigma : \Psi; \Gamma \vdash \Delta$. This proof can now replace our original redex. Since all new occurrences of cuts have B as their cut formula and since the rank part of the measure of redexes does not consider the subproof of the left premise of *cut*! and cut_l, the measure of the cut-rules in Ξ_l'' is strictly smaller than $|\Xi|$. ◀

The previous lemma removed a *cut*! by converting some $decide_m$ rules into cut_k rules. The treatment of the cut_l rule is not so easily handled. In particular, we will use the following lemma to show that the "side cut" case can be treated by moving a cut_l rule over an entire left-introduction phase.

▶ **Lemma 24** (Side cut_l case). *Let Ξ be a cut_l-redex such that a decide rule is the last inference rule of the proof of the right premise. If the formula selected is not the cut formula, then there exists a $\Downarrow^+\mathcal{L}_2$ proof with the same endsequent in which all instances of cuts have a measure strictly less than $|\Xi|$.*

Proof. The decide rule that ends the proof of the right premise must select its focus from either the bounded or unbound zone on the left. We consider these two cases below.

Case: The decide rule selects from the bounded zone. Let Ξ be the following proof.

$$\dfrac{\begin{array}{c}\Xi_l\\ \Sigma:\Psi;\Gamma\vdash C,\Delta\end{array} \qquad \dfrac{\begin{array}{c}\Xi_r\\ \Sigma:\Psi;\Gamma',C\Downarrow B\vdash\cdot\Downarrow\mathcal{A}\end{array}}{\Sigma:\Psi;\Gamma',B,C\vdash\mathcal{A}}\ \mathit{decide}_m}{\Sigma:\Psi;\Gamma,\Gamma',B\vdash\Delta,\mathcal{A}}\ \mathit{cut}_l$$

Here, the only occurrences of cut rules in the subproofs Ξ_l and Ξ_r are atomic key cuts, and $\mathcal{A}$ is a multiset of atomic formulas. By Proposition 8,[1] the sequent $\Sigma:\Psi;\Gamma',C\Downarrow B\vdash\cdot\Downarrow\mathcal{A}$ is the endsequent of a left-introduction phase with a multiset of premises $\mathcal{P}$ such that there is a path P in B with the associated sequent

$$\Sigma':C_1,\ldots,C_n;B_1,\ldots,B_m\vdash A_1,\ldots,A_p,$$

and there is a substitution θ that maps the variables in Σ' to Σ-terms such that

1. $\mathcal{A}$ is equal to the multiset union $\{A_1\theta,\ldots,A_p\theta\}\cup\mathcal{A}_1\cup\cdots\cup\mathcal{A}_m$;
2. $\Gamma'\cup\{C\}$ is the multiset union $\Gamma_1\cup\cdots\cup\Gamma_m$; and
3. $\mathcal{P}$ is $\{\Sigma:\Psi;\cdot\vdash C_i\theta\}_{i=1}^n\cup\{\Sigma:\Psi;\Gamma_i\vdash B_i\theta,\mathcal{A}_i\}_{i=1}^m$.

Since the left-phase is multiplicative, there is a unique $k\in\{1,\ldots,m\}$ such that C occurs in Γ_k. Let Γ'_k be the result of removing one occurrence of C from Γ_k. Thus, one of the premises in $\mathcal{P}$ is $\Sigma:\Psi;\Gamma'_k,C\vdash B_k\theta,\mathcal{A}_k$. By using the cut_l rule we have, together with a proof of the above sequent, the following proof.

$$\dfrac{\begin{array}{c}\Xi_l\\ \Sigma:\Psi;\Gamma'\vdash C,\mathcal{A}\end{array} \quad \Sigma:\Psi;\Gamma'_k,C\vdash B_k\theta,\mathcal{A}_k}{\Sigma:\Psi;\Gamma',\Gamma'_k\vdash B_k\theta,\mathcal{A}_k,\mathcal{A}}\ \mathit{cut}_l$$

By using the same path above, we can move this left-introduction phase below the cut_l rule. Thus, the original cut_l rule has been moved up, and its measure has decreased.

Case: The decide rule selects from the unbounded zone. Let Ξ be the following proof, and assume that B is a member of Ψ.

$$\dfrac{\begin{array}{c}\Xi_l\\ \Sigma:\Psi;\Gamma\vdash C,\Delta\end{array} \qquad \dfrac{\begin{array}{c}\Xi_r\\ \Sigma:\Psi;\Gamma',C\Downarrow B\vdash\cdot\Downarrow\mathcal{A}\end{array}}{\Sigma:\Psi;\Gamma',C\vdash\mathcal{A}}\ \mathit{decide}_m}{\Sigma:\Psi;\Gamma,\Gamma'\vdash\Delta,\mathcal{A}}\ \mathit{cut}_l$$

Here, the only occurrences of cut rules in the subproofs Ξ_l and Ξ_r are atomic key cuts, and $\mathcal{A}$ is a multiset of atomic formulas. This case is treated the same as the previous case. ◀

$$\dfrac{\begin{array}{c}\Xi_i\\ \cdots\quad \Sigma_i:\Psi_i;\Gamma_i,B\vdash\mathcal{A}_i\quad\cdots\end{array}}{\dfrac{\vdots}{\Sigma:\Psi;\Gamma,B\vdash\Delta}}$$

Remark: Let Ξ be a $\Downarrow^{+}\mathcal{L}_2$ proof of $\Sigma:\Psi;\Gamma,B\vdash\Delta$. If Δ contains a logical connective, then this proof is of the form displayed to the right. Here, $\mathcal{A}_i$ is a multiset of atomic formulas; Γ is a sub-multiset of Γ_i; Ψ is a sub-multiset of Ψ_i; all the inference rules elided here are either right-introduction rules or atomic key cuts; and the last inference rule of the subproofs Ξ_i's are one of the decide rules. An instance of cut_l on B in the endsequent can then be lifted to several instances of cut_l with Ξ_i. This does not change the measure of any cuts. Next we resolve the cut/decide pairing as described in the following proof.

[1] While Proposition 8, was proved for $\Downarrow\mathcal{L}_2$ proofs, it also holds in the presence of cut rules since no cut rule contains a $\Downarrow$ in its conclusion.

▶ **Lemma 25** (Replace cut_l with cut_k). *Let Ξ be a cut_l redex. Then there exists a proof of the same endsequent in which the only instances of cut rules are cut_k, and all such instances of cuts have a measure strictly less than $|\Xi|$.*

Proof. Consider the following cut_l-redex Ξ.

$$\dfrac{\dfrac{\Xi_l}{\Sigma : \Psi; \Gamma_1 \vdash B, \Delta_1} \qquad \dfrac{\Xi_r}{\Sigma : \Psi; \Gamma_2, B \vdash \Delta_2}}{\Sigma : \Psi; \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ cut_l$$

Here, the only occurrences of cut rules in the subproofs Ξ_l and Ξ_r are atomic key cuts. Given the remark above, we only need to consider the situation where the right-bounded context contains only atomic formulas and that the last inference rule of Ξ_r is a decide rule.

Case: Ξ_r ends in the $decide_m$ rule. If the formula selected for the focus is B, then the proof Ξ_r has the form

$$\dfrac{\dfrac{\Xi'_r}{\Sigma : \Psi; \Gamma'_2 \Downarrow B \vdash \cdot \Downarrow \Delta'_2}}{\Sigma : \Psi; \Gamma'_2, B \vdash \Delta'_2}\ decide_m.$$

This instance of the cut_l rule above can be replaced with the following instance of cut_k.

$$\dfrac{\dfrac{\Xi_l}{\Sigma : \Psi; \Gamma_1 \vdash B, \Delta_1} \qquad \dfrac{\Xi'_r}{\Sigma : \Psi; \Gamma_2 \Downarrow B \vdash \cdot \Downarrow \Delta_2}}{\Sigma : \Psi; \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ cut_k$$

If the formula selected for the focus is some other formula than B, then the proof Ξ_r has the form (Γ_2 is of the form C, Γ'_2)

$$\dfrac{\dfrac{\Xi'_r}{\Sigma : \Psi; \Gamma'_2, B \Downarrow C \vdash \cdot \Downarrow \Delta_2}}{\Sigma : \Psi; \Gamma'_2, B, C \vdash \Delta_2}\ decide_m.$$

We now use Lemma 24 to construct a $\Downarrow^+\mathcal{L}_2$ proof of $\Sigma : \Psi; \Gamma'_2, C \vdash \Delta_2$ of lower right rank.

Case: Ξ_r ends in the $decide_m$ rule. Then the redex Ξ necessarily ends in a side cut, so Lemma 24 provides the necessary rewriting of this redex. ◀

▶ **Lemma 26** (Reduce cut_k). *Let Ξ be a non-atomic cut_k redex. Then there exists a proof of the same endsequent in which the redexes it has are cut_l and $cut!$-redexes all with a measure strictly less than $|\Xi|$.*

Proof. Consider a cut_k-redex Ξ of the form

$$\dfrac{\dfrac{\Xi_l}{\Sigma : \Psi; \Gamma_1 \vdash B, \Delta} \quad \dfrac{\Xi_r}{\Sigma : \Psi; \Gamma_2 \Downarrow B \vdash \cdot \Downarrow \mathcal{A}}}{\Sigma : \Psi; \Gamma_1, \Gamma_2 \vdash \Delta, \mathcal{A}}\ cut_k,$$

where Ξ_l and Ξ_r are $\Downarrow^a\mathcal{L}_2$ proofs. Since B is not atomic, Ξ_l ends in a right-introduction phase and Ξ_r ends in a left-introduction phase. By Proposition 8, there is a path P in B that has the associated sequent representation

$$\Sigma' : C_1, \ldots, C_n; B_1, \ldots, B_m \vdash A_1, \ldots, A_p,$$

and there is a substitution θ that maps the variables in Σ' to Σ-terms such that $\mathcal{A}$ is the multiset union $\{A_1\theta, \ldots, A_p\theta\} \cup \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_m$, Γ is the multiset union $\Gamma_1 \cup \cdots \cup \Gamma_m$, and this phase has $n + m$ premises $\{\Sigma : \Psi; \cdot \vdash C_i\theta\}_{i=1}^n \cup \{\Sigma : \Psi; \Gamma_i \vdash B_i\theta, \mathcal{A}_i\}_{i=1}^m$. By Proposition 7, Ξ_l ends with a right-introduction phase that contains a premise of the form

$$\overset{\Xi_0}{\Sigma, \Sigma' : \Psi, C_1, \ldots, C_n; \Gamma, B_1, \ldots, B_m \vdash \mathcal{A}', A_1, \ldots, A_p}.$$

By repeated application of Proposition 21, we know that the sequent

$$\overset{\Xi_0'}{\Sigma : \Psi, C_1\theta, \ldots, C_n\theta; \Gamma, B_1\theta, \ldots, B_m\theta \vdash \mathcal{A}', A_1\theta, \ldots, A_p\theta}$$

has a $\Downarrow^a\mathcal{L}_2$ proof. We can take Ξ_0' and use cut_l and $cut\,!$ with the proofs of the $n+m$ premises above to yield a proof with $n + m$ occurrences of these cut rules to provide a proof without occurrences of cut_k of the endsequent $\Sigma : \Psi; \Gamma, \Gamma' \vdash \Delta, \mathcal{A}$. Note that the size of each of the cut formulas $C_1\theta, \ldots, C_n\theta, B_1\theta, \ldots, B_m\theta$ is strictly smaller than the size of the original cut formula B. ◀

We are now in a position to prove the cut-elimination theorem for $\Downarrow^+\mathcal{L}_2$ proofs.

◀ **Theorem 10** (Cut elimination for $\Downarrow^+\mathcal{L}_2$) *Let B be an $\mathcal{L}_2$ Σ-formula. If the sequent $\Sigma : \cdot; \cdot \vdash B$ has a $\Downarrow^+\mathcal{L}_2$ proof, then it has an $\Downarrow\mathcal{L}_2$ proof.*

Proof. We divide this proof into two parts. The first part proves that if a sequent has a $\Downarrow^+\mathcal{L}_2$ proof, then it has a $\Downarrow^a\mathcal{L}_2$-proof. The second part proves that if a sequent has a $\Downarrow^a\mathcal{L}_2$-proof then it has a (cut-free) $\Downarrow\mathcal{L}_2$ proof.

Thus, assume that we have a $\Downarrow^+\mathcal{L}_2$ proof. We proceed by induction on the number of occurrences of cut rules in that proof that are not atomic key cuts. If the number of such redexes is zero, we are finished with the first part of this proof. Otherwise, select a redex Ξ that is not an atomic key cut redex. We prove by induction on the measure $|\Xi|$ that this redex can be replaced by a $\Downarrow^a\mathcal{L}_2$-proof of the same endsequent. If Ξ is a $cut\,!$-redex then apply Lemma 23; if Ξ is a cut_l-redex then apply Lemma 25; and, finally, if Ξ is a non-atomic cut_k-redex then apply Lemma 26. The results of such applications are proofs of the same endsequent as Ξ in which all redexes have a measure strictly less than $|\Xi|$. Thus, by induction, all of these can be replaced by $\Downarrow^a\mathcal{L}_2$-proofs.

To complete the second part of this proof, we proceed to prove by induction that if the $\Downarrow^a\mathcal{L}_2$-proof Ξ contains $n \geq 0$ occurrences of atomic key cases, then there is a $\Downarrow\mathcal{L}_2$ proof of the same endsequent. Pick any atomic key cut occurrence in Ξ. That occurrence resembles the *Rep* rule, which is trivial to remove. ◀

A.3 The completeness of $(\cdot)^\circ$

For convenience, define $B^\bullet$ for positive intuitionistic formulas B as follows: $\mathsf{f}^\bullet = \top$, $(B \vee C)^\bullet = (B^\circ \Rightarrow \bot) \,\&\, (C^\circ \Rightarrow \bot)$, $(\exists x.B)^\bullet = \forall x.(B^\circ \Rightarrow \bot)$, $(\mathsf{t}^+)^\bullet = \bot \multimap \bot$, $(B \wedge^+ C)^\bullet = B^\circ \Rightarrow C^\circ \Rightarrow \bot$. Thus, for B a positive intuitionistic formula, B° is the same formulas as $B^\bullet \multimap \bot$.

◀ **Proposition 17** (Completeness of $(\cdot)^\circ$) *Let B be a formula over the connectives in* $\mathsf{Neg} \cup \mathsf{Pos}$*. If B° is provable in linear logic, then B is provable in the* $\mathsf{LJT}^\pm$ *proof system.*

Proof. Let B be a formula over the connectives in $\mathsf{Neg} \cup \mathsf{Pos}$. If B° is provable in linear logic, then $\Sigma : \cdot; \cdot \vdash B^\circ$ has a $\Downarrow\mathcal{L}_2$ proof. There are a few different kinds of sequents that can appear in such a $\Downarrow\mathcal{L}_2$ proof, and we need to consider $\Downarrow\mathcal{L}_2$ proofs of sequents which are in one of the following shapes: $\Sigma:\Psi^\circ;\cdot\vdash B^\circ$ or $\Sigma:\Psi^\circ;B^\bullet\vdash\cdot$ or $\Sigma:\Psi^\circ;\cdot\Downarrow B^\circ\vdash\cdot\Downarrow A$ or $\Sigma:\Psi^\circ;\cdot\Downarrow B^\bullet\vdash\cdot\Downarrow\cdot$. Note that if the left-bounded zone is non-empty, then that zone contains one formula which is the result of $(\cdot)^\bullet$ of a positive formula, and the right zone is empty.

Remark: If the sequent $\Sigma:\Psi;B^\bullet\Downarrow(C)^\circ\vdash\cdot\Downarrow\cdot$ has a proof (when B is a positive formula) then C is also positive. This remark is easily proved by induction on the structure of C.

We can now translate $\Downarrow\mathcal{L}_2$ proofs of these four kinds of sequents directly into $\mathsf{LJT}^\pm$ proofs. We proceed by induction on the structure of an $\Downarrow\mathcal{L}_2$ proof Ξ of these kinds of sequents.

Case: Ξ is a proof of $\Sigma:\Psi^\circ;\cdot\vdash B^\circ$. If B is positive, then Ξ has a subproof of $\Sigma:\Psi^\circ;B^\bullet\vdash\cdot$: the translation of that proof (see below) is the needed $\mathsf{LJT}^\pm$ proof. If B is negative, we consider the last inference rule of Ξ, which is either $\top R$, $\& R$, $\Rightarrow R$, or $\forall R$. In each of these cases, the translation is achieved by first translating the immediate subproof(s) and then adding the corresponding $\mathsf{LJT}^\pm$ rules of $\mathsf{t}R$, $\wedge R$, $\supset R$, and $\forall R$. The right introduction rules for the negative connectives arise this way.

Case: Ξ is a proof of $\Sigma:\Psi^\circ;B^\bullet\vdash\cdot$, where B is a positive formula. This sequent is the conclusion of a decide rule that selects either $B^\bullet$ or a member of Ψ°. The former case is considered below. In the latter case, this is only possible (by the remark above) if the selected member C of Ψ is a positive formula. Ξ contains a subproof of the sequent $\Sigma:\Psi^\circ;B^\bullet\Downarrow C^\bullet\multimap\bot\vdash\cdot\Downarrow\cdot$ and this has a subproof of $\Sigma:\Psi^\circ;B^\bullet\vdash C^\bullet$. By considering all cases for the positive formula C, Ξ will contain subproofs of the shape $\Sigma':\Psi'^\circ;B^\bullet\vdash\cdot$. The translation of those subproofs and the corresponding left-introduction rules, yields the required translation.

Case: Ξ is a proof of $\Sigma:\Psi^\circ;\cdot\Downarrow B^\circ\vdash\cdot\Downarrow A$. If B is a negative formula, then Ξ must be the right introduction of either $\top$, $\&$, $\Rightarrow$, or $\forall$. The required $\mathsf{LJT}^\pm$ proof results from applying the right introduction rules for t, $\wedge$, $\supset$, or $\forall$ to the transformations of the associated subproofs of Ξ. If B is a positive formula, then Ξ must end with

$$\dfrac{\dfrac{\Sigma:\Psi^\circ;\cdot\vdash B^\bullet, A}{\Sigma:\Psi^\circ;\cdot\Downarrow\cdot\vdash B^\bullet\Downarrow A}\qquad\Sigma:\Psi^\circ;\cdot\Downarrow\bot\vdash\cdot\Downarrow\cdot}{\Sigma:\Psi^\circ;\cdot\Downarrow B^\bullet\multimap\bot\vdash\cdot\Downarrow A}\;.$$

If we now consider each case for the positive formula B, we see that invertibility will yield direct translations of the corresponding left introduction. For example, if B is $B_1\vee B_2$ then the Ξ proof of $\Sigma:\Psi^\circ;\cdot\Downarrow(B_1\vee B_2)^\circ\vdash\cdot\Downarrow A$ contains a subproof of

$$\Sigma:\Psi^\circ;\cdot\vdash(B_1^\circ\Rightarrow\bot)\,\&\,(B_2^\circ\Rightarrow\bot), A,$$

which in turn contains subproofs of $\Sigma:\Psi^\circ,B_i^\circ;\cdot\vdash A$, for $i\in\{1,2\}$. The full translation uses the $\vee L$ rule of $\mathsf{LJT}^\pm$.

Case: Ξ is a proof of $\Sigma:\Psi^\circ;\cdot\Downarrow B^\bullet\vdash\cdot\Downarrow\cdot$. This case emulates the right introduction rule of $\mathsf{LJT}^\pm$ for the positive connectives. For example, if B is $B_1\vee B_2$ then Ξ must have the form $\Sigma:\Psi^\circ;\cdot\Downarrow(B_1^\circ\Rightarrow\bot)\,\&\,(B_2^\circ\Rightarrow\bot)\vdash\cdot\Downarrow\cdot$ and this means that there must be a subproof of Ξ of $\Sigma:\Psi^\circ;\cdot\vdash B_i^\circ$.

Note that the abstraction mechanism of synthetic inference rules allows hiding the internal presence of multiple conclusion sequents even within an intuitionistic proof.

◀

Right phase rules

$$\frac{}{\Sigma : \Psi; \Gamma \vdash \top, \Delta}\ \top R \qquad \frac{\Sigma : \Psi; \Gamma \vdash B, \Delta \quad \Sigma : \Psi; \Gamma \vdash C, \Delta}{\Sigma : \Psi; \Gamma \vdash B \mathbin{\&} C, \Delta}\ \&\mathrm{R}$$

$$\frac{\Sigma : \Psi; \Gamma \vdash \Delta}{\Sigma : \Psi; \Gamma \vdash \bot, \Delta}\ \bot R \qquad \frac{\Sigma : \Psi; \Gamma \vdash B, C, \Delta}{\Sigma : \Psi; \Gamma \vdash B \parr C, \Delta}\ \parr R \qquad \frac{\Sigma : \Psi; B, \Gamma \vdash C, \Delta}{\Sigma : \Psi; \Gamma \vdash B \multimap C, \Delta}\ \multimap R$$

$$\frac{\Sigma : B, \Psi; \Gamma \vdash C, \Delta}{\Sigma : \Psi; \Gamma \vdash B \Rightarrow C, \Delta}\ \Rightarrow R \qquad \frac{y : \tau, \Sigma : \Psi; \Gamma \vdash B[y/x], \Delta}{\Sigma : \Psi; \Gamma \vdash \forall_\tau x.B, \Delta}\ \forall R$$

Left phase rules

$$\frac{}{\Sigma : \Psi; \cdot \Downarrow \bot \vdash \cdot \Downarrow \cdot}\ \bot L \qquad \frac{\Sigma : \Psi; \Gamma_1 \Downarrow B, \Theta_1 \vdash \Theta_3 \Downarrow \Delta_1 \quad \Sigma : \Psi; \Gamma_2 \Downarrow C, \Theta_2 \vdash \Theta_4 \Downarrow \Delta_2}{\Sigma : \Psi; \Gamma_1, \Gamma_2 \Downarrow B \parr C, \Theta_1, \Theta_2 \vdash \Theta_3, \Theta_4 \Downarrow \Delta_1, \Delta_2}\ \parr\mathrm{L}$$

$$\frac{\Sigma : \Psi; \Gamma \Downarrow B_i, \Theta \vdash \Theta' \Downarrow \Delta}{\Sigma : \Psi; \Gamma \Downarrow B_1 \mathbin{\&} B_2, \Theta \vdash \Theta' \Downarrow \Delta}\ \&\mathrm{L}_i,\ i \in \{1,2\} \qquad \frac{\Sigma : \Psi; \Gamma \Downarrow B[t/x], \Theta \vdash \Theta' \Downarrow \Delta}{\Sigma : \Psi; \Gamma \Downarrow \forall_\tau x.B, \Theta \vdash \Theta' \Downarrow \Delta}\ \forall L$$

$$\frac{\Sigma : \Psi; \Gamma_1 \Downarrow \Theta_1 \vdash \Theta_3, B \Downarrow \Delta_1 \quad \Sigma : \Psi; \Gamma_2 \Downarrow C, \Theta_2 \vdash \Theta_4 \Downarrow \Delta_2}{\Sigma : \Psi; \Gamma_1, \Gamma_2 \Downarrow B \multimap C, \Theta_1, \Theta_2 \vdash \Theta_3, \Theta_4 \Downarrow \Delta_1, \Delta_2}\ \multimap\mathrm{L}$$

$$\frac{\Sigma : \Psi; \cdot \vdash B \quad \Sigma : \Psi; \Gamma \Downarrow C, \Theta \vdash \Theta' \Downarrow \Delta}{\Sigma : \Psi; \Gamma \Downarrow B \Rightarrow C, \Theta \vdash \Theta' \Downarrow \Delta}\ \Rightarrow\mathrm{L} \qquad \frac{}{\Sigma : \Psi; \cdot \Downarrow B \vdash \cdot \Downarrow B}\ \mathit{init}$$

Phase switching rules

$$\frac{\Sigma : \Psi_1, \Psi_2; \Gamma_1 \Downarrow \hat{\Psi}_2, \Gamma_2 \vdash \cdot \Downarrow \Delta}{\Sigma : \Psi_1, \Psi_2; \Gamma_1, \Gamma_2 \vdash \Delta}\ \mathit{decide}_m \qquad \frac{\Sigma : \Psi; \Gamma \vdash \Theta, \Delta}{\Sigma : \Psi; \Gamma \Downarrow \cdot \vdash \Theta \Downarrow \Delta}\ \mathit{release}$$

The decide_m rule is restricted so that (i) the union $\hat{\Psi}_2, \Gamma_2$ is non-empty, (ii) Δ is a multiset of atomic formulas, and (iii) Ψ_2 and $\hat{\Psi}_2$ are instantiated with multisets of formulas so that every formula with a non-zero multiplicity in one of them also has a non-zero multiplicity (not necessarily equal) in the other. The quantifier rules have the usual provisos: $y \notin \Sigma$ in $\forall$R, and t is a Σ-term of type τ in $\forall$L.

Figure 1 The $\Downarrow\mathcal{L}_2$ focused proof system.

$$\frac{}{\Sigma : \Psi; \Gamma \vdash \top}\ \top R \qquad \frac{\Sigma : \Psi; \Gamma \vdash B \quad \Sigma : \Psi; \Gamma \vdash C}{\Sigma : \Psi; \Gamma \vdash B \,\&\, C}\ \&R$$

$$\frac{\Sigma : \Psi; B, \Gamma \vdash C}{\Sigma : \Psi; \Gamma \vdash B \multimap C} \multimap R \qquad \frac{\Sigma : B, \Psi; \Gamma \vdash C}{\Sigma : \Psi; \Gamma \vdash B \Rightarrow C} \Rightarrow R \qquad \frac{y : \tau, \Sigma : \Psi; \Gamma \vdash B[y/x]}{\Sigma : \Psi; \Gamma \vdash \forall_\tau x.B}\ \forall R$$

$$\frac{\Sigma : \Psi; \Gamma \Downarrow B_i \vdash A}{\Sigma : \Psi; \Gamma \Downarrow B_1 \,\&\, B_2 \vdash A}\ \&L_i \qquad \frac{\Sigma : \Psi; \Gamma \Downarrow B[t/x] \vdash A}{\Sigma : \Psi; \Gamma \Downarrow \forall_\tau x.B \vdash A}\ \forall L \qquad \frac{}{\Sigma : \Psi; \cdot \Downarrow A \vdash A}\ \mathit{init}$$

$$\frac{\Sigma : \Psi; \cdot \vdash B \quad \Sigma : \Psi; \Gamma \Downarrow C \vdash A}{\Sigma : \Psi; \Gamma \Downarrow B \Rightarrow C \vdash A} \Rightarrow L \qquad \frac{\Sigma : \Psi; \Gamma_1 \vdash B \quad \Sigma : \Psi; \Gamma_2 \Downarrow C \vdash A}{\Sigma : \Psi; \Gamma_1, \Gamma_2 \Downarrow B \multimap C \vdash A} \multimap L$$

$$\frac{\Sigma : \Psi, B; \Gamma \Downarrow B \vdash A}{\Sigma : \Psi, B; \Gamma \vdash A}\ \mathit{decide}! \qquad \frac{\Sigma : \Psi; \Gamma \Downarrow B \vdash A}{\Sigma : \Psi; \Gamma, B \vdash A}\ \mathit{decide}_l$$

Figure 2 The $\Downarrow\mathcal{L}_1$ proof system

$$\frac{}{\Sigma : \Psi \vdash \top}\ \top R \qquad \frac{\Sigma : \Psi \vdash B \quad \Sigma : \Psi \vdash C}{\Sigma : \Psi \vdash B \,\&\, C}\ \&R \qquad \frac{\Sigma : B, \Psi \vdash C}{\Sigma : \Psi \vdash B \Rightarrow C} \Rightarrow R \qquad \frac{y : \tau, \Sigma : \Psi \vdash B[y/x]}{\Sigma : \Psi \vdash \forall_\tau x.B}\ \forall R$$

$$\frac{\Sigma : \Psi \Downarrow B_i \vdash A}{\Sigma : \Psi \Downarrow B_1 \,\&\, B_2 \vdash A}\ \&L_i \qquad \frac{\Sigma : \Psi \Downarrow B[t/x] \vdash A}{\Sigma : \Psi \Downarrow \forall_\tau x.B \vdash A}\ \forall L \qquad \frac{}{\Sigma : \Psi \Downarrow A \vdash A}\ \mathit{init}$$

$$\frac{\Sigma : \Psi \vdash B \quad \Sigma : \Psi \Downarrow C \vdash A}{\Sigma : \Psi \Downarrow B \Rightarrow C \vdash A} \Rightarrow L \qquad \frac{\Sigma : \Psi, B \Downarrow B \vdash A}{\Sigma : \Psi, B \vdash A}\ \mathit{decide}!$$

Figure 3 The rules that result from restricting $\Downarrow\mathcal{L}_2$ to $\mathcal{L}_0$ sequents.

$$\frac{}{\Sigma : \Psi \vdash \mathsf{t}}\ \mathsf{t}R \qquad \frac{\Sigma : \Psi \vdash B \quad \Sigma : \Psi \vdash C}{\Sigma : \Psi \vdash B \wedge C}\ \wedge R \qquad \frac{\Sigma : B, \Psi \vdash C}{\Sigma : \Psi \vdash B \supset C}\ \supset\!\mathrm{R} \qquad \frac{y : \tau, \Sigma : \Psi \vdash B[y/x]}{\Sigma : \Psi \vdash \forall_\tau x.B}\ \forall R$$

$$\frac{\Sigma : \Psi, N \Downarrow N \vdash P_a}{\Sigma : \Psi, N \vdash P_a}\ decide_m \qquad \frac{}{\Sigma : \Psi \Downarrow A \vdash A}\ init$$

$$\frac{\Sigma : \Psi \Downarrow B_i \vdash A}{\Sigma : \Psi \Downarrow B_1 \wedge B_2 \vdash A}\ \wedge L_i \qquad \frac{\Sigma : \Psi \Downarrow B[t/x] \vdash A}{\Sigma : \Psi \Downarrow \forall_\tau x.B \vdash A}\ \forall L \qquad \frac{\Sigma : \Psi \vdash B \quad \Sigma : \Psi \Downarrow C \vdash A}{\Sigma : \Psi \Downarrow B \supset C \vdash A}\ \supset\!\mathrm{L}$$

Figure 4 The LJT^- proof system

$$\frac{\Sigma : \Psi \vdash B_i}{\Sigma : \Psi \vdash B_1 \vee B_2}\ \vee R \qquad \frac{\Sigma : \Psi \vdash B[t/x]}{\Sigma : \Psi \vdash \exists x.B}\ \vee R \qquad \frac{}{\Sigma : \Psi \vdash \mathsf{t}^+}\ \mathsf{t}^+R \qquad \frac{\Sigma : \Psi \vdash B \quad \Sigma : \Psi \vdash C}{\Sigma : \Psi \vdash B \wedge^+ C}\ \wedge^+ R$$

$$\frac{\Sigma : \Psi, \mathcal{P} \Uparrow \mathcal{P} \vdash P_a}{\Sigma : \Psi, \mathcal{P} \vdash P_a}\ invert \qquad \frac{\Sigma : \Psi, \mathcal{N} \vdash P_a}{\Sigma : \Psi \Uparrow \mathcal{N} \vdash P_a}\ done$$

$$\frac{\Sigma : \Psi \Uparrow B, \Gamma \vdash P_a \quad \Sigma : \Psi \Uparrow C, \Gamma \vdash P_a}{\Sigma : \Psi \Uparrow B \vee C, \Gamma \vdash P_a}\ \vee L \qquad \frac{}{\Sigma : \Psi \Uparrow \mathsf{f}, \Gamma \vdash P_a}\ \mathsf{f}L$$

$$\frac{\Sigma : \Psi \Uparrow B, C, \Gamma \vdash P_a}{\Sigma : \Psi \Uparrow B \wedge^+ C, \Gamma \vdash P_a}\ \wedge^+ L \qquad \frac{\Sigma : \Psi \Uparrow \Gamma \vdash P_a}{\Sigma : \Psi \Uparrow \mathsf{t}^+, \Gamma \vdash P_a}\ \mathsf{t}^+ L \qquad \frac{\Sigma : \Psi \Uparrow B[t/x], \Gamma \vdash P_a}{\Sigma : \Psi \Uparrow \exists x.B, \Gamma \vdash P_a}\ \exists L$$

Here, P_A ranges over either positive formulas or atomic formulas, and $\mathcal{P}$ (in the *invert* rule) is a non-empty multiset of positive formulas.

Figure 5 The additional rules for the $\mathsf{LJT}^\pm$ proof system.