Linear Logic Using Negative Connectives

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Invertibility

A rule is *invertible* if whenever the conclusion is provable, the premises are provable.

$$\frac{\Gamma, B \vdash C}{\Gamma \vdash B \supset C} \qquad \frac{\Gamma, B \vdash E \quad \Gamma, C \vdash E}{\Gamma, B \vee C \vdash E}$$

Invertibility is an important property of inference rules to observe.

- When searching for proofs, invertible rules yield *don't-care non-determinism* (no backtracking needed).
- Gentzen never considered this property in his publications.
- Ketonen [1944] recognized its importance and restructured Gentzen's LK calculus around invertible rules.
- The popular G3 sequent calculus proof system is designed to maximize the presence of invertible rules.

Polarity

A logical connective is

- *negative* if its right-introduction rule is invertible.
- *positive* if its left-introduction rule is invertible.

This classification can be *ambiguous* (a connective can be both positive and negative) or *partial* (a connective might not be either).

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Two striking facts about linear logic.

- This classification is unambiguous and total.

Negative	Positive
$\bot$, $⅋$, $\top$, $\&$, $\forall$, $?$	$\mathbf{1}$, $\otimes$, $\mathbf{0}$, $\oplus$, $\exists$, $!$

- De Morgan duality flips polarity.

Polarity and focused proofs

Focused proofs organize proofs into two phases.

- the invertible or negative phase
- the non-invertible or positive phase

Uniform proofs [LICS 1987] used goal reduction and backward chaining as two phases (for a subset of intuitionistic logic).

Andreoli [JLC 1992] provided the first comprehensive focused proof system by using polarization and working with linear logic.

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Focused proof systems for classical and intuitionistic logics.

- LKT, LKQ: Danos, Joinet, and Schellinx [1993]
- LJT, LJQ: Heberlin's PhD [1995]
- LJF, LKF: Liang & M [CSL 2007]: generalizes the others

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- $\mathcal{L}_0 = \{\top, \&, \Rightarrow, \forall\}$ captures the core of intuitionistic logic.
- $\mathcal{L}_1 = \mathcal{L}_0 \cup \{\multimap\}$ corresponds to a linear intuitionistic logic.
- $\mathcal{L}_2 = \mathcal{L}_1 \cup \{\bot, \invamp\}$ is a complete set for linear logic.

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- $\mathcal{L}_2 = \mathcal{L}_1 \cup \{\bot, \parr\}$ is a complete set for linear logic.

$$
\begin{array}{rl@{\qquad}rl}
\mathbf{0} & \equiv \top \multimap \bot & B \oplus C & \equiv ((B \multimap \bot) \mathbin{\&} (C \multimap \bot)) \multimap \bot \\
\mathbf{1} & \equiv \bot \multimap \bot & B \otimes C & \equiv (B \multimap \bot) \multimap (C \multimap \bot) \multimap \bot \\
?\, B & \equiv (B \multimap \bot) \Rightarrow \bot & !\, B & \equiv (B \Rightarrow \bot) \multimap \bot \\
 & & \exists x.B & \equiv (\forall x.B \multimap \bot) \multimap \bot
\end{array}
$$

The $\parr$ connective is redundant: $B \parr C \equiv (B \multimap \bot) \multimap C$.

The focused proof system $\Downarrow\mathcal{L}_0$

$$\frac{}{\Psi \vdash \top}\ \top R \qquad \frac{\Psi \vdash B \quad \Psi \vdash C}{\Psi \vdash B \,\&\, C}\ \&\mathrm{R}$$

$$\frac{B, \Psi \vdash C}{\Psi \vdash B \Rightarrow C}\ \Rightarrow\!\mathrm{R}$$

$$\frac{\Psi, B \Downarrow B \vdash A}{\Psi, B \vdash A}\ \mathit{decide}! \qquad (A \text{ is atomic})$$

$$\frac{}{\Psi \Downarrow A \vdash A}\ \mathit{init} \qquad \frac{\Psi \vdash B \quad \Psi \Downarrow C \vdash A}{\Psi \Downarrow B \Rightarrow C \vdash A}\ \Rightarrow\!\mathrm{L}$$

$$\frac{\Psi \Downarrow B_i \vdash A}{\Psi \Downarrow B_1 \,\&\, B_2 \vdash A}\ \&\mathrm{L}_i$$

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$\beta\eta$-long normal form of simply typed λ-terms

To account for simply typed λ-terms, ignore $\top$, &, and $\forall$.

Think of Ψ as a typing context and assume that

$$h\colon \beta_1 \Rightarrow \cdots \Rightarrow \beta_m \Rightarrow \beta_0 \in \Psi,$$

where β_0 atomic.

$$\dfrac{\dfrac{\dfrac{\Psi, \bar{\alpha} \vdash t_1\colon \beta_1 \quad \cdots \quad \Psi, \bar{\alpha} \vdash t_m\colon \beta_m \quad \dfrac{(\alpha_0 = \beta_0)}{\Psi, \bar{\alpha} \Downarrow \beta_0 \vdash \alpha_0}}{\Psi, \bar{x}\colon \bar{\alpha} \Downarrow h\colon \beta_1 \Rightarrow \cdots \Rightarrow \beta_m \Rightarrow \beta_0 \vdash h\bar{t}\colon \alpha_0} \Rightarrow L*}{\Psi, x_1 : \alpha_1, \ldots, x_1 : \alpha_n \vdash h\bar{t}\colon \alpha_0}\ \mathit{decide}!}{\Psi \vdash \lambda\bar{x}.h\bar{t}\colon \alpha_1 \Rightarrow \cdots \Rightarrow \alpha_n \Rightarrow \alpha_0} \Rightarrow R*$$

The focused proof system $\Downarrow\mathcal{L}_1$

We add a new linear zone to sequents.

$$\frac{}{\Psi; \Gamma \vdash \top} \qquad \frac{\Psi; \Gamma \vdash B \quad \Psi; \Gamma \vdash C}{\Psi; \Gamma \vdash B \mathbin{\&} C}$$

$$\frac{\Psi; B, \Gamma \vdash C}{\Psi; \Gamma \vdash B \multimap C} \qquad \frac{B, \Psi; \Gamma \vdash C}{\Psi; \Gamma \vdash B \Rightarrow C}$$

$$\frac{\Psi, B; \Gamma \Downarrow B \vdash A}{\Psi, B; \Gamma \vdash A} \qquad \frac{\Psi; \Gamma \Downarrow B \vdash A}{\Psi; \Gamma, B \vdash A} \qquad \frac{}{\Psi; \cdot \Downarrow A \vdash A}$$

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Synthetic inference rules

Synthetic inference rule are defined using focused proof systems.

Border sequents are of the form $\Psi; \Gamma \vdash A$, where A is atomic.

A proof of a border sequent must end with a decide rule.

Synthetic rules have border sequents as conclusions and premises.

$$
\dfrac{\cdots \qquad \dfrac{\dfrac{\Psi_1; \Gamma_1 \vdash A_1 \quad \cdots \quad \Psi_n; \Gamma_n \vdash A_n}{\vdots}\ \textit{right rules}}{} \qquad \cdots}{\dfrac{\Psi; \Gamma' \Downarrow B \vdash A}{\Psi; \Gamma \vdash A}\ \textit{decide}}\ \textit{left rules}
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Cut elimination holds automatically for synthetic rules.

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Synthetic rules (examples)

Let a, b, c be propositional constants.

Assume that Ψ contains the formula $a \multimap b \multimap c$.

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Instead, assume that Ψ contains the $a \Rightarrow b \Rightarrow c$.

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Primer on multiset rewriting

$\alpha_1 : \{a, b\} \to \{c, d\}$

$\alpha_2 : \{e, e\} \to \{f\}$

$$\{a, b, e, e, \Gamma\} \quad \begin{array}{c} \xrightarrow{\alpha_1;\alpha_2} \\ \xrightarrow{\alpha_2;\alpha_1} \\ \xrightarrow{\alpha_1|\alpha_2} \end{array} \quad \{c, d, f, \Gamma\}$$

$\alpha_3 : \{a, k\} \to \{b, k\}$

$\alpha_4 : \{c, k\} \to \{d, k\}$

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The parallel application $\alpha_3|\alpha_4$ is not possible here.
The symbol k acts as a lock.

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Rewriting the multiset on the left: $a \otimes b \multimap c \otimes d$

Since we do not have $\otimes$, we can write it using

$$a \otimes b \equiv (a \multimap b \multimap \bot) \multimap \bot$$

This rule can be written as

$$((a \multimap b \multimap \bot) \multimap \bot) \multimap ((c \multimap d \multimap \bot) \multimap \bot)$$
$$(c \multimap d \multimap \bot) \multimap (a \multimap b \multimap \bot)$$

Since $\mathcal{L}_1$ does not have $\bot$, introduce a new symbol, say q, and encode the rewriting rule as

$$(c \multimap d \multimap q) \multimap (a \multimap b \multimap q) \in \Psi$$

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Full linear logic $\mathcal{L}_2$

Multiple conclusion sequents and multifocused sequents.

$$\Psi; \Gamma \vdash \Delta \qquad \Psi; \Gamma \Downarrow \Theta_1 \vdash \Theta_2 \Downarrow \Delta$$

- Γ, Δ, Θ_1, Θ_2 are multisets
- Θ_1, Θ_2 are the focused formulas: $\Theta_1 \cup \Theta_2$ is non-empty.

- Ψ is the *unbounded* (left) zone (a multiset treated as a set)
- Γ is the *left-bounded* zone
- Δ is the *right-bounded* zone

Border sequents are now of the form $\Psi; \Gamma \vdash \mathcal{A}$, where $\mathcal{A}$ is a multiset of atomic formulas.

The focused proof system $\Downarrow\mathcal{L}_2$: negative phase

$$\frac{}{\Psi; \Gamma \vdash \top, \Delta} \qquad \frac{\Psi; \Gamma \vdash B, \Delta \quad \Psi; \Gamma \vdash C, \Delta}{\Psi; \Gamma \vdash B \,\&\, C, \Delta}$$

$$\frac{B, \Psi; \Gamma \vdash C, \Delta}{\Psi; \Gamma \vdash B \Rightarrow C, \Delta} \qquad \frac{\Psi; B, \Gamma \vdash C, \Delta}{\Psi; \Gamma \vdash B \multimap C, \Delta}$$

$$\frac{\Psi; \Gamma \vdash \Delta}{\Psi; \Gamma \vdash \bot, \Delta} \qquad \frac{\Psi; \Gamma \vdash B, C, \Delta}{\Psi; \Gamma \vdash B \parr C, \Delta}$$

$$\frac{\Psi_1, \Psi_2; \Gamma_1 \Downarrow \Psi_2, \Gamma_2 \vdash \cdot \Downarrow \mathcal{A}}{\Psi_1, \Psi_2; \Gamma_1, \Gamma_2 \vdash \mathcal{A}}\ decide_m$$

The restrictions on $decide_m$:

- the union Ψ_2, Γ_2 is non-empty, and
- $\mathcal{A}$ is a multiset of atomic formulas.

The focused proof system $\Downarrow\mathcal{L}_2$: negative phase

$$\frac{}{\Psi; \Gamma \vdash \top, \Delta} \qquad \frac{\Psi; \Gamma \vdash B, \Delta \quad \Psi; \Gamma \vdash C, \Delta}{\Psi; \Gamma \vdash B \,\&\, C, \Delta}$$

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The focused proof system $\Downarrow\mathcal{L}_2$: positive phase

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$$\frac{\Psi; \cdot \vdash B \qquad \Psi; \Gamma \Downarrow C, \Theta \vdash \Theta' \Downarrow \mathcal{A}}{\Psi; \Gamma \Downarrow B \Rightarrow C, \Theta \vdash \Theta' \Downarrow \mathcal{A}}$$

$$\frac{\Psi; \Gamma_1 \Downarrow \Theta_1 \vdash \Theta_3, B \Downarrow \mathcal{A}_1 \quad \Psi; \Gamma_2 \Downarrow C, \Theta_2 \vdash \Theta_4 \Downarrow \mathcal{A}_2}{\Psi; \Gamma_1, \Gamma_2 \Downarrow B \multimap C, \Theta_1, \Theta_2 \vdash \Theta_3, \Theta_4 \Downarrow \mathcal{A}_1, \mathcal{A}_2}$$

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$$\frac{}{\Psi; \cdot \Downarrow \bot \vdash \cdot \Downarrow \cdot} \quad \frac{\Psi; \Gamma \Downarrow B_i, \Theta \vdash \Theta' \Downarrow \mathcal{A}}{\Psi; \Gamma \Downarrow B_1 \,\&\, B_2, \Theta \vdash \Theta' \Downarrow \mathcal{A}}$$

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The focused zones are treated linearly. Release is not incremental.

Conservativity results

Proposition: Let Ξ be a $\Downarrow\mathcal{L}_2$ proof of the sequent $\cdot ; \cdot \vdash B$.

- If B is in $\mathcal{L}_1$ then every sequent in Ξ is a *single-conclusion* and *single-focused* sequent.
- If B is in $\mathcal{L}_0$ then every sequent in Ξ has an empty left-bounded zone.

Thus, $\Downarrow\mathcal{L}_0$ and $\Downarrow\mathcal{L}_1$ arise as simple restrictions on $\Downarrow\mathcal{L}_2$.

The single-conclusion nature of proofs in $\Downarrow\mathcal{L}_1$ is not imposed (as Gentzen did to get LJ from LK) but is a consequence.

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Cut admissibility

Theorem: The following cut rules are admissible in $\Downarrow\mathcal{L}_2$.

$$\frac{\Psi; \cdot \vdash B \qquad \Psi, B; \Gamma \vdash \Delta}{\Psi; \Gamma \vdash \Delta} \qquad \frac{\Psi; \Gamma_1 \vdash B, \Delta_1 \qquad \Psi; \Gamma_2, B \vdash \Delta_2}{\Psi; \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}$$

This theorem is proved as a cut-elimination theorem in the extended version of the paper and in Chapter 7 of [Miller, 2025].

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Completeness of $\Downarrow\mathcal{L}_2$: If B is an $\mathcal{L}_2$ formula provable in linear logic, then the sequent $\cdot; \cdot \vdash B$ has a $\Downarrow\mathcal{L}_2$-proof.

Proved as a simple consequence of the cut-elimination theorem.

Rewriting the right-bounded zone

$\alpha_1 : \{a, b\} \to \{c, d\}$

$\alpha_2 : \{e, e\} \to \{f\}$

Encoding α_1 using $c \parr d \multimap a \parr b$ yields the synthetic rule.

$$\dfrac{\dfrac{\dfrac{\Psi; \Gamma \vdash \Delta, c, d}{\Psi; \Gamma \vdash \Delta, c \parr d} \quad \dfrac{\dfrac{}{\Psi; \cdot \Downarrow a \vdash a} \quad \dfrac{}{\Psi; \cdot \Downarrow b \vdash b}}{\Psi; \cdot \Downarrow a \parr b \vdash a, b}}{\Psi; \Gamma \Downarrow c \parr d \multimap a \parr b \vdash \Delta, a, b}}{\Psi; \Gamma \vdash \Delta, a, b}$$

Focusing on α_1 and α_2 simultaneously yields the synthetic rule

$$\frac{\Psi; \Gamma \vdash \Delta, c, d, f}{\Psi; \Gamma \vdash \Delta, a, b, e, e}\ \alpha_1 | \alpha_2.$$

Parallel rule application yields new synthetic rules.

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Maximally multifocusing proofs

The interesting results surrounding multifocusing proofs are related to *maximally multifocused proofs* (MMF): these often correspond to *canonical proof structures*.

- $\beta\eta$-long normal λ-terms as MMF proofs (since single-focused proofs in $\mathcal{L}_0$ are multifocused proofs)
- proof nets for MALL as MMF proofs [Chaudhuri, M, & Saurin, 2008]
- expansion proof for classical first-order logic as MMF LKF proofs [Chaudhuri, Hetzel, & M, 2012].

A canonical treatment of $\vee$ and $\exists$ in natural deduction

The negative connectives of intuitionistic logic directly translate into $\mathcal{L}_0$.

$$\mathrm{t}^\circ = \top \qquad (B \wedge C)^\circ = B^\circ \mathbin{\&} C^\circ \qquad (B \supset C)^\circ = B^\circ \Rightarrow C^\circ$$

$$(\forall x.B)^\circ = \forall x.B^\circ \qquad A^\circ = A \text{ for atomic formulas } A$$

Focused proofs for negative connectives correspond to *natural deduction*: Herbelin [CSL 1994] & Espírito Santo [TLCA 2007].

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Focused proofs for negative connectives correspond to *natural deduction*: Herbelin [CSL 1994] & Espírito Santo [TLCA 2007].

We can translate the positive connectives into $\mathcal{L}_2$ using $\bot$.

$$\begin{aligned} \mathsf{f}^\circ &= \top \multimap \bot \\ (B \vee C)^\circ &= ((B^\circ \Rightarrow \bot) \,\&\, (C^\circ \Rightarrow \bot)) \multimap \bot \\ (\exists x.B)^\circ &= (\forall x.(B^\circ \Rightarrow \bot)) \multimap \bot \end{aligned}$$

Extending the correspondence to natural deduction permits a new treatment for positive connectives.

Parallel elimination rules

Let $p \geq 1$ and $a_1, \ldots, a_p, b_1, \ldots, b_p$ be atomic formulas.

An example: The parallel application of the $\vee$-elimination rule is:

$$\dfrac{a_1 \vee b_1 \quad \cdots \quad a_p \vee b_p \quad \left(\begin{array}{c} \{a_i \mid i \in I\} \cup \{b_i \mid i \notin I\} \\ \vdots \\ D \end{array} \right)_{I \subseteq \{1,\ldots,p\}}}{D}$$

This rule has $p + 2^p$ premises.

Maximal multifocusing (maximal use of parallel elimination rules) corresponds to Prawitz's *maximal segments*.

Anyone know a citation for this style rule?

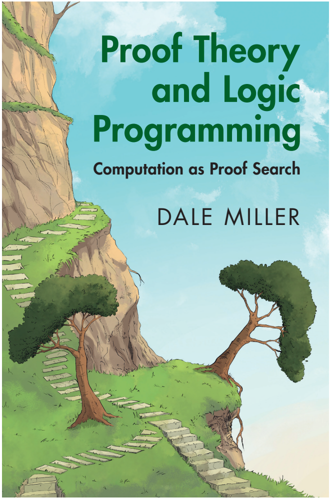
Proof Theory and Logic Programming: Computation as proof search, by Dale Miller

To be published by Cambridge University Press by December 2025.

Preprint available from my web page. `https://www.lix.polytechnique.fr/Labo/Dale.Miller/ptlp/` (317 pages, 90 exercises).

Organizes everything I learned about the intersection of proof theory and logic programming during four decades (1985-2025).

Uses classical, intuitionistic, and linear logic (first-order and higher-order) to design and reason about logic programs.



Art by Nadia Miller

Conclusions

1. Decomposed linear logic into $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_2$.
2. Proved that the multifocused, multiple-conclusion sequent proof system $\Downarrow\mathcal{L}_2$
   - satisfies cut-elimination and is complete for linear logic;
   - captures both $\Downarrow\mathcal{L}_0$ and $\Downarrow\mathcal{L}_1$; and
   - supports parallel rule application via the presence of $\bot$.
3. Showed that adding multifocusing or multiple conclusions to $\Downarrow\mathcal{L}_0$ and $\Downarrow\mathcal{L}_1$ does not change provability.
4. Demonstrated that synthetic rules
   - are formally defined using focused proofs,
   - automatically satisfy cut-elimination, and
   - can capture parallel rule application.
5. Provided a candidate for canonical proof format for natural deduction with positive connectives.