Non-Hausdorff manifolds over locally ordered spaces via sheaf theory

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joint work with Emmanuel Haucourt

Outline

Why blowing up?

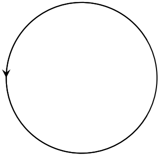
(Concurrent) programs

(tensor product of) graphs ⇝ (realization) locally ordered topological spaces ⇝ (blowup) manifolds

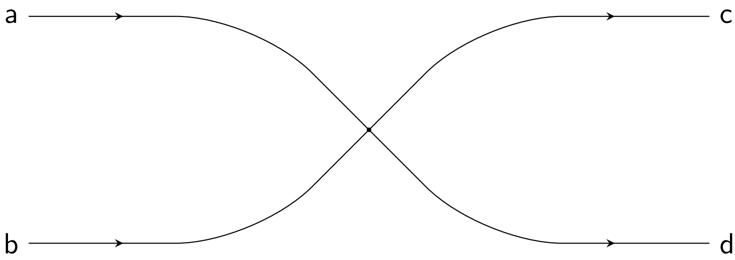
Why blowing up?

(Concurrent) programs

(tensor product of) graphs — realization → locally ordered topological spaces — blowup → manifolds



Graph blowup



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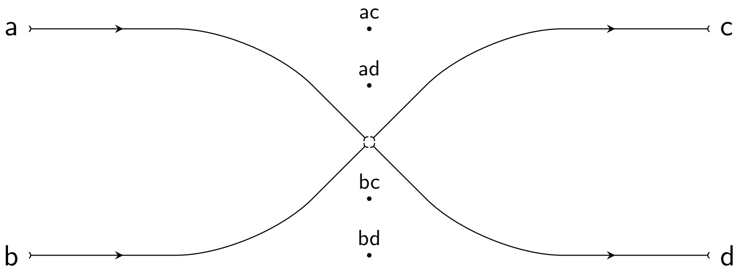

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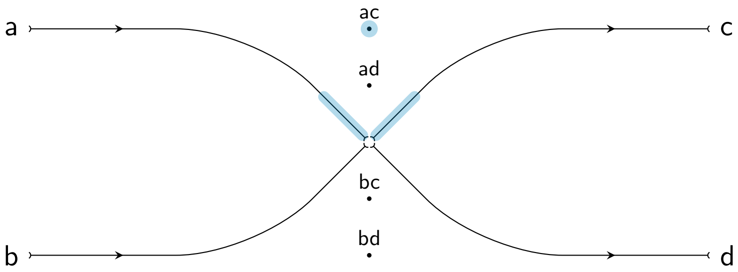

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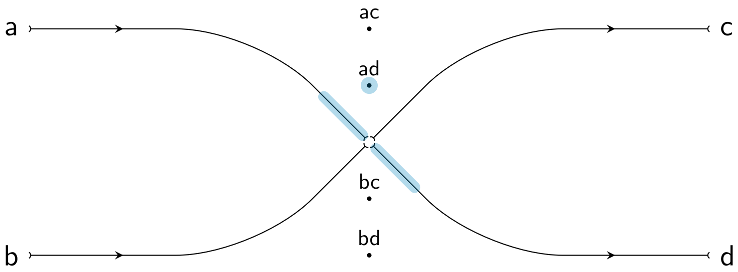

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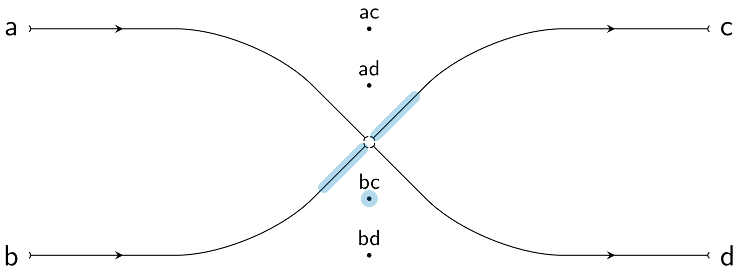

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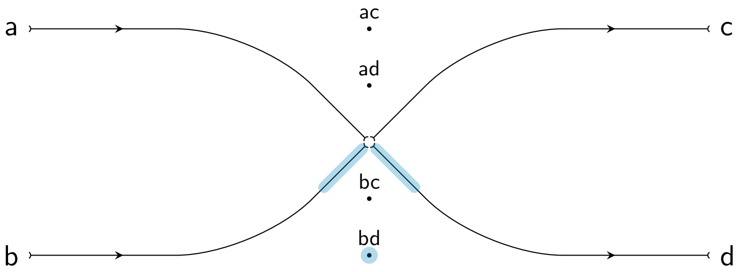
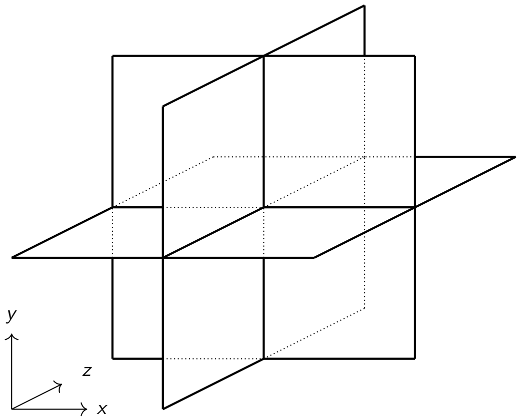


An example which is not a graph product



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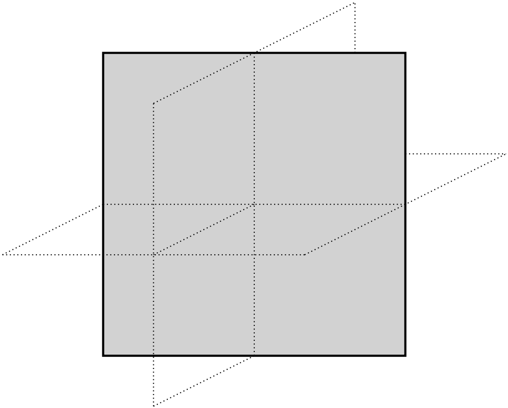

$\{z = 0\}$

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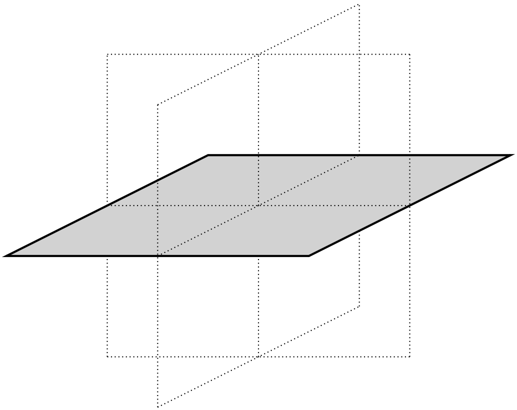

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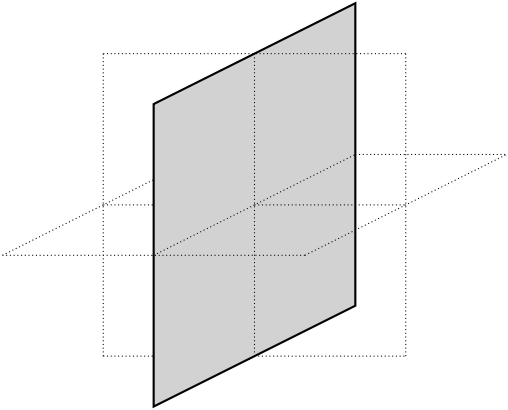

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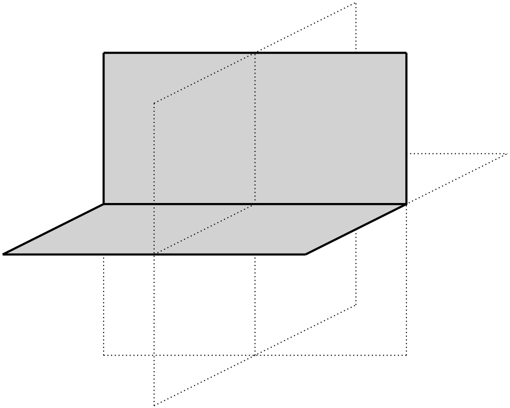

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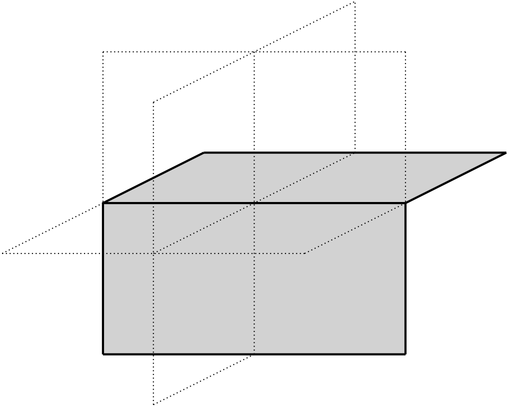

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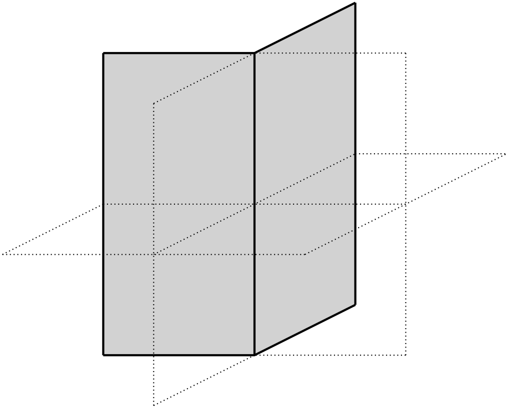

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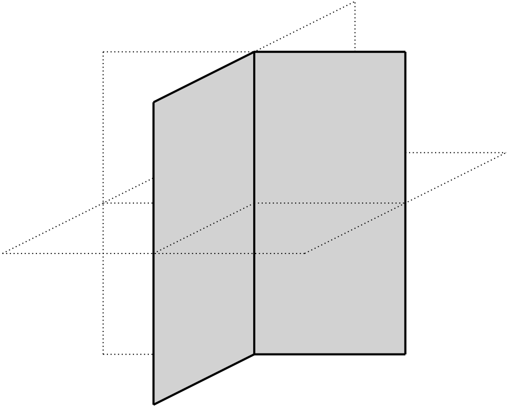

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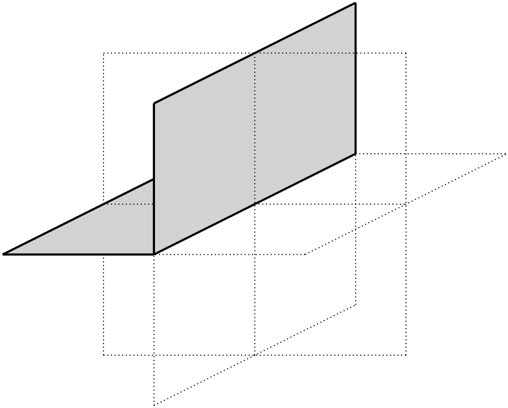

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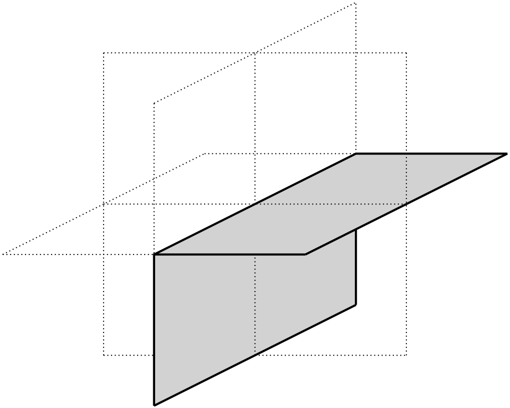

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Outline

Some sheaf theory

Let X be a topological space.

Definition

- $O(X)$ is the poset category of open sets of X.
- A *presheaf* on X is a functor $O(X)^{op} \to \mathbf{Set}$.
- A presheaf F is a *sheaf* if the following diagram is an equalizer:

$$F(U) \longrightarrow \Pi_{i \in I} F(U_i) \rightrightarrows \Pi_{i,j \in I} F(U_i \cap U_j)$$

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Example

$F : U \mapsto \mathcal{C}^0(U, \mathbb{R})$ with restriction defines a sheaf on X. In this case, the sheaf condition corresponds to the fact that a family $f_i : U_i \to \mathbb{R}$ satisfying

$$(f_i)_{|U_i \cap U_j} = (f_j)_{|U_i \cap U_j}$$

can be amalgamated into a unique $f : \bigcup_i U_i \to \mathbb{R}$.

The fundamental correspondance

Definition

$p : E \to X$ is a *local homeomorphism* if every $x \in E$ has an open neighborhood U such that $p_{|U}$ is a homeomorphism. $Et(X)$ is the category of local homeomorphisms over X.

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There is an adjunction between $Psh(X)$ *and* Top/X *that restricts to an equivalence*

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Example

Let M be a manifold. The above correspondance sends the tangent bundle $p : TM \to M$ of M to the sheaf of vector fields of M defined by:

$$U \mapsto \{s : U \to TM \,|\, s \text{ continuous}, \; p \circ s = \text{id}\}$$

In the other direction...

Let F be a sheaf on X.

Definition

Let $x \in X$. The *set of germs of* F *at* x is given by

$$F_x := \operatorname{colim}_{U \ni x} F(U)$$

For $s \in F(U)$, the *germ of* s *at* x, noted s_x, is the equivalence class of s in F_x.

To build a bundle out of F, just take the projection $p : \bigsqcup_{x \in X} F_x \to X$. The open sets are generated by

$$U_s := \{s_x \,|\, x \in U\}$$

for $s \in F(U)$.

Outline

The blowup construction

Let X be a locally ordered space.

Definition

The *sheaf of n-traversals* $O_X^{(n)}$ is defined by:

$$O_X^{(n)}(U) := \{A \subseteq U \,|\, A \cong E \text{ with } E \text{ } n\text{-euclidean local order}\}$$

for an open set $U \subseteq X$, the restriction map $O_X^{(n)}(V) \to O_X^{(n)}(U)$ for $U \subseteq V$ being given by intersection with U.

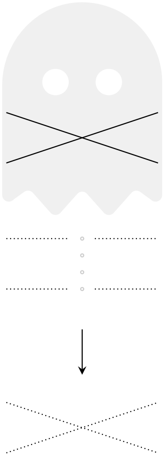
Definition

The *n-blowup of X* is defined by

$$\tilde{X} := \sqcup_{x \in X} \{A_x \in O_{X,x}^{(n)} \,|\, x \in A\} \subseteq \phi(O_X^{(n)})$$

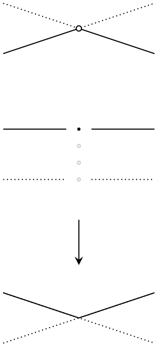
(with the induced topology and local order). The *n-blowup map* is the projection $\beta_X : \tilde{X} \to X$.

Back to the example



The ghost space,
i.e. the elements $\varnothing_x$, $x \in X$

Back to the example



U_A

$A \in O_n^{(X)}(U)$

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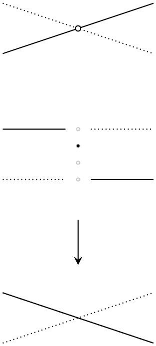

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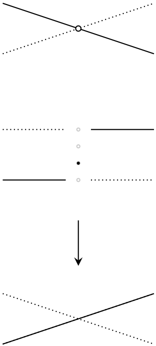

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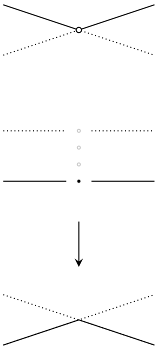

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The universal property

Theorem

The n-blowup $\tilde{X}$ satisfies the following universal property: for any n-euclidean local order E, for any local embedding $f : E \to X$, there is a unique continuous map $\tilde{f} : E \to \tilde{X}$ making the following diagram commute

$$\begin{array}{ccc} & & \tilde{X} \\ & \overset{\tilde{f}}{\nearrow} & \downarrow \beta_X \\ E & \xrightarrow[f]{} & X \end{array}$$

Moreover, $\tilde{f}$ is a local embedding.

Corollary

The category of n-euclidean local orders and local embeddings is a coreflective subcategory of the category of local orders and weakly euclidean morphisms (or local embeddings).

Outline

The concrete case

Definition

- A *graph* is a presheaf on the category $0 \rightrightarrows 1$.
- $\square$ is the free monoidal category on $0 \rightrightarrows 1$ with unit 0.
- A *precubical set* is a presheaf on $\square$.

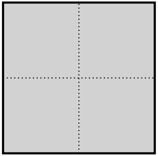
Informally, a precubical set is a set of n-cubes for every $n \in \mathbb{N}$, glued together on their bounderies. A precubical set can be realized as a locally ordered space.

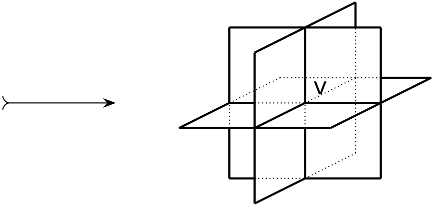
Proposition

The blowup of the locally ordered realization of a precubical set admits a combinatorial expression.

This description is encapsulated in a presheaf over the category of elements of the precubical set.

Back to the (other) example...





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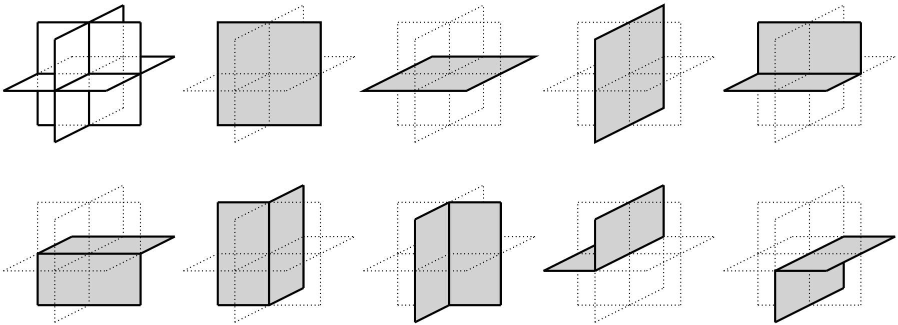

Final words

- The tools developped can be used to prove rigidity results on locally ordered realizations.
- We can see this contruction as an instance of an abstract construction on *geometric contexts*.
- We conjecture the existence of a smooth atlas on the blowup of a precubical set.

References

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