Concurrency 4

CCS - Simulation and bisimulation. Coinduction.

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James Leifer (INRIA Rocquencourt)
Eric Goubault (CEA)

http://pauillac.inria.fr/~leifer/teaching/mpri-concurrency-2005/

Announcement

The class of Wednesday 26 October will follow the usual schedule (16h15 - 19h15).

Outline

The semaphore

Define in CCS a semaphore with initial value n

First Solution

$$\mathrm{rec}_{S_n}\, down.\mathrm{rec}_{S_{n-1}}(up.S_n + down.\mathrm{rec}_{S_{n-2}}(\ldots(up.S_2 + down.\mathrm{rec}_{S_0}\, up.S_1)\ldots))$$

Second solution

- Let $S = \mathrm{rec}_X down.up.X$
- Then $S_n = S \mid S \mid \ \ldots \mid S \quad n$ times

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Maximal trace equivalence is not a congruence

Consider the following processes

- $P = a.(b.0 + c.0)$
- $Q = a.b.0 + a.c.0$
- $R = \bar{a}.\bar{b}.\bar{d}.0$

P and Q have the same traces, but $(\nu a)(\nu b)(\nu c)(P \mid R)$ and $(\nu a)(\nu b)(\nu c)(Q \mid R)$ have different maximal traces.

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Syntax of "modern" CCS

- (channel, port) names: $a, b, c, \ldots$
- co-names: $\bar{a}, \bar{b}, \bar{c}, \ldots$ Note: $\bar{\bar{a}} = a$
- silent action: τ
- actions, prefixes: $\mu ::= a \mid \bar{a} \mid \tau$
- processes:

$$
\begin{array}{llll}
P, Q & ::= & 0 & \text{inaction} \\
 & \mid & \mu.P & \text{prefix} \\
 & \mid & P \mid Q & \text{parallel} \\
 & \mid & P + Q & \text{(external) choice} \\
 & \mid & (\nu a)P & \text{restriction} \\
 & \mid & K(\vec{a}) & \text{process name with parameters}
\end{array}
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- Process definitions:

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Labeled transition system for “modern” CCS

We assume a given set of definitions D

$$[\text{Act}]\quad \frac{}{\mu.P \xrightarrow{\mu} P} \qquad\qquad [\text{Res}]\quad \frac{P \xrightarrow{\mu} P' \quad \mu{\neq}a,\overline{a}}{(\nu a)P \xrightarrow{\mu} (\nu a)P'}$$

$$[\text{Sum1}]\quad \frac{P \xrightarrow{\mu} P'}{P{+}Q \xrightarrow{\mu} P'} \qquad\qquad [\text{Sum2}]\quad \frac{Q \xrightarrow{\mu} Q'}{P{+}Q \xrightarrow{\mu} Q'}$$

$$[\text{Par1}]\quad \frac{P \xrightarrow{\mu} P'}{P|Q \xrightarrow{\mu} P'|Q} \qquad\qquad [\text{Par2}]\quad \frac{Q \xrightarrow{\mu} Q'}{P|Q \xrightarrow{\mu} P|Q'}$$

$$[\text{Com}]\quad \frac{P \xrightarrow{a} P' \quad Q \xrightarrow{\overline{a}} Q'}{P|Q \xrightarrow{\tau} P'|Q'} \qquad\qquad [\text{Rec}]\quad \frac{P[\vec{a}/\vec{x}] \xrightarrow{\mu} P' \quad K(\vec{x}) \overset{def}{=} P \in D}{K(\vec{a}) \xrightarrow{\mu} P'}$$

The reason for moving to “modern” CCS was to get static scope (thanks to the presence of the parameters). The old version had dynamic scope.

Outline

Simulation

- **Definition** We say that a relation R on processes is a *simulation* if

$$P \mathcal{R} Q \quad \text{implies that} \quad \textit{if } P \xrightarrow{\mu} P' \textit{ then } \exists Q' \textit{ s.t. } Q \xrightarrow{\mu} Q' \textit{ and } P' \mathcal{R} Q'$$

- Note that this property does not uniquely defines $\mathcal{R}$. There may be several relations that satisfy it.
- Define $\lesssim \; = \; \bigcup\{\mathcal{R} \mid \mathcal{R} \text{ is a simulation}\}$
- **Theorem** $\lesssim$ is a bisimulation (Proof: Exercise)
- $P \lesssim Q$ intuitively means that Q can do everything that P can do. Q simulates P.

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Proof methods

- Simulation and bisimulation are coinductive definitions.
- In order to prove that $P \lesssim Q$ it is sufficient to find a simulation $\mathcal{R}$ such that $P \mathcal{R} Q$
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Examples and exercises

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  - $P = a.(b.0 + c.0)$
  - $Q = a.b.0 + a.c.0$

  Prove that $Q \lesssim P$ but $P \not\lesssim Q$ and $Q \not\sim P$
- Assume that $Q \lesssim P$ and $P \lesssim Q$ (for two generic P and Q). Does it follow that $P \sim Q$?
- Consider the following processes
  - $R = a.(b.0 + b.0)$
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  Prove that $Q \sim P$
- Consider the two definitions of semaphore given at the beginning of this lecture. Prove that they are bisimilar.
- Consider the processes $H(a)$ and $K(a)$ defined by $H(x) \stackrel{def}{=} x.H(x)$ and $K(x) \stackrel{def}{=} x.K(x) \mid x.K(x)$. Are they bisimilar?
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Bisimulation as greatest fixpoint

- Consider the set of relations on processes (that is, on the powerset of the cartesian product on processes) ordered by set inclusion. Obviously, this is a complete lattice.
- **Definition** Let $\mathcal{F}$ be a function on relation defined in the following way:

$$\begin{array}{lll} P\ \mathcal{F}(\mathcal{R})\ Q & \text{iff} & \textit{if } P \xrightarrow{\mu} P' \textit{ then } \exists Q' \textit{ s.t. } Q \xrightarrow{\mu} Q' \textit{ and } P'\ \mathcal{R}\ Q' \\ & & \textit{if } Q \xrightarrow{\mu} Q' \textit{ then } \exists P' \textit{ s.t. } P \xrightarrow{\mu} P' \textit{ and } P'\ \mathcal{R}\ Q' \end{array}$$

- **Lemma** $\mathcal{F}$ is monotonic
- **Theorem (Knaster-Tarski)** F has (unique) least and greatest fixpoints, and

$$\begin{array}{rcl} \mathit{lfp}(\mathcal{F}) & = & \bigcap\{\mathcal{R} \mid \mathcal{F}(\mathcal{R}) \subseteq \mathcal{R}\} \\ \mathit{gfp}(\mathcal{F}) & = & \bigcup\{\mathcal{R} \mid \mathcal{R} \subseteq \mathcal{F}(\mathcal{R})\} \end{array}$$

- **Corollary** $\sim\ =\ \mathit{gfp}(\mathcal{F})$
- A similar characterization, of course, holds for $\lesssim$ as well.

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- A similar characterization, of course, holds for $\lesssim$ as well.

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- Consider the set of relations on processes (that is, on the powerset of the cartesian product on processes) ordered by set inclusion. Obviously, this is a complete lattice.

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Outline

Bisimulation in CCS is a congruence

- **Definition** A relation R on a language is called *congruence* if
  - $\mathcal{R}$ is an equivalence relation (i.e. it is reflexive, symmetric, and transitive), and
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Exercises

- Complete the proof that bisimulation in CCS is a congruence
- Prove that if $P \lesssim Q$ then the traces of P are contained in the traces of Q
- Prove that if $P \sim Q$ then $P \lesssim Q$ and $Q \lesssim P$
- Prove that
  - $P + 0 \sim P$ and $P|0 \sim P$
  - $P + P \sim P$ but (in general) $P|P \not\sim P$
  - $P + Q \sim Q + P$ and $P|Q \sim Q|P$
  - $(P + Q) + R \sim P + (Q + R)$ and $(P|Q)|R \sim P|(Q|R)$