Weak Relative Pseudo-Complements of Closure Operators

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Abstract

We define the notion of weak relative pseudo-complement, and we show that, for an arbitrary lattice, the property of weak relative pseudo-complementation is strictly weaker than relative pseudo-complementation, but stronger than pseudo-complementation. Our main interest for this notion is in relation with the theory of closure operators. We prove that if a complete lattice $\mathcal{L}$ is completely inf-distributive with respect to chains, then every closure operator on $\mathcal{L}$ admits weak relative pseudo-complements with respect to continuous closure operators on $\mathcal{L}$.

1 Introduction

Closure operators have been widely used in various fields of theoretical computer science, like semantics of declarative languages and program analysis. In particular, Cousot and Cousot proved in [2] that the lattice of abstract interpretations, or equivalently the lattice of semantics derivable by abstraction ([3]), for an arbitrary program whose data-objects have a lattice structure $\mathcal{L}$, is isomorphic to the lattice of the upper closure operators on $\mathcal{L}$, $uco(\mathcal{L})$. The correspondence between the ordering relations is that a more abstract semantics corresponds to a bigger closure operator, with respect to the standard functional ordering. The bottom element in $uco(\mathcal{L})$ corresponds to the *identical abstraction*, which leaves unchanged the domain of the computation, while the top element corresponds to the *straightforward abstraction* which forgets about the data structure, mapping $\mathcal{L}$ into its top-element $\top$.

One facility offered by this result is that the meet operator in $uco(\mathcal{L})$ can be applied to compose semantics, thus getting more concrete (precise) information about the program behavior. A natural question that arises in this setting is whether it is possible to define the inverse operation, namely an operation which, starting from two semantics ρ and η for the same program, with ρ more abstract than η, gives as result the most abstract semantics φ whose composition with ρ is exactly η. Such an operation would be useful to isolate properties of ρ which are not shared by η. In terms of closure operators this amounts to define an operation analogous to the *relative pseudo-complement*, the only difference is that we require φ to be the maximum of the set C of closure operators whose meet with ρ is *equal to* η, instead than *smaller than or equal to* η. It is easy to see that if a maximum exists for the latter case, then its meet with ρ is equal to η, therefore it is also the maximum of C. In other words, the relative pseudo-complement of ρ with respect to η, when it exists, is the intended element. The converse is not true, i.e. it is possible that the intended element exists and the relative pseudo-complement does not exist, due to the fact that there can be a closure operator whose meet with ρ is smaller than η, and which is not smaller than the maximum of C.

In this paper we define formally the notion described above as a partial operation on arbitrary meet semi-lattices. We call such an operation *weak relative pseudo-complement*. We investigate conditions on $\mathcal{L}$ under which pairs of elements in $uco(\mathcal{L})$ admit the weak relative pseudo-complement. In particular, we are interested in the case in which $\mathcal{L}$ is a complete lattice (in which case also $uco(\mathcal{L})$ is a complete lattice). Our main result is that the weak relative pseudo-complement of a closure operator ρ with respect to η, for η continuous, always exists when $\mathcal{L}$ is completely inf-distributive with respect to chains (i.e. it is *chain inf-distributive*). We note that the pseudo-complement is a particular case of weak relative pseudo-complement, hence this condition also guarantees that $uco(\mathcal{L})$ is pseudo-complemented.

The chain inf-distributivity condition is much less restrictive than the condition under which $uco(\mathcal{L})$ is relatively pseudo-complemented: in fact it is known that, for a complete lattice, relative pseudo-complementation is equivalent to complete inf-distributivity ([1]), and complete inf-distributivity in $uco(\mathcal{L})$ is equivalent to distributivity ([7]). Now, for $uco(\mathcal{L})$ to be distributive, it is necessary (and sufficient) that $\mathcal{L}$ is a complete well-ordered chain ([4]). This condition is also necessary and sufficient for $uco(\mathcal{L})$ to be complemented ([8]).

The plan of the paper is as follows: In § 2 we recall some basic notions and properties about (relative) pseudo-complementation and closure operators. In § 3 we define the notion of weak relative pseudo-complement, and we study its relation with the notions of pseudo-complement and relative pseudo-complement. In § 4 we present the main result of this paper, namely we prove that if $\mathcal{L}$ is chain inf-distributive, then $uco(\mathcal{L})$ is weakly relatively pseudo-complemented with respect to continuous closure operators. In § 5 we exploit this result to give sufficient conditions for $uco(\mathcal{L})$ to be pseudo-complemented.

2 Preliminaries

In this section we summarize some definitions and well known properties concerning (relative) pseudo-complementation and closure operators of complete lattices. For more details see [1, 6, 11].

Let $\mathcal{L} = \langle L, \preceq, \wedge, \vee, \top, \bot \rangle$ be a complete lattice, where L is a set, $\preceq$ is the partial ordering relation, $\wedge$ is the meet, $\vee$ is the join, $\top$ is the top element and $\bot$ is the bottom element. In the following, unless stated otherwise, $\mathcal{L}$ will stand for the structure $\langle L, \preceq, \wedge, \vee, \top, \bot \rangle$. We will use $\bigwedge$ and $\bigvee$ for the meet and the join of sets of elements in L, respectively, and $x \prec y$ to denote that $x \preceq y$ and $x \neq y$.

The lattice $\mathcal{L}$ is *completely inf-distributive* iff it satisfies the infinite distributive law:

$$\text{if } x \in L \text{ and } Y \subseteq L \text{ then } x \wedge \bigvee Y = \bigvee_{y \in Y} (x \wedge y).$$

Given $x, y \in L$, the *relative pseudo-complement* of x with respect to y, if it exists, is the (unique) element $x * y \in L$ such that

(i) $x \wedge x * y \preceq y$,

(ii) for each $z \in L$, if $x \wedge z \preceq y$ then $z \preceq x * y$.

If $x * y$ exists for every $x, y \in L$, then $\mathcal{L}$ is *relatively pseudo-complemented*. If $\mathcal{L}$ is completely inf-distributive then $x * y$ always exists, and coincides with $\bigvee\{z \in L : x \wedge z \preceq y\}$. The converse also holds, i.e. if $\mathcal{L}$ is relatively pseudo-complemented then it is completely inf-distributive.

The element $x * \bot$, if it exists, is called the *pseudo-complement* of x, and it is denoted by x^*. If x^* exists for every $x \in L$, then $\mathcal{L}$ is *pseudo-complemented*, and we say that $\mathcal{L}$ satisfies the *Stone identity* iff for every $x \in L$, $x^* \vee x^{**} = \top$ holds.

An *(upper) closure operator* on $\mathcal{L}$ is a mapping $\rho : L \to L$ which satisfies the following conditions: for every $x, y \in L$,

(i) if $x \preceq y$ then $\rho(x) \preceq \rho(y)$,

(ii) $x \preceq \rho(x)$,

(iii) $\rho(\rho(x)) = \rho(x)$.

A closure operator ρ is uniquely determined by the set of its fixpoints, which coincides with its image $\rho(L)$. A set $X \subseteq L$ is the set of fixpoints of some closure operator iff X contains $\top$ and it is meet-closed, i.e. for any $Y \subseteq X$, $\bigwedge Y \in X$ holds.

Let $uco(L)$ be the set of all closure operators on L. We will denote by $uco(\mathcal{L})$ the complete lattice $\langle uco(L), \sqsubseteq, \sqcap, \sqcup, \mho, \iota \rangle$ where, for every $\rho, \eta \in uco(L)$, $\{\rho_i\}_{i \in I} \subseteq uco(L)$, and $x \in L$,

(i) $\rho \sqsubseteq \eta$ iff for each $y \in L$, $\rho(y) \preceq \eta(y)$,

(ii) $(\sqcap_{i \in I} \rho_i)(x) = \bigwedge_{i \in I} \rho_i(x)$,

(iii) $(\sqcup_{i \in I} \rho_i)(x) = x$ iff for each $i \in I$, $\rho_i(x) = x$,

(iv) $\mho(x) = \top$ and $\iota(x) = x$.

Note that $\rho \sqsubseteq \eta$ iff $\eta(L) \subseteq \rho(L)$. Note also that the set of fixpoints of $\sqcap_{i \in I} \rho_i$ is the smallest meet-closed set which contains all the fixpoints of each ρ_i, and that the set of fixpoints of $\sqcup_{i \in I} \rho_i$ is the intersection of all the sets of fixpoints of the ρ_i's, i.e. $(\sqcup_{i \in I} \rho_i)(L) = \bigcap_{i \in I} \rho_i(L)$.

3 Weak relative pseudo-complement

In this section we define the notion of weak relative pseudo-complement and we study its relation with the notions of relative pseudo-complement and of pseudo-complement. We will show that the property of weak relative pseudo-complementation is strictly weaker than the property of relative pseudo-complementation, but strictly stronger than the property of pseudo-complementation.

Intuitively, in a meet semi-lattice, the weak relative pseudo-complement of an element x with respect to an element y smaller than x, is defined as the greatest element z whose meet with x gives y (if such an element exists).

Definition 3.1 Let $\mathcal{L} = \langle L, \preceq, \wedge \rangle$ be a meet semi-lattice, and consider $x, y \in L$ such that $y \preceq x$. The *weak relative pseudo-complement* of x with respect to y, if it exists, is the (unique) element $wr(x, y) \in L$ such that

(i) $x \wedge wr(x, y) = y$,

(ii) for each $z \in L$, if $x \wedge z = y$ then $z \preceq wr(x, y)$.

If $wr(x, y)$ exists for every $x, y \in L$ such that $y \preceq x$, then $\mathcal{L}$ is *weakly relatively pseudo-complemented.* □

The reason why we require $y \preceq x$, in the definition of $wr(x, y)$, is that for $y \not\preceq x$ condition (i) would never be satisfied.

If we compare the above definition with the definition of relative pseudo-complement, we see that for $y \preceq x$ they are technically rather similar. The main difference is that here we are more restrictive in the set of elements z considered in the premise of condition (ii) ($x \wedge z = y$ instead of $x \wedge z \preceq y$).

If the lattice is completely inf-distributive, then it is weakly relatively pseudo-complemented and $wr(x, y)$ can be defined as the join of all the elements z satisfying the premise of condition (ii).

Remark 3.2 *If $\mathcal{L}$ is completely inf-distributive then $wr(x, y)$ exists for every $x, y \in L$ such that $y \preceq x$, and it coincides with $\bigvee\{z \in L : x \wedge z = y\}$.*

The proof is analogous to that of the corresponding property for the relative pseudo-complement (see for instance [1, 5]).

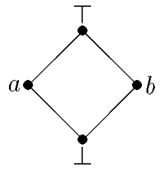
In general, if $y \preceq x$ then the existence of $x * y$ implies the existence of $wr(x, y)$.

Remark 3.3 *Let $y \preceq x$. If $x * y$ exists, then also $wr(x, y)$ exists, and $wr(x, y) = x * y$ holds.*

Proof. By condition (ii) on $x * y$ we derive $y \preceq x * y$. Hence, by condition (i) on $x * y$ and from $y \preceq x$, we get $x \wedge x * y = y$. Therefore, condition (i) in the definition of $wr(x, y)$ is satisfied. Condition (ii) for $wr(x, y)$ follows immediately from condition (ii) on $x * y$. ■

On the other hand, the existence of $wr(x, y)$ does not imply the existence of $x * y$. The following example shows this fact in a finite lattice of closure operators.

Example 3.4 Consider the lattice $\mathcal{L}$ given below.

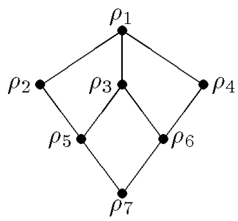


The closure operators on $\mathcal{L}$ are the following:

$$\rho_1 = \{\top\},\ \rho_2 = \{\top, a\},\ \rho_3 = \{\top, \bot\},\ \rho_4 = \{\top, b\},$$

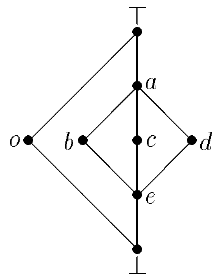
$$\rho_5 = \{\top, a, \bot\},\ \rho_6 = \{\top, b, \bot\},\ \rho_7 = \{\top, a, b, \bot\}.$$

Thus, $uco(\mathcal{L})$ is the lattice depicted below.



It is easy to see that $uco(\mathcal{L})$ is weakly relatively pseudo-complemented (and pseudo-complemented). But $\rho_2 * \rho_5$, for instance, does not exist (while $wr(\rho_2, \rho_5) = \rho_3$). □

We show now that weak relative pseudo-complementation is a stronger property than pseudo-complementation.
If $\mathcal{L}$ has a bottom element $\bot$ and it is weakly relatively pseudo-complemented, then it is pseudo-complemented. In fact, as a consequence of Remark 3.3, we have that if x^* exists, then it coincides with $wr(x, \bot)$. However, a pseudo-complemented lattice may not be weakly relatively pseudo-complemented. Consider for example the following lattice:



This lattice is pseudo-complemented, being $\top^* = \bot$, $\bot^* = \top$, $a^* = b^* = c^* = d^* = e^* = o$, and $o^* = a$. However, it is not weak relatively pseudo-complemented, as for instance $wr(b, e)$ does not exist.

4 Weak relative pseudo-complements of closure operators

In this section we study the conditions for the existence of weak relative pseudo-complements in $uco(\mathcal{L})$. In particular, we show that if $\mathcal{L}$ is a chain inf-distributive complete lattice then $uco(\mathcal{L})$ admits weak relative pseudo-complements with respect to continuous closure operators.

Definition 4.1 Let $\mathcal{L}$ be a complete lattice. A closure operator $\rho \in uco(L)$ is *continuous* iff for every chain $C \subseteq L$, $\rho(\bigvee C) = \bigvee_{x \in C} \rho(x)$ holds. □

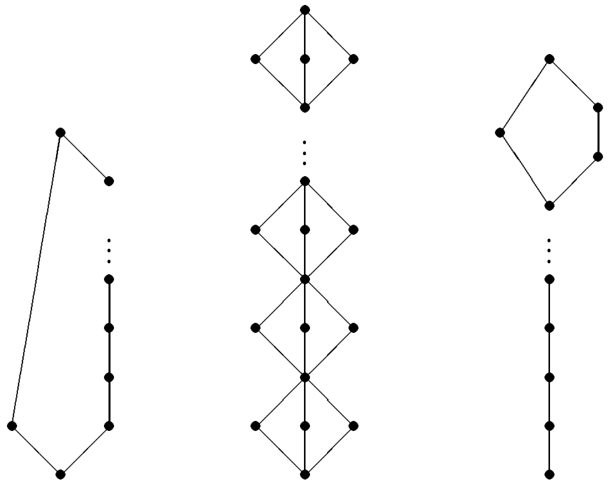


Figure 1: Chain inf-distributive lattices which are not completely inf-distributive.

Definition 4.2 A complete lattice $\mathcal{L}$ is *chain inf-distributive* iff for any chain $C \subseteq L$, and for each element $x \in L$, $x \wedge \bigvee C = \bigvee_{y \in C}(x \wedge y)$ holds. $\square$

It is worth noting that the chain inf-distributivity property is strictly weaker than the complete inf-distributivity property. For instance, any finite lattice is chain inf-distributive, but not necessarily completely inf-distributive (which, for a finite lattice, amounts to be distributive). Moreover, also in the infinite case there are lattices which are chain inf-distributive and not completely inf-distributive. Examples of such lattices are shown in Figure 1.

The following theorem expresses the main result of this paper. In the proof we use the theory of ordinal and cardinal numbers, for which we refer to [9]. The ordinal numbers will be denoted by greek letters $\alpha, \beta, \ldots$.

Theorem 4.3 *If $\mathcal{L}$ is a complete, chain inf-distributive lattice then $uco(\mathcal{L})$ is weakly relatively pseudo-complemented with respect to continuous closure operators, i.e. for every $\rho, \eta \in uco(L)$ such that $\eta \sqsubseteq \rho$ and η is continuous, there exists $wr(\rho, \eta)$.*

Proof. For $\rho, \eta \in uco(L)$ such that $\eta \sqsubseteq \rho$ and η is continuous, let $P_{\rho\eta} = \{\varphi \in uco(L) : \rho \sqcap \varphi = \eta\}$, and $\rho^* = \sqcup P_{\rho\eta}$. Notice that $\rho^*(L) = \bigcap\{\varphi(L) : \varphi \in P_{\rho\eta}\}$.
Clearly $\varphi \sqsubseteq \rho^*$ for every $\varphi \in P_{\rho\eta}$. Furthermore, $\eta \sqsubseteq \rho \sqcap \rho^*$ holds, because $P_{\rho\eta}$ contains η. Hence, we have only to show that $\rho \sqcap \rho^* \sqsubseteq \eta$, which amounts to prove that $\eta(L) \subseteq (\rho \sqcap \rho^*)(L)$.
Let $\bar{\alpha}$ be the least ordinal whose cardinality is strictly greater than the cardinalities of all chains in $\eta(L)$ (which are bounded from above by the cardinality of $\eta(L)$). We show that, if $\eta(L) \not\subseteq (\rho \sqcap \rho^*)(L)$, then we are able to construct a strictly increasing chain $\{x_\beta\}_{\beta \leq \bar{\alpha}}$ in $\eta(L)$, whose cardinality is $\bar{\alpha}$, thus obtaining a contradiction.
Assume that there exists $\bar{x} \in \eta(L)$ such that $\bar{x} \notin (\rho \sqcap \rho^*)(L)$. We prove by transfinite induction that for each ordinal α it is possible to construct in $\eta(L)$ two sets $\{x_\beta\}_{\beta \leq \alpha}$, with $x_0 = \bar{x}$, and $\{y_\beta\}_{\beta < \alpha}$, such that:

(i) $\{x_\beta\}_{\beta \leq \alpha}$ is a strictly increasing chain in $\eta(L) \setminus (\rho \sqcap \rho^*)(L)$,

(ii) $\forall \beta < \alpha.\ y_\beta \in \rho(L)$,

(iii) $x_0 = x_\alpha \wedge (\bigwedge_{\beta < \alpha} y_\beta)$, and

(iv) $\forall \beta, \gamma.\ \beta \leq \gamma < \alpha \Rightarrow x_\beta \prec y_\gamma$.

Let us consider the three cases $\alpha = 0$, α successor ordinal, and α limit ordinal.

($\alpha = 0$) Define $x_0 = \bar{x}$. Then $x_0 \in \eta(L) \setminus (\rho \sqcap \rho^*)(L)$ by definition, thus (i) is satisfied. Conditions (ii), (iii) and (iv) are vacuously satisfied.

($\alpha + 1$) By the induction hypothesis (i), $x_\alpha \notin \rho^*(L)$, hence there exists a closure operator $\varphi \in P_{\rho\eta}$ such that $x_\alpha \notin \varphi(L)$. Also, again by the induction hypothesis (i), $x_\alpha \notin \rho(L)$ and $x_\alpha \in \eta(L)$. By the latter and the fact that $\rho \sqcap \varphi = \eta$, we derive $x_\alpha \in (\rho \sqcap \varphi)(L)$. Therefore, there must exist $x \in \varphi(L) \setminus \rho(L)$ and $y \in \rho(L) \setminus \varphi(L)$ such that $x_\alpha = x \wedge y$. Define $x_{\alpha+1} = x$ and $y_\alpha = y$, and observe that (ii) and (iv) are satisfied.
We prove now (iii). By the induction hypothesis (iii), we have $x_0 = x_\alpha \wedge (\bigwedge_{\beta<\alpha} y_\beta)$, hence we derive $x_0 = x_{\alpha+1} \wedge y_\alpha \wedge (\bigwedge_{\beta<\alpha} y_\beta) = x_{\alpha+1} \wedge (\bigwedge_{\beta<\alpha+1} y_\beta)$.
Concerning (i), observe that $x_\alpha \preceq x_{\alpha+1}$, and $x_\alpha \neq x_{\alpha+1}$, since $x_\alpha \notin \varphi(L)$ while $x_{\alpha+1} \in \varphi(L)$. Furthermore, $x_{\alpha+1} \in \eta(L)$ since $\varphi(L) \subseteq \eta(L)$. Finally, note that the term $\bigwedge_{\beta<\alpha+1} y_\beta$ in (iii) is an element of $\rho(L)$, being $\rho(L)$ meet-closed. Since $\rho(L) \subseteq (\rho \sqcap \rho^*)(L)$, and $(\rho \sqcap \rho^*)(L)$ is meet-closed, we deduce that $x_{\alpha+1}$ cannot belong to $(\rho \sqcap \rho^*)(L)$, otherwise also x_0 would.

(α limit ordinal) Define $x_\alpha = \bigvee_{\beta<\alpha} x_\beta$. Conditions (ii) and (iv) are trivially satisfied by the corresponding induction hypothesis.
Consider now (iii). By the induction hypothesis (iii), we have that, for any $\beta < \alpha$, $x_0 = x_\beta \wedge (\bigwedge_{\gamma<\beta} y_\gamma)$ holds. Hence, we derive $x_0 = \bigvee_{\beta<\alpha}(x_\beta \wedge (\bigwedge_{\gamma<\beta} y_\gamma))$. By the induction hypothesis (iv), for $\beta \leq \gamma < \alpha$, $x_\beta \preceq y_\gamma$ holds. Therefore, we have $x_0 = \bigvee_{\beta<\alpha}(x_\beta \wedge (\bigwedge_{\beta\leq\gamma<\alpha} y_\gamma) \wedge (\bigwedge_{\gamma<\beta} y_\gamma)) = \bigvee_{\beta<\alpha}(x_\beta \wedge (\bigwedge_{\gamma<\alpha} y_\gamma))$, from which we get, applying the chain inf-distributivity property, $x_0 = (\bigvee_{\beta<\alpha} x_\beta) \wedge (\bigwedge_{\gamma<\alpha} y_\gamma) = x_\alpha \wedge (\bigwedge_{\gamma<\alpha} y_\gamma)$.
Finally, we show (i). Since all the elements of $\{x_\beta\}_{\beta<\alpha}$ are distinct, also x_α is strictly greater than x_β, for any $\beta < \alpha$. Furthermore, $x_\alpha \in \eta(L)$: in fact $\eta(x_\alpha) = \eta(\bigvee_{\beta<\alpha} x_\beta) =$ (by continuity of η) $= \bigvee_{\beta<\alpha} \eta(x_\beta) =$ (since, by the induction hypothesis (i), $x_\beta \in \eta(L)$, i.e. $\eta(x_\beta) = x_\beta$) $= \bigvee_{\beta<\alpha} x_\beta = x_\alpha$. The proof that $x_\alpha \notin (\rho \sqcap \rho^*)(L)$ is like in the case of α successor ordinal. ∎

In the above theorem, the chain inf-distributivity of $\mathcal{L}$ and the continuity of η play a fundamental role. In fact, in general, if $\mathcal{L}$ is not chain inf-distributive, then $wr(\rho, \eta)$ might not exist, even if η is continuous. Moreover, if η is not continuous, then $wr(\rho, \eta)$ might not exist, even if $\mathcal{L}$ is chain inf-distributive. This is shown in the following two examples.

Example 4.4 Let $\mathcal{L} = \langle L, \preceq, \wedge, \vee, \top, \bot \rangle$ be the lattice where L is defined as

$$L = \{\langle m, n \rangle : (m = 0 \text{ or } m = 1),\ n \in \mathbb{N}\} \cup \{\top\},$$

where $\mathbb{N}$ is the set of natural numbers, and $\preceq$ is defined as

$$\langle m, n \rangle \preceq \langle m', n' \rangle \ \text{ iff } \ m \leq m' \ \&\ n \leq n', \text{ and for each } x \in L,\ x \preceq \top,$$

where $\leq$ is the standard ordering on natural numbers (see Figure 2). Notice that for each $\langle m, n \rangle, \langle m', n' \rangle \in L$, $\langle m, n \rangle \wedge \langle m', n' \rangle = \langle min\{m, m'\}, min\{n, n'\} \rangle$.
$\mathcal{L}$ is complete and not chain inf-distributive: in fact, $\langle 1, 0 \rangle \wedge (\bigvee_{k\in\mathbb{N}} \langle 0, k \rangle) = \langle 1, 0 \rangle \wedge \top = \langle 1, 0 \rangle$, while $\bigvee_{k\in\mathbb{N}}(\langle 1, 0 \rangle \wedge \langle 0, k \rangle) = \langle 0, 0 \rangle$.
This same lattice $\mathcal{L}$ is used in [10, page 16] as an example of a distributive complete lattice which is not pseudo-complemented.
Define ρ as the closure operator whose set of fixpoints is $\{\langle 1, n \rangle : n \in \mathbb{N}\} \cup \{\top\}$, and consider the bottom closure operator ι, which is of course continuous. We show that $wr(\rho, \iota)$, i.e. the pseudo-complement of ρ, does not exist.
For any $k \in \mathbb{N}$, define the closure operator ρ_k such that $\rho_k(L) = \{\langle 0, n \rangle : n \in \mathbb{N},\ k \leq n\} \cup \{\top\}$. Then, for any k we have $\rho \sqcap \rho_k = \iota$. Define $\hat{\rho} = \sqcup_{k\in\mathbb{N}} \rho_k$ and observe that, if the pseudo-complement of ρ exists, then it must be greater than or equal to $\hat{\rho}$. But $\hat{\rho}(L) = \bigcap_{k\in\mathbb{N}} \rho_k(L) = \{\top\}$, and therefore $\rho \sqcap \hat{\rho} = \rho$, which is strictly grater than ι. Since $\hat{\rho}$ is the top element in $uco(\mathcal{L})$, we conclude that ρ does not have the pseudo-complement. □

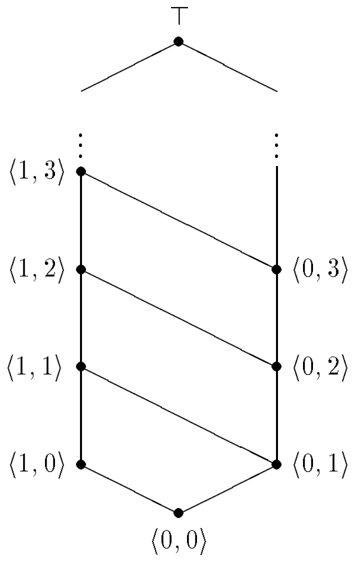


Figure 2: The non chain inf-distributive lattice $\mathcal{L}$ of Example 4.4.

Example 4.5 Consider the lattice $\mathcal{L}$ defined as in Example 4.4 with an additional element $\langle 0, \omega\rangle$ such that for each natural number k, $\langle 0, k\rangle \preceq \langle 0, \omega\rangle$, and $\langle 0, \omega\rangle \preceq \top$. It is easy to check that $\mathcal{L}$ is complete and chain inf-distributive in this case.
Consider the same closure operator ρ of the previous example, viz. $\rho(L) = \{\langle 1, n\rangle : n \in \mathbb{N}\} \cup \{\top\}$, and the closure operator η such that $\eta(L) = L \setminus \{\langle 0, \omega\rangle\}$. Clearly η is not continuous, because $\eta(\bigvee_{k\in\mathbb{N}}\langle 0, k\rangle) = \eta(\langle 0, \omega\rangle) = \top$, while $\bigvee_{k\in\mathbb{N}} \eta(\langle 0, k\rangle) = \bigvee_{k\in\mathbb{N}}\langle 0, k\rangle = \langle 0, \omega\rangle$.
For any $k \in \mathbb{N}$, consider the closure operator ρ_k defined as in the previous example, i.e. $\rho_k(L) = \{\langle 0, n\rangle : n \in \mathbb{N},\ k \leq n\} \cup \{\top\}$, and observe that for each k, $\rho \sqcap \rho_k = \eta$. Define, as before, $\hat{\rho} = \sqcup_{k\in\mathbb{N}}\rho_k$, and observe that $wr(\rho, \eta)$, if it exists, must be greater than or equal to $\hat{\rho}$. But again we have $\hat{\rho}(L) = \{\top\}$, and therefore $\rho \sqcap \hat{\rho} = \rho$, which is strictly greater than η. □

We conclude this section with the following observation, which relates the two notions of continuity for closure operators and chain inf-distributivity for lattices.
Given a complete lattice $\mathcal{L} = \langle L, \preceq, \wedge, \vee, \top, \bot\rangle$, and $\rho \in uco(L)$, the structure $\rho(\mathcal{L}) = \langle \rho(L), \preceq, \wedge\rangle$ is a complete meet-sublattice of $\mathcal{L}$. Furthermore, $\rho(\mathcal{L})$ is indeed a complete lattice, but, in general, not a complete sublattice of $\mathcal{L}$, because the join in $\rho(\mathcal{L})$ may be different from $\vee$. Thus, one might wonder whether the chain inf-distributivity property of $\mathcal{L}$ is inherited by $\rho(\mathcal{L})$. The following remark shows that this is the case whenever ρ is continuous.

Remark 4.6 *Let $\mathcal{L}$ be chain inf-distributive. If $\rho \in uco(L)$ is continuous then $\rho(\mathcal{L})$ is chain inf-distributive.*

Proof. Let $\{y_i\}_{i\in I} \subseteq \rho(L)$ be a chain and $x \in \rho(L)$. By continuity of ρ and by the fact that the y_i's are fixpoints of ρ we have:

$$\rho(\bigvee_{i\in I} y_i) = \bigvee_{i\in I} \rho(y_i) = \bigvee_{i\in I} y_i.$$

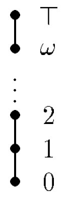
Hence, $\bigvee_{i\in I} y_i \in \rho(L)$, and this implies that it is the join in $\rho(\mathcal{L})$ of $\{y_i\}_{i\in I}$. Analogously, the join in $\rho(\mathcal{L})$ of the chain $\{x \wedge y_i\}_{i\in I}$ is $\bigvee_{i\in I}(x \wedge y_i)$. Since the meet in $\rho(\mathcal{L})$ coincides with the meet in $\mathcal{L}$, proving the chain inf-distributivity of $\rho(\mathcal{L})$ amounts to prove that

$$x \wedge \bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \wedge y_i),$$

which holds by chain inf-distributivity of $\mathcal{L}$. ■

The converse of the above result does not hold, as shown by the following example.

Example 4.7 Consider the complete lattice $\mathcal{L}$ depicted below.



Obviously, $\mathcal{L}$ is chain inf-distributive. Consider the closure operator $\rho \in uco(L)$ whose set of fixpoints is $\rho(L) = \mathbb{N} \cup \{\top\}$ (i.e. all the elements of L except ω). Then, $\rho(\mathcal{L})$ is clearly chain inf-distributive, while ρ is not continuous. In fact, $\rho(\bigvee_{n \in \mathbb{N}} n) = \rho(\omega) = \top$, while $\bigvee_{n \in \mathbb{N}} \rho(n) = \bigvee_{n \in \mathbb{N}} n = \omega$. □

5 Pseudo-complements of closure operators

Assume $\mathcal{L}$ is a complete lattice. An immediate consequence of Theorem 4.3 is that $uco(\mathcal{L})$ is pseudo-complemented if $\mathcal{L}$ is chain inf-distributive. The existence of the pseudo-complement, in fact, is just a special case of the existence of the weak relative pseudo-complement.

Corollary 5.1 *If $\mathcal{L}$ is chain inf-distributive then $uco(\mathcal{L})$ is pseudo-complemented.*

Proof. The bottom closure operator ι is obviously continuous, hence, by Theorem 4.3, for any $\rho \in uco(L)$ there exists $wr(\rho, \iota)$, that is the pseudo-complement ρ^*. ■

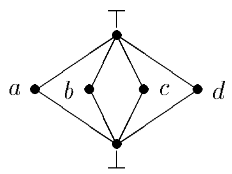
If $\mathcal{L}$ satisfies the ascending chain condition ([1]), namely if $\mathcal{L}$ does not have infinite ascending chains, then, obviously, $\mathcal{L}$ is chain inf-distributive, and each closure operator on $\mathcal{L}$ is continuous. Therefore, we have the following two corollaries.

Corollary 5.2 *If $\mathcal{L}$ satisfies the ascending chain condition then for any $\rho, \eta \in uco(L)$, with $\eta \sqsubseteq \rho$, there exists $wr(\rho, \eta)$.*

Corollary 5.3 *If $\mathcal{L}$ satisfies the ascending chain condition then $uco(\mathcal{L})$ is pseudo-complemented.*

One might, at this point, raise the question whether $uco(\mathcal{L})$ satisfies the Stone identity, in case the conditions for it to be pseudo-complemented hold. Next example answers this question negatively.

Example 5.4 Let $\mathcal{L}$ be the following lattice:



Consider the closure operator ρ on $\mathcal{L}$ for which $\rho(L) = \{\top, a, b, \bot\}$. It is easy to verify that $\rho^*(L) = \{\top, c, d, \bot\}$ and $\rho^{**} = \rho$. Thus, the Stone identity does not hold since $(\rho^* \sqcup \rho^{**})(L) = \{\top, \bot\}$. □

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