Elementarily Computable Functions Over the Real Numbers and $\mathbb{R}$-Sub-Recursive Functions

Olivier Bournez and Emmanuel Hainry

LORIA/INRIA, 615 Rue du Jardin Botanique, BP101
54602 Villers lès Nancy, France
{Olivier.Bournez,Emmanuel.Hainry}@loria.fr

Abstract We present an *analog* and *machine-independent* algebraic characterization of elementarily computable functions over the real numbers in the sense of recursive analysis: we prove that they correspond to the smallest class of functions that contains some basic functions, and closed by composition, linear integration, and a simple limit schema.
We generalize this result to all higher levels of the Grzegorczyk Hierarchy.
This paper improves several previous partial characterizations and has a dual interest:

- Concerning *recursive analysis*, our results provide machine-independent characterizations of natural classes of computable functions over the real numbers, allowing to define these classes without usual considerations on higher-order (type 2) Turing machines.
- Concerning *analog models*, our results provide a characterization of the power of a natural class of analog models over the real numbers and provide new insights for understanding the relations between several analog computational models.

1 Introduction

Several approaches have been proposed to model computations over real numbers. *Recursive analysis* or *computable analysis*, was introduced by Turing [38], Grzegorczyk [17], Lacombe [21]. Many works have been devoted to giving computable foundations to most of the concepts of mathematical analysis in this framework : see *e.g.* monograph [39].

Alternative views exist. Among them, we can mention the model proposed by Blum *et al.*, sometimes called *real Turing machine*, measuring the algebraic complexity of problems independently of real number representation considerations defined in [5] and extended to arbitrary structures in [33]. Several papers have been devoted to understanding complexity classes and their relations in this framework: see monographs [4,33].

These models concern *discrete time* computability. Models of machines where the time is *continuous* can also be considered. The first ever built computers were continuous time machines: *e.g.* *Blaise Pascal's pascaline* or Lord Kelvin's model of *Differential Analyzer* [20], that gave birth to a real machine, built in 1931 at the MIT to solve differential equations [9], and which motivated Shannon's *General Purpose Analog Computer (GPAC) model* [36], whose computational power was characterized algebraically in terms of solutions of polynomial differential equations [36,34,22,16]. Continuous time machines also include analog neural networks [32,37], hybrid systems [3,6], or theoretical physical models [31,19,15]: see also survey [32].

The relations between all the models are not fully understood. One can say, that the theory of analog computations has not yet experienced the unification that digital discrete time computations have experienced through Turing work and the so-called *Church thesis* [13,32].

This however becomes a crucial matter since the progress of electronics makes the construction of some of the machines realistic, whereas some models were recently proved very (far too?) powerful: using the so-called *Zeno's paradox*, some models make it possible to compute non-Turing computable functions in a constant time: see *e.g.* [23,7,3,19,15].

Notice that understanding whether there exist analog continuous time models that do not suffer from Zeno's paradox problems is also closely related to the important problems of finding criteria for so-called *robustness* for continuous (hybrid) time models: see *e.g.* [18,2].

In [23], Moore introduced a class of functions over the reals inspired from the classical characterization of computable functions over integers: observing that the continuous analog of a primitive recursion is a differential equation, Moore proposes to consider the class of $\mathbb{R}$-*recursive functions*, defined as the the smallest class of functions containing some basic functions, and closed by composition, differential equation solving (called *integration*), and minimization.

This class of functions, also investigated in [24,25,26,27,28,29], can be related to GPAC computable functions: see [23], corrected by [16].

Putting aside possible objections about the physical feasibility of the μ-operator considered in paper [23], the original definitions of this class in [23] suffer from several technical problems[1]. At least some of them make it possible to use a *"compression trick"* (another incarnation of Zeno's paradox) to simulate in a bounded time an unbounded number of discrete transitions in order to recognize arithmetical reals [23].

In [11,12,13], Campagnolo, Costa and Moore propose to consider the (better-defined) subclass $\mathcal{L}$ of $\mathbb{R}$-recursive functions corresponding to the smallest class of functions containing some basic functions and closed by composition and *linear* integration. Class $\mathcal{L}$ is related to functions elementarily computable over integers in classical recursion theory and functions elementarily computable over the real numbers in recursive analysis (discussed in [40]): any function of class $\mathcal{L}$ is elementarily computable in the sense of recursive analysis, and conversely, any function over the integers computable in the sense of classical recursion theory is the restriction to integers of a function that belongs to $\mathcal{L}$ [12,13].

[1] For example not well defined functions are considered, $\infty \times 0$ is always considered as 0, etc.... Some of them are discussed in [11,12,13] and even in the original paper [23].

However, the previous results do not provide a characterization of *all* functions over the reals that are computable in the sense of recursive analysis.

This paper provides one:

Theorem 1. *For functions over the reals of class $\mathcal{C}^2$ defined on a product of compact intervals with rational endpoints, f is elementarily computable in the sense of recursive analysis iff it belongs to the smallest class of functions containing some basic functions and closed by composition, linear integration and a simple limit schema.*

We extend this theorem to a characterization of all higher levels of the Grzegorczyk hierarchy (observe that previous theorem is a consequence of this theorem).

Theorem 2. *For functions over the reals of class $\mathcal{C}^2$ defined on a product of compact intervals with rational endpoints, f is computable in the sense of recursive analysis in level $n \geq 3$ of the Grzegorczyk hierarchy iff f belongs to the smallest class of functions containing some (other) basic functions and closed by composition, linear integration and a simple limit schema.*

Concerning *analog models*, these results have several impacts: first, they contribute to understand analog models, in particular the relations between GPAC computable functions, $\mathbb{R}$-recursive functions, and computable functions in the sense of recursive analysis. Furthermore, they prove that no *Super-Turing* phenomenon can occur for these classes of functions. In particular we have a "robust" class of functions in the sense of [18,2].

Concerning *recursive analysis*, our theorems provide *a purely algebraic* and *machine independent* characterization of elementarily computable functions over the reals. Observe the potential benefits offered by these characterizations compared to classical definitions of these classes in recursive analysis, involving discussions about higher-order (type 2) Turing machines (see *e.g.* [39]), or compared to characterizations in the spirit of [10].

In Section 2, we start by some mathematical preliminaries. In Section 3, we recall some notions from classical recursion theory. We present basic definitions of recursive analysis in Section 4. Previous known results are recalled in Section 5. Our characterizations are presented in Section 6. The proofs are given in Sections 7 and 8. Some extensions are presented in Section 9 and 10.

2 Mathematical preliminaries

Let $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}^{>0}$ denote the set of natural integers, the set of rational numbers, the set of real numbers, and the set of positive real numbers respectively. Given $x \in \mathbb{R}^n$, we write $\overrightarrow{x}$ to emphasize that x is a vector.

We will use the following simple mathematical result

Lemma 1. *Let $F : \mathbb{R} \times \mathcal{V} \subset \mathbb{R}^{k+1} \to \mathbb{R}^l$ be a function of class*[2] *$\mathcal{C}^1$, and $\beta(x) : \mathcal{V} \to \mathbb{R}$, $K(x) : \mathcal{V} \to \mathbb{R}$ be some continuous functions.*

- *Assume that for all t and $\overrightarrow{x}$, $\|\frac{\partial F}{\partial t}(t, \overrightarrow{x})\| \leq K(\overrightarrow{x}) \exp(-t\beta(\overrightarrow{x}))$.*
  Let $\mathcal{D}$ be the subset of the $\overrightarrow{x} \in \mathcal{V}$ with $\beta(\overrightarrow{x}) > 0$.
  Then,
  - *for all $\overrightarrow{x} \in \mathcal{D}$, $F(t, \overrightarrow{x})$ has a limit $L(\overrightarrow{x})$ in $t = +\infty$.*
  - *Function $L(\overrightarrow{x})$ is a continuous function.*
  - *Furthermore*

$$\|F(t, \overrightarrow{x}) - L(\overrightarrow{x})\| \leq \frac{K(\overrightarrow{x}) \exp(-t\beta(\overrightarrow{x}))}{\beta(\overrightarrow{x})}$$

- *Assume that, in addition, for all t and $\overrightarrow{x}$, $\frac{\partial^2 F}{\partial t \partial x_i}(t, \overrightarrow{x})$ exists and $\|\frac{\partial^2 F}{\partial t \partial x_i}(t, \overrightarrow{x})\| \leq K(\overrightarrow{x}) \exp(-t\beta(\overrightarrow{x}))$.*
  Then:
  - *Function $L(\overrightarrow{x})$ is of class $\mathcal{C}^1$.*
  - *Its partial derivative $\frac{\partial L}{\partial x_i}$ are the limit of $\frac{\partial F}{\partial x_i}(t, \overrightarrow{x})$ in $t = +\infty$.*
  - *Furthermore*

$$\|\frac{\partial F}{\partial x_i}(t, \overrightarrow{x}) - \frac{\partial L}{\partial x_i}(\overrightarrow{x})\| \leq \frac{K(\overrightarrow{x}) \exp(-t\beta(\overrightarrow{x}))}{\beta(\overrightarrow{x})}.$$

Proof. By mean value theorem,

$$\begin{aligned} \|F(t, \overrightarrow{x}) - F(t', \overrightarrow{x})\| &\leq \int_t^{t'} K \exp(-t\beta(\overrightarrow{x}))dt \\ &\leq K \int_t^{+\infty} \exp(-t\beta(\overrightarrow{x}))dt = K \frac{\exp(-t\beta(\overrightarrow{x}))}{\beta(\overrightarrow{x})} \end{aligned}$$

This implies that $F(t, \overrightarrow{x})$ satisfies Cauchy criterion, and hence converges in $t = +\infty$. This implies the existence of function L. The first inequality of the lemma is obtained by letting t' go to $+\infty$ in previous inequality. Observe that it implies that the convergence is uniform in $\overrightarrow{x}$ in every compact domain.

L is continuous since the limit of a uniformly convergent sequence of continuous function is continuous.

Replacing $F(t, \overrightarrow{x})$ by $\frac{\partial F}{\partial x_i}(t, \overrightarrow{x})$ in previous arguments proves the uniform convergence of $\frac{\partial F}{\partial x_i}(t, \overrightarrow{x})$ in $t = +\infty$ on every compact domain under the additional hypothesis.

Observing that the derivative of a converging sequence of functions, whose sequence of derivatives converges uniformly, exists and is the limit of the derivatives, and that the limit of a uniformly converging sequence of continuous functions is continuous, the other assertions follow.

[2] Recall that function $f : \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$, $k, l \in \mathbb{N}$, is said to be of class $\mathcal{C}^r$ if it is r-times continuously differentiable on $\mathcal{D}$. It is said to be of class $\mathcal{C}^\infty$ if it is of class $\mathcal{C}^r$ for all r.

The following result[3], with previous lemma, is a key to provide upper bounds on the growth of functions of our classes (c.f. Lemma 7).

Lemma 2 (Bounding Lemma for Linear Differential Equations [1]). *For linear differential equation $\overrightarrow{x}' = A(t)\overrightarrow{x}$, if A is defined and continuous on interval $I = [a, b]$, where $a \leq 0 \leq b$, then, for all $\overrightarrow{x}_0$, the solution of $\overrightarrow{x}' = A(t)\overrightarrow{x}$ with initial condition $\overrightarrow{x}(0) = \overrightarrow{x}_0$ is defined and unique on I. Furthermore, the solution satisfies*

$$\|\overrightarrow{x}(t)\| \leq \|\overrightarrow{x}_0\| \exp(\sup_{\tau\in[0,t]} \|A(\tau)\|t).$$

Remark 1. Recall that the solution of any differential equation of type $\overrightarrow{x}' = A(t)\overrightarrow{x} + B(t)$, $\overrightarrow{x}(0) = \overrightarrow{x}_0$, where $A(t)$ is a $n \times n$ matrix and $B(t)$ is a n dimension vector can be obtained by the solution of linear differential equation $\overrightarrow{y}' = C(t)\overrightarrow{y}$, $\overrightarrow{y}(0) = \overrightarrow{y}_0$ by working in dimension $n + 1$ and considering

$$y(t) = \begin{pmatrix} x(t) \\ 1 \end{pmatrix}, y_0 = \begin{pmatrix} x_0 \\ 1 \end{pmatrix}, \text{ and } C = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}.$$

3 Classical Recursion Theory

Classical recursion theory deals with functions over integers. Most classes of classical recursion theory can be characterized as closures of a set of basic functions by a finite number of basic rules to build new functions [35,30]: given a set $\mathcal{F}$ of functions and a set $\mathcal{O}$ of operators on functions (an operator is an operation that maps one or more functions to a new function), $[\mathcal{F}; \mathcal{O}]$ will denote the closure of $\mathcal{F}$ by $\mathcal{O}$.

Proposition 1 (Classical settings: see *e.g.* **[35,30]).** *Let f be a function from $\mathbb{N}^k$ to $\mathbb{N}$ for $k \in \mathbb{N}$. Function f is*

- *elementary iff it belongs to* $\mathcal{E} = [0, S, U, +, \ominus; \text{COMP}, \text{BSUM}, \text{BPROD}]$*;*
- *in class $\mathcal{E}_n$ of the Grzegorczyk Hierarchy ($n \geq 3$) iff it belongs to* $\mathcal{E}_n = [0, S, U, +, \ominus, E_{n-1}; \text{COMP}, \text{BSUM}, \text{BPROD}]$*;*
- *primitive recursive iff it belongs to* $\mathcal{PR} = [0, U, S; \text{COMP}, \text{REC}]$*;*
- *recursive iff it belongs to* $\mathcal{R}ec = [0, U, S; \text{COMP}, \text{REC}, \text{MU}]$.

A function $f : \mathbb{N}^k \to \mathbb{N}^l$ is elementary (resp: primitive recursive, recursive) iff its projections are elementary (resp: primitive recursive, recursive).

The base functions $0, (U_i^m)_{i,m\in\mathbb{N}}, S, +, \ominus$ and the operators COMP*,* BSUM*,* BPROD*,* REC*,* MU *are given by*

1. $0 : \mathbb{N} \to \mathbb{N}$, $0 : n \mapsto 0$; $U_i^m : \mathbb{N}^m \to \mathbb{N}$, $U_i^m : (n_1, \ldots, n_m) \mapsto n_i$; $S : \mathbb{N} \to \mathbb{N}, S : n \mapsto n + 1$; $+ : \mathbb{N}^2 \to \mathbb{N}$, $+ : (n_1, n_2) \mapsto n_1 + n_2$; $\ominus : \mathbb{N}^2 \to \mathbb{N}$, $\ominus : (n_1, n_2) \mapsto max(0, n_1 - n_2)$;

[3] As it was already the case in [11,12,13].

2. BSUM *: bounded sum. Given* f*,* $h = \text{BSUM}(f)$ *is defined by* $h : (\overrightarrow{x}, y) \mapsto \sum_{z<y} f(\overrightarrow{x}, z)$*;* BPROD *: bounded product. Given* f*,* $h = \text{BPROD}(f)$ *is defined by* $h : (\overrightarrow{x}, y) \mapsto \prod_{z<y} f(\overrightarrow{x}, z)$*;*
3. COMP *: composition. Given* f *and* g*,* $h = \text{COMP}(f, g)$ *is defined as the function verifying* $h(\overrightarrow{x}) = g(f(\overrightarrow{x}))$*;*
4. REC *: primitive recursion . Given* f *and* g*,* $h = \text{REC}(f, g)$ *is defined as the function verifying* $h(\overrightarrow{x}, 0) = f(\overrightarrow{x})$ *and* $h(\overrightarrow{x}, n+1) = g(\overrightarrow{x}, n, h(\overrightarrow{x}, n))$*.*
5. MU *: minimization. The minimization of* f *is* $h : \overrightarrow{x} \mapsto \inf\{y : f(\overrightarrow{x}, y) = 0\}$*.*

Functions E_n*, involved in the definition of the classes* $\mathcal{E}_n$ *of the Grzegorczyk Hierarchy, are defined by induction as follows (when* f *is a function,* $f^{[d]}$ *denotes its d-th iterate:* $f^{[0]}(\overrightarrow{x}) = x$*,* $f^{[d+1]}(\overrightarrow{x}) = f(f^{[d]}(\overrightarrow{x}))$*):*

1. $E_0(x, y) = x + y$*,*
2. $E_1(x, y) = (x + 1) \times (y + 1)$*,*
3. $E_2(x) = 2^x$*,*
4. $E_{n+1}(x) = E_n^{[x]}(1)$ *for* $n \geq 2$*.*

$\mathcal{PR}$ corresponds to functions computable using *loop programs*. $\mathcal{E}$ corresponds to computable functions bounded by some iterate of the exponential function [35,30].

The following facts are known:

Proposition 2 ([35,30]).

- $\mathcal{E}_3 = \mathcal{E} \subsetneq \mathcal{PR} \subsetneq \mathcal{R}ec$
- $\mathcal{E}_n \subsetneq \mathcal{E}_{n+1}$ *for* $n \geq 3$*.*
- $\mathcal{PR} = \bigcup_i \mathcal{E}_i$

Previous classes can also be related to complexity classes. If TIME(t) and SPACE(t) denote the classes of functions that are computable with time and space t, then:

Proposition 3 ([35,30]). *For all* $n \geq 3$*,*

- $\mathcal{E}_n = \text{TIME}(\mathcal{E}_n) = \text{SPACE}(\mathcal{E}_n)$*,*
- $\mathcal{PR} = \text{TIME}(\mathcal{PR}) = \text{SPACE}(\mathcal{PR})$*.*

In classical computability, more general objects than functions over the integers can be considered, in particular functionals, i.e. functions $\Phi : (\mathbb{N}^{\mathbb{N}})^m \times \mathbb{N}^k \to \mathbb{N}^l$. A functional will be said to be *elementary* (respectively. $\mathcal{E}_n$, *primitive recursive*, *recursive*) when it belongs to the corresponding[4] class.

[4] Formally, a function f over the integers can be considered as functional $\overline{f} : (V_1, \dots, V_m, \overrightarrow{n}) \mapsto f(\overrightarrow{n})$. Similarly, an operator Op on functions $f_1, \dots, f_m$ over the integers can be extended to $\overline{Op}(F_1, \dots, F_m) : (V_1, \dots, V_m, \overrightarrow{n}) \mapsto Op(F_1(V_1, \dots, V_m, .), \dots, F_m(V_1, \dots, V_m, .))(\overrightarrow{n})$. We will still (abusively) denote by $[f_1, \dots, f_p; O_1, \dots, O_q]$ for the smallest class of functionals that contains basic functions $\overline{f_1}, \dots, \overline{f_p}$, plus the functionals $Map_i : (V_1, \dots, V_m, n) \to (V_i)_n$, the nth element of sequence V_i, and which is closed by the operators $\overline{O_1}, \dots, \overline{O_q}$. For example, a functional will be said to be elementary iff it belongs to $\mathcal{E} = [Map, \overline{0}, \overline{S}, \overline{U}, \overline{+}, \overline{\ominus}; \overline{\text{COMP}}, \overline{\text{BSUM}}, \overline{\text{BPROD}}]$.

4 Computable Analysis

The idea sustaining *Computable analysis*, also called *recursive analysis*, is to define computable functions over real numbers by considering functionals over fast-converging sequences of rationals [38,21,17,39].

Let $\nu_{\mathbb{Q}} : \mathbb{N} \to \mathbb{Q}$ be the following representation[5] of rational numbers by integers: $\nu_{\mathbb{Q}}(\langle p, r, q\rangle) \mapsto \frac{p-r}{q+1}$, where $\langle ., ., .\rangle : \mathbb{N}^3 \to \mathbb{N}$ is an elementarily computable bijection.

A sequence of integers $(x_i) \in \mathbb{N}^{\mathbb{N}}$ *represents a real number* x if it converges quickly toward x (denoted by $(x_i) \rightsquigarrow x$) in the following sense:

$$\forall i, |\nu_{\mathbb{Q}}(x_i) - x| < \exp(-i).$$

For $X = ((x_1), \ldots, (x_k)) \in (\mathbb{N}^{\mathbb{N}})^k$, $\overrightarrow{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$, we write $X \rightsquigarrow \overrightarrow{x}$ for $(x_i) \rightsquigarrow x_i$ for $i = 1, \ldots, k$.

Definition 1 (Recursive analysis). *A function $f : \mathcal{D} \to \mathbb{R}$, where $\mathcal{D}$ is a closed subset of $\mathbb{R}^k$ for some integer k, is said to be computable (in the sense of recursive analysis) if there exists a recursive functional $\phi : (\mathbb{N}^k)^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ such that for all $\overrightarrow{x} \in \mathcal{D}$, for all $X \in (\mathbb{N}^k)^{\mathbb{N}}$, we have $(\phi(X, j))_j \rightsquigarrow f(\overrightarrow{x})$ whenever $X \rightsquigarrow \overrightarrow{x}$.*

A function $f : \mathcal{D} \to \mathbb{R}^l$, with $l > 1$, is said to be computable if all its projections are.

A function f will be said to be *elementarily* (respectively $\mathcal{E}_n$) computable whenever the corresponding functional ϕ is. The class of elementarily (respectively $\mathcal{E}_n$) computable functions over the reals will be denoted by $\mathcal{E}(\mathbb{R})$ (resp. $\mathcal{E}_n(\mathbb{R})$).

Elementarily computable functions have been discussed in [40]. Observing that classical proofs for computable functions (see *e.g.* [39]) use only elementary functionals one can state:

Proposition 4. *Functions $+$, $-$, $\times$, e^x, $\sin(x)$, $\cos(x)$, $1/x$ are elementarily computable*[6] *in the sense of recursive analysis.*

The following result is also well-known:

Proposition 5 (see e.g. [39]). *All (elementarily) computable functions in the sense of recursive analysis are continuous.*

Actually, one can go further: adapting to the elementary case the classical statements and proofs of recursive analysis (see *e.g.* [39]), one can state that elementarily computable functions are uniformly continuous on all compact subsets of their domains with an elementarily computable modulus of continuity.

[5] Many other natural representations of rational numbers can be chosen and provide the same class of computable functions: see [39].

[6] More precisely, with our definition, $1/x$ restricted to any closed domain, is elementarily computable in the sense of recursive analysis.

Definition 2. *A* modulus of continuity *of a function* $f : \mathcal{D} \to \mathbb{R}^l$ *defined over a closed domain is a function* $M : \mathbb{N} \to \mathbb{N}$ *such that for all* $i \in \mathbb{N}$*, for all* x, y*,*

$$\|x - y\| < \exp(-M(i)) \Rightarrow \|f(x) - f(y)\| < \exp(-i).$$

Adapting the arguments of [39] to elementarily computable functions, one gets easily:

Proposition 6. *If* $f \in \mathcal{E}(\mathbb{R})$ *is defined over a product of compact intervals, then* f *has a modulus of continuity in* $\mathcal{E}$*.*

Actually, more generally, we have:

Proposition 7. *If* $f \in \mathcal{E}(\mathbb{R})$ *is defined over a product of closed intervals* $\mathcal{D} \subset \mathbb{R}^k$*, then there is a function* $M : \mathbb{N}^2 \to \mathbb{N}$ *in* $\mathcal{E}$*, such that for all integer* K*,* $M(K, \_)$ *is a modulus of continuity of* f *over* $\mathcal{D} \cap [-K, K]^k$*.*

When f is (elementarily) computable, then its derivative f' is not necessarily computable. However, this holds for functions of class $\mathcal{C}^2$ over a compact domain (we are still adapting to the elementary case the classical proofs of recursive analysis: see *e.g.* [39]):

Lemma 3. *Let* $f : \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$ *be a function of class* $\mathcal{C}^2$ *defined over compact domain* $\mathcal{D}$*.*

If f *is elementarily computable, then its partial derivatives are.*

Proof. We give the proof for a function f defined on interval $[0, 1]$ to $\mathbb{R}$. The general case is easy to obtain.

Since f'' is continuous on a compact set, f'' is bounded by some constant M. By mean value theorem, we have $|f'(x) - f'(y)| \leq M|x - y|$ for all x, y.

Given $x \in [0, 1]$, and $i \in \mathbb{N}$, an approximation z of $f'(x)$ at precision $\exp(-i)$ can be computed as follows: compute n with $M \exp(-n) \leq \exp(-i)/2$. Compute y_1 a rational at most $\exp(-i - n - 2)$ far from $f(x)$, and y_2 a rational at most $\exp(-i - n - 2)$ far from $f(x + \exp(-n))$. Take $z = (y_1 - y_2)/\exp(-n)$.

This is indeed a value at most $\exp(-i)$ far from $f'(x)$ since by mean value theorem there exists $\chi \in [x, x + \exp(-n)]$ such that $f'(\chi_j) = \frac{f(x+\exp(-n))-f(x)}{\exp(-n)}$. Now

$$\begin{aligned}
|z - f'(x)| &\leq \tfrac{|y_1 - f(x)|}{\exp(-n)} + \tfrac{|y_2 - f(x+\exp(-n))|}{\exp(-n)} + |\tfrac{f(x+\exp(-n))-f(x)}{\exp(-n)} - f'(x)| \\
&\leq \exp(-i - n - 2)\exp(n) + \exp(-i - n - 2)\exp(n) \\
&\quad + |f'(\chi_j) - f'(x)| \\
&\leq 2\exp(-i - 2) + M\exp(-n) \\
&\leq \exp(-i)/2 + \exp(-i)/2 \\
&\leq \exp(-i).
\end{aligned}$$

5 Real-recursive and recursive functions

Following the original ideas from [23], but observing that the minimization schema considered in [23] is the source of many technical problems, Campagnolo, Costa and Moore proposed in [11,12,13] not to consider classes of functions over the reals defined in analogy with the full class of recursive functions, but with subclasses. Indeed, the considered classes are built in analogy with class of elementary functions and the classes of the Grzegorczyk hierarchy . Furthermore, they proposed to restrict the integration schema to a simpler (and better defined) linear integration schemata LI [13,12].

We call *real extension of a function* $f : \mathbb{N}^k \to \mathbb{N}^l$ a function $\tilde{f}$ from $\mathbb{R}^k$ to $\mathbb{R}^l$ whose restriction to $\mathbb{N}^k$ is f.

Definition 3 ([13,12]). *Let $\mathcal{L}$ and $\mathcal{L}_n$ be the classes of functions $f : \mathbb{R}^k \to \mathbb{R}^l$, for some $k, l \in \mathbb{N}$, defined by*

$$\mathcal{L} = [0, 1, -1, \pi, U, \theta_3; \mathrm{COMP}, \mathrm{LI}]$$

and

$$\mathcal{L}_n = [0, 1, -1, \pi, U, \theta_3, \overline{E}_{n-1}; \mathrm{COMP}, \mathrm{LI}]$$

where the base functions 0, 1, -1, π, $(U_i^m)_{i,m\in\mathbb{N}}$, θ_3, $\overline{E}_n$ *and the schemata* COMP *and* LI *are defined as follows:*

1. $0, 1, -1, \pi$ *are the corresponding constant functions;* $U_i^m : \mathbb{R}^m \to \mathbb{R}$ *are, as in the classical settings, projections:* $U_i^m : (x_1, \ldots, x_m) \mapsto x_i$*;*
2. $\theta_3 : \mathbb{R} \to \mathbb{R}$ *is defined as* $\theta_3 : x \mapsto x^3$ *if* $x \geq 0, 0$ *otherwise.*
3. $\overline{E}_n$*: for* $n \geq 3$*, let* $\overline{E}_n$ *denote a monotone real extension of the function* $\exp_n$ *over the integers defined inductively by* $\exp_2(x) = 2^x$, $\exp_{i+1}(x) = \exp_i^{[x]}(1)$.
4. COMP*: composition is defined as in the classical settings: Given* f *and* g*,* $h = \mathrm{COMP}(f, g)$ *is the function verifying* $h(\overrightarrow{x}) = g(f(\overrightarrow{x}))$*;*
5. LI*: linear integration. From* g *and* h*,* $\mathrm{LI}(g, h)$ *is the maximal solution of the linear differential equation* $\frac{\partial f}{\partial y}(\overrightarrow{x}, y) = h(\overrightarrow{x}, y) f(\overrightarrow{x}, y)$ *with* $f(\overrightarrow{x}, 0) = g(\overrightarrow{x})$.
   In this schema, if g *goes to* $\mathbb{R}^n$*,* $f = \mathrm{LI}(g, h)$ *also goes to* $\mathbb{R}^n$ *and* $h(\overrightarrow{x}, y)$ *is a* $n \times n$ *matrix with elements in* $\mathcal{L}$.

Lemma 4. *These classes contain functions* $id : x \mapsto x$, $\sin$, $\cos$, $\exp$, $+, \times$, $x \mapsto r$ *for all rational* r*, as well as for all* $f \in \mathcal{L}$*, or* $f \in \mathcal{L}^*$*, its primitive function* F *equal to* $\overrightarrow{0}$ *at* $\overrightarrow{0}$*, denoted by* $\int(f)$.

Proof. Indeed, $\int(f)$ can be defined by $\begin{pmatrix} F \\ 1 \end{pmatrix} = \mathrm{LI}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix} \right)$.

Function id is given by $\int(1)$.

Function $\Theta : t \mapsto (\sin(t), \cos(t))$ can be defined by $\mathrm{LI}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right)$. Project this function on each of its two variables to get sinus and cosinus function.

Function exp is given by $\mathrm{LI}(0, 1)$.

Addition is given by $x+0 = x$, $\frac{\partial x+y}{\partial y} = 1$. Multiplication is given by $x \times 0 = 0$, $\frac{\partial x \times y}{\partial y} = x$.

Given $p, q \in \mathbb{N}$ with $q > 0$, Function $x \mapsto p$, is $1 + 1 + \ldots + 1$, function $x \mapsto x^{q-1}$ is $x \times \ldots \times x$, and $p \times \int(x \mapsto x^{q-1})$ is $x \mapsto px^q/q$ whose value in 1 is p/q.

However, non total functions like $x \mapsto 1/x$ can not belong to the class since all functions from $\mathcal{L}$ are total:

Proposition 8 ([12,13]). *All functions from $\mathcal{L}$ and $\mathcal{L}_n$ are continuous, defined everywhere, and of class $\mathcal{C}^2$.*

The previous classes can be partially related to classes $\mathcal{E}$, $\mathcal{E}_n$ over integers and to classes $\mathcal{E}(\mathbb{R})$ and $\mathcal{E}_n(\mathbb{R})$ over real numbers. Indeed, in order to compare functions over the reals with functions over the integers, we introduce the following notation: given some class $\mathcal{C}$ of functions from $\mathbb{R}^k$ to $\mathbb{R}^l$, we write $\mathrm{DP}(\mathcal{C})$ (DP stands for discrete part) for the class of functions from $\mathbb{N}^k$ to $\mathbb{N}^l$ which have a real extension in $\mathcal{C}$.

One main contribution of [12,13] is:

Proposition 9 ([12,13]).

- $\mathrm{DP}(\mathcal{L}) = \mathcal{E}$;
- $\mathrm{DP}(\mathcal{L}_\mathrm{n}) = \mathcal{E}_\mathrm{n}$.

Actually, stronger inclusions were proved in [12,13]:

Proposition 10 ([12,13]).

- $\mathcal{L} \subset \mathcal{E}(\mathbb{R})$.
- $\mathcal{L}_n \subset \mathcal{E}_n(\mathbb{R})$.

However there is no hope to get the other inclusion: these inclusions are strict. Indeed, $x \mapsto 1/x$ is elementarily computable while Proposition 8 says that all functions from $\mathcal{L}$ are defined everywhere. A similar argument works for $\mathcal{E}_n(\mathbb{R})$. We conjecture the inclusions to be strict even when restricting to total functions.

Remark 2. Let θ_k be the function defined by $\theta_k(x) = \begin{cases} x^k \text{ if } x > 0 \\ 0 \text{ otherwise} \end{cases}$.

If one replace θ_3 by θ_k for a $k > 3$ in the definitions of $\mathcal{L}$ and $\mathcal{L}_n$, the classes $\mathcal{L}$ and $\mathcal{L}_n$ may differ from previous ones.

However:

- Propositions 9 and 10 still hold for the obtained classes
- Proposition 8 is changed into "All functions from $\mathcal{L}$ and $\mathcal{L}_n$, are continuous, defined everywhere, and of class $\mathcal{C}^{k-1}$"

Remark 3. Note that all base functions except θ_3 (and the θ_k) are analytic, and that all previous schemes preserve analyticity: in other words, the use of such a function θ_k is necessary in order to be able not to consider only analytic functions.

6 Real-recursive and recursive functions revisited

We now propose to consider new classes of functions that we will prove to correspond precisely to $\mathcal{E}(\mathbb{R})$ and $\mathcal{E}_n(\mathbb{R})$.

First, we restrict to functions defined over closed domains. These functions include in particular functions defined over $\mathbb{R}^k$ for some k, that is total functions, but also functions defined on closed subsets of $\mathbb{R}^k$.

The motivation is the following (observe that in this paper we defined computability in the sense of recursive analysis only for our class of functions, but computability over more general domains can also be defined: see e.g. [39]).

Lemma 5. *General elementarily computable functions are not stable by composition*[7].

To do so, we slightly modify LI schema, by allowing not-necessarily maximal solutions of linear differential equations to be considered. By abuse of notation, LI will denote this schema in what follows.

Definition 4 (LI **schema).** *From g and h, $\mathrm{LI}(g,h)$ is any solution defined on a product of closed intervals of the linear differential equation $\frac{\partial f}{\partial y}(\overrightarrow{x},y) = h(\overrightarrow{x},y)f(\overrightarrow{x},y)$ with $f(\overrightarrow{x},0)=g(\overrightarrow{x})$.*

In this schema, if g goes to $\mathbb{R}^n$, $f=\mathrm{LI}(g,h)$ also goes to $\mathbb{R}^n$ and $h(\overrightarrow{x},y)$ is a $n\times n$ matrix with elements in $\mathcal{L}$.

Now, we suggest to add a limit operator.

Remark 4. The idea of adding a limit operator has already been investigated in papers like [28,24]. However, since we are interested in $\mathbb{R}$-sub-recursive functions, and not to build a whole hierarchy above recursive functions as in [28,24], our limit schema will not be as general: as the LI schema of [11,12,13] is a restrained version of Moore's integration operator, our LIM may be seen as a restrained version of the operators of [28,24].

The conditions we impose on LIM are inspired from Lemma 1: a polynomial β over $x\in\mathbb{R}$ is a function of the form $\beta:\mathbb{R}\to\mathbb{R}$, $\beta: x\mapsto\sum_{i=0}^n a_i x^i$ for some $a_0,\dots,a_n\in\mathbb{R}$. A polynomial β over $\overrightarrow{x}=(x_1,\dots,x_{k+1})\in\mathbb{R}^{k+1}$ is a function of the form $\beta:\mathbb{R}^{k+1}\to\mathbb{R}$, $\beta:\overrightarrow{x}\mapsto\sum_{i=0}^n a_i x_{k+1}^i$ for some $a_0,\dots,a_n$ polynomial over $(x_1,\dots,x_k)\in\mathbb{R}^k$.

Definition 5 (LIM **schema).** *Let $f:\mathbb{R}\times\mathcal{D}\subset\mathbb{R}^{k+1}\to\mathbb{R}^l$, $K:\mathcal{D}\to\mathbb{R}$ and $\beta:\mathcal{D}\to\mathbb{R}$ a polynomial with the following hypothesis: such that for all $\overrightarrow{x}=(x_1,\dots,x_k)$, for all $t\geq\|\overrightarrow{x}\|$,*

$$\|\frac{\partial f}{\partial t}(t,\overrightarrow{x})\|\leq K(\overrightarrow{x})\exp(-t\beta(\overrightarrow{x})),$$

[7] The proof uses non-total functions, defined on open domains. Computable functions defined over closed domains can be shown stable by composition.

$\frac{\partial^2 f}{\partial t \partial x_i}(t, x_i)$ *exists for all* $1 \leq i \leq k$*, and*

$$\|\frac{\partial^2 f}{\partial t \partial x_i}(t, \overrightarrow{x})\| \leq K(\overrightarrow{x}) \exp(-t\beta(\overrightarrow{x})).$$

Then, on every product of closed intervals $I \subset \mathbb{R}^k$ *on which* $\beta(\overrightarrow{x}) > 0$, $F(\overrightarrow{x}) = \lim_{t \to +\infty} f(t, \overrightarrow{x})$ *exists by Lemma 1. If* F *is of class*[8] $\mathcal{C}^2$*, then we define* $\mathrm{LIM}(f, K, \beta)$ *as this function* $F : I \to \mathbb{R}$.

We are ready to define our classes:

Definition 6 (Classes $\mathcal{L}^*$, $\mathcal{L}^*_n$). *The class* $\mathcal{L}^*$*, and* $\mathcal{L}^*_n$*, for* $n \geq 3$*, of functions from* $\mathbb{R}^k$ *to* $\mathbb{R}^l$*, for* $k, l \in \mathbb{N}$*, are the following classes:*

- $\mathcal{L}^* = [0, 1, -1, U, \theta_3; \mathrm{COMP}, \mathrm{LI}, \mathrm{LIM}]$.
- $\mathcal{L}^*_n = [0, 1, -1, U, \theta_3, \overline{E}_{n-1}; \mathrm{COMP}, \mathrm{LI}, \mathrm{LIM}]$.

Remark 5. Previous classes can easily be shown stable by the primitive operator that sends a function f to its primitive $\int(f)$ equal to $\overrightarrow{0}$ at $\overrightarrow{0}$.

Indeed, $\int(f)$ can still be defined by $\begin{pmatrix} F \\ 1 \end{pmatrix} = \mathrm{LI}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix}\right)$.

Remark 6. Unlike classes from previous sections, class $\mathcal{L}^*$ also includes some non-total functions.

In particular any restriction to a closed domain of function $\frac{1}{x} : \begin{cases} \mathbb{R}^{>0} \to \mathbb{R} \\ x \mapsto \frac{1}{x} \end{cases}$.
Indeed, $E(t, x) = \int(\exp(-tx))$ is such that $E(t, x) = \begin{cases} \frac{(1-\exp(-tx))}{x} & \text{for } x \neq 0 \\ t & \text{for } x = 0 \end{cases}$
(of class $\mathcal{C}^k$ for all k). Now $\frac{1}{x} = \mathrm{LIM}(E, K, id)$ for some suitably chosen constant K (depending on the domain).

Our classes are supersets of previous classes:

Proposition 11. $\mathcal{L} \subsetneq \mathcal{L}^*$, $\mathcal{L}_n \subsetneq \mathcal{L}^*_n$ *for all* $n \geq 3$.

Proof. The function $x \mapsto \pi$ is actually in $\mathcal{L}^*$. Indeed, from $x \mapsto \frac{1}{1+x^2}$ in the class, we have $\arctan x = \int(\frac{1}{1+x^2})$, and $\pi = 4 \arctan(1)$.

The main results of this paper are the following (proved in following two sections):

Theorem 1 (Characterization of $\mathcal{E}(\mathbb{R})$). *Let* $f : \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$ *be some function over the reals of class* $\mathcal{C}^2$*, with* $\mathcal{D}$ *product of compact intervals.*

$$f \text{ is in } \mathcal{E}(\mathbb{R}) \text{ iff it belongs to } \mathcal{L}^*.$$

[8] By Lemma 1, if f is of class $\mathcal{C}^1$, function F is at least of class $\mathcal{C}^1$.

Theorem 2 (Characterization of $\mathcal{E}_n(\mathbb{R})$). *Let $f : \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$ be some function over the reals of class $\mathcal{C}^2$, with $\mathcal{D}$ product of compact intervals. Let $n \geq 3$.*

$$f \text{ is in } \mathcal{E}_n(\mathbb{R}) \text{ iff it belongs to } \mathcal{L}_n^*.$$

Observe that Theorem 1 is clearly the particular case $n = 3$ of Theorem 2.

Remark 7. If we replace θ_3 by θ_k for a $k \geq 3$ in the definitions of $\mathcal{L}^*$ and $\mathcal{L}_n^*$, and impose the result of a LIM operation to be of class $\mathcal{C}^{k-1}$ in Definition 5 (instead of $\mathcal{C}^2$), the classes $\mathcal{L}^*$ and $\mathcal{L}_n^*$ may differ. However, we have almost the same theorems for the corresponding classes: replace $\mathcal{C}^2$ by $\mathcal{C}^{k-1}$ in the statements of the theorems.

7 Upper bounds

We now prove the upper bound $\mathcal{L}^* \subset \mathcal{E}(\mathbb{R})$. As one may expect, this direction of the proof has many similarities with the proof $\mathcal{L} \subset \mathcal{E}$ in [12,13]: main differences lie in the presence of non-total functions and of schema LIM.

We first discuss the domain of the considered functions.

Lemma 6. *All functions from $\mathcal{L}^*$ are of class $\mathcal{C}^2$ and defined on a domain of the form $I_1 \times I_2 \ldots \times I_k$ where each I_i is a closed interval.*

Proof. By structural induction

- This is clear for basic functions (1, 0, -1, U, and θ_3).
- Composition preserves this property.
- Linear differential equations preserve class $\mathcal{C}^2$ [1,14]. They also preserve the domain property by definition.
- If $g = \mathrm{LIM}(f, K, \beta)$, from definition of LIM schema, this is clear.

We propose to introduce the following notation: given $a \in \mathbb{R}$, let ρ_a be the function $x \mapsto \frac{1}{x-a}$. Let $\rho_{+\infty}$ and $\rho_{-\infty}$ be the function identity $x \mapsto x$.

Given I real interval with bounds $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$, $\rho_I(x) = |\rho_a(x)| + |\rho_b(x)|$. For $\mathcal{D} = I_1 \times I_2 \ldots \times I_k$, let $\rho_{\mathcal{D}}(x) = \rho_{I_1}(U_1^k(x)) + \ldots + \rho_{I_k}(U_k^k(x))$. In any case, $\rho_{\mathcal{D}}(x)$ is elementarily computable and grows to $+\infty$ when x gets close to a bound of domain $\mathcal{D}$.

The following Lemma is an extension of a Lemma of [11,12,13].

Lemma 7. *Let $f : \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$ be a function of $\mathcal{L}^*$. There exist some integer d, and some constants A and B such that for all $\overrightarrow{x} \in \mathcal{D}$, $\|f(\overrightarrow{x})\| \leq A \exp^{[d]}(B \rho_{\mathcal{D}}(\overrightarrow{x}))$. Call the smallest such integer d the degree of f (denoted by $\deg f$). All partial derivatives of f also have a finite degree.*

Proof. By some elementary algebra and elementary properties of the exponential function, observe that by adjusting constants A, B, it is always possible to assume for all functions f and g, $\deg fg \leq max(\deg f, \deg g)$, and $\deg(f + g) \leq max(\deg f, \deg g)$.

Now, by structural induction:

– 0, 1, −1, U and all their derivatives have degree at most 1.
– $\theta_3(x)$ and its derivative have degree 1.
– The degree of $\mathrm{COMP}(f,g)$ is less than $\deg(f)+\deg(g)$, since $\deg(f\circ g) \leq \deg(f)+\deg(g)$ can easily be established using basic properties of exponential function. By the chain rule, the degree of any of the derivative of the composition $f(g)$ is bounded by $\max_i(\deg \frac{\partial g}{\partial i}, \deg \frac{\partial f}{\partial i} + \deg g)$.
– For $f = \mathrm{LI}(g,h)$ as in Definition 3, Lemma 2 allows us to write

$$\|f(x,y)\| \leq \|g(x)\| \exp(\sup_{\tau\in[0,y]} \|h(x,\tau)\| y).$$

It follows that the degree of f is less than $\max(\deg g, \deg h + 1)$.
The derivative of f relative to y is $h(\overrightarrow{x},y)f(\overrightarrow{x},y)$. Hence its degree is also bounded by $\max(\deg g, \deg h+1)$. By [1,14], we know that the other derivative relative to variable x is solution of linear differential equation $d' = hd + \frac{\partial h}{\partial i} f$ with initial condition $d(x,0) = \frac{\partial g}{\partial x}$. The bound given by Lemma 2 for this linear differential equation allows us to state that the degree of this derivative is less than $\max(\deg \frac{\partial g}{\partial x}, \deg h+1, \deg \frac{\partial h}{\partial x}+1, \deg f+1)$.
– Let $g = \mathrm{LIM}(f,K,\beta)$ as in Definition 5. By Lemma 1, we know that $g(\overrightarrow{x}) = \lim_{i\to\infty} f(i,\overrightarrow{x})$, g is of class $\mathcal{C}^1$, $\|g(\overrightarrow{x})\| \leq \|f(0,\overrightarrow{x})\| + K(\overrightarrow{x})/\beta(\overrightarrow{x})$ and $\|\frac{\partial g}{\partial x_i}\| \leq \|\frac{\partial f}{\partial x_i}(0,\overrightarrow{x})\| + K(\overrightarrow{x})/\beta(\overrightarrow{x})$. Now, the degree of $\frac{1}{\beta(\overrightarrow{x})}$ for any polynomial β can easily be shown to be less than 1. Hence, the degree of g and of $\frac{\partial g}{\partial x}$ is smaller than $max(\deg f, \deg K)$.

We are ready to prove the upper bound.

Proposition 12. $\mathcal{L}^* \subseteq \mathcal{E}(\mathbb{R})$.

Proof. By structural induction:

– The basic functions $0, 1, -1, U, \theta_3$ are easily shown elementarily computable.
– When $h = \mathrm{COMP}(f,g)$, f and g elementarily computable, then h is also elementarily computable: the constructions in [39] preserve elementarily computability.
– Let $g = \mathrm{LIM}(f,K,\beta)$, with f computed by elementary functional ϕ. We give the proof for f defined on $\mathbb{R}\times\mathcal{C}$ to $\mathbb{R}$ where $\mathcal{C}$ is a compact interval of $\mathbb{R}$. The general case is easy to obtain.
Let $x \in \mathbb{R}$, with $\beta(x) > 0$. Since $\beta(x)$ is a polynomial, $1/\beta(x)$ can be bounded elementarily by some computable integer N in some computable neighborhood of x.
$K(x)$ can be bounded elementarily by some computable integer K in some computable neighborhood of x. In a same way, the norm of x can be bounded by some computable integer X.
Let $(x_n) \rightsquigarrow x$. For all $i,j \in \mathbb{N}$, if we write abusively i for the constant sequence $k \mapsto i$, we have $|\nu_{\mathbb{Q}}(\phi(((i,x_n),j)) - f(i,x)| < \exp(-j)$.
By Lemma 1, if i is big enough ($i > \|x\|$), we have

$$\begin{aligned}|f(i,x) - g(x)| &\leq \tfrac{K\exp(-\beta(x)i)}{\beta(x)}\\ &\leq KN\exp(-\beta(x)i).\end{aligned}$$

Hence,

$$|\nu_{\mathbb{Q}}(\phi((i,x_n),j)) - g(x)| < \exp(-j) + KN\exp(-\beta(x)i).$$

If we take $j' \geq j+1$, $i' \geq N(j+1+\lceil\ln(KN)\rceil)$, we have $\exp(-j') \leq \frac{1}{2}\exp(-j)$, and $KN\exp(-\beta(x)i') \leq \frac{1}{2}\exp(-j)$. Hence g is computed by the functional $\psi : ((x_n),j) \mapsto \phi((\max(X, N(j+1+\lceil\ln(KN)\rceil), x_n)), j+1)$. since for all j,

$$\|\nu_{\mathbb{Q}}(\psi((x_n),j)) - g(x)\| \leq \frac{exp(-j)}{2} + \frac{exp(-j)}{2} \leq exp(-j).$$

– Let $f = \mathrm{LI}(g,h)$. We give the proof for $g : [0,1] \to \mathbb{R}$ and $h : [0,1]\times[c,d] \to \mathbb{R}$. The general case is easy to obtain.
This proof is copied from [12,13]. The idea is that, to find ϕ elementary computing f, one uses a numeric integration algorithm (Euler's Method).
First, let us note that f is twice differentiable with respect to its second variable since its derivative is the product of f and h that are differentiable.
To compute $f(x,y)$, we will slice $[0,y]$ into segments of length λ and compute approximations of $f(x,\tau_i)$ for τ_i multiple of λ.
$h \in \mathcal{E}(\mathbb{R})$. Let ϕ_h computing h. Let $(\phi) \rightsquigarrow (x,\tau_i)$. Let us define $\omega_i = \frac{(\phi_h(\phi))_n}{n+1}$ for n to be chosen.
$f \in \mathcal{E}(\mathbb{R})$. Let ϕ_g computing g. Let $(\phi_x) \rightsquigarrow x$. We will approach $f(x,\tau_i)$ by ψ_i defined by

$$\psi_0 = \frac{(\phi_g(\phi_x))_m}{m+1}$$
$$\psi_{i+1} = \psi_i + \lambda\psi_i\omega_i$$

Let us now compute the error induced by our approximation. Let $\varepsilon_i = f(x,\tau_i) - \psi_i$.
$\forall i, \exists\chi \in [\tau_i, \tau_{i+1}]; f(x,\tau_{i+1}) = f(x,\tau_i) + \lambda f(x,\tau_i)h(x,\tau_i) + \frac{\lambda^2}{2}\frac{\partial f}{\partial y}(x,\chi)$.

$$\begin{aligned}
\varepsilon_{i+1} &= f(x,\tau_i) - \psi_i + \lambda f(x,\tau_i)h(x,\tau_i) - \lambda\psi_i\omega_i + \tfrac{\lambda^2}{2}\tfrac{\partial f}{\partial y}(x,\chi)\\
|\varepsilon_{i+1}| &\leqslant |\varepsilon_i| + |\lambda h(x,\tau_i)(f(x,\tau_i) - \psi_i)| + |\lambda\psi_i(\omega_i - h(x,\tau_i))| + |\tfrac{\lambda^2}{2}\tfrac{\partial f}{\partial y}(x,\chi)|\\
&\leqslant |\varepsilon_i| \times |1 + \lambda h(x,\tau_i)| + \lambda\psi_i|\omega_i - h(x,\tau_i)| + \tfrac{\lambda^2}{2}\beta\\
&< |\varepsilon_i| \times |1 + \lambda h(x,\tau_i)| + \lambda\psi_i\tfrac{1}{n+1} + \tfrac{\lambda^2}{2}\beta
\end{aligned}$$

With $\beta = \max_{\chi\in[0,y]}(\frac{\partial f}{\partial y}(x,\chi))$.

$$\varepsilon_{i+1} < |\varepsilon_i| \times |1 + \lambda\overline{y}| + \tfrac{\lambda}{n+1}\overline{y} + \tfrac{\lambda^2}{2}\beta$$

With $\overline{y}$ set as a bound for h that can be elementarily computed as shown by the preceding lemma.
Some little algebra shows then

$$\begin{aligned}
|\varepsilon_i| &< |\varepsilon_0|\,[1 + \lambda\overline{y}]^i + \left(\lambda\overline{y}\tfrac{1}{n+1} + \tfrac{\lambda^2}{2}\beta\right)\tfrac{(1+\lambda\overline{y})^i - 1}{\lambda\overline{y}}\\
&< \left[\tfrac{1}{m+1} + \tfrac{\lambda\beta}{2\overline{y}} + \tfrac{\psi_i}{\overline{y}(n+1)}\right]\exp(\lambda i\overline{y})\\
&< \left[\tfrac{1}{m+1} + \tfrac{1}{n+1} + \tfrac{\lambda\beta}{2\overline{y}}\right]\exp(\lambda i\overline{y})
\end{aligned}$$

So, if we choose m, n, and i adequately (this choice can be made elementarily), we can make the error as little as wanted. This proves that f is elementarily computable and terminates our proof.

This ends the proof.

Replacing in previous proofs the bounds of Lemma 7 by bounds of type $\|f(\overrightarrow{x})\| \leq A\overline{E}_{n-1}^{[d]}(B\rho_{\mathcal{D}}(\overrightarrow{x}))$, one can also obtain:

Proposition 13. *$\forall n \geq 3$, $\mathcal{L}_n^* \subseteq \mathcal{E}_n(\mathbb{R})$.*

8 Lower bounds

We will now consider the opposite inclusion: $\mathcal{E}(\mathbb{R}) \subseteq \mathcal{L}^*$, proved for functions of class $\mathcal{C}^2$ on compact domains with rational endpoints.

Let $\epsilon > 0$ be some real. We write $\mathbb{N}\epsilon$ for the set of reals of the form $i\epsilon$ for some integer i. Given $y \in \mathbb{R}$, write $\lfloor y \rfloor_\epsilon$ for the unique $j\epsilon$ with j integer and $y \in [j\epsilon, j\epsilon + \epsilon)$.

Lemma 8. *Let $\epsilon : \mathbb{R} \to \mathbb{R}$ be some decreasing function of $\mathcal{L}^*$, with $\epsilon(x) > 0$ for all x and going to 0 when x goes to $+\infty$, and $1/\epsilon(x) \in \mathcal{L}^*$. Write ϵ_i for $\epsilon(\lfloor i \rfloor)$.*

Given $f : \mathbb{R}^2 \to \mathbb{R}^l$ in $\mathcal{L}^*$, there exists $F : \mathbb{R}^2 \to \mathbb{R}^l$ in $\mathcal{L}^*$ with the following properties:

- *For all $i \in \mathbb{N}$, $x \in \mathbb{N}\epsilon_i$, $F(i,x) = f(i,x)$*
- *For all $i \in \mathbb{N}$, $x \in \mathbb{R}$, $\|F(i,x) - f(i, \lfloor x \rfloor_{\epsilon_i})\| \leq \|f(i, \lfloor x \rfloor_{\epsilon_i} + \epsilon_i) - f(i, \lfloor x \rfloor_{\epsilon_i})\|$*
- *For all $i \in \mathbb{R}$, $x \in \mathbb{R}$, $\|\frac{\partial F}{\partial i}(i,x)\| \leq 5\|f(\lfloor i+1 \rfloor, \lfloor x \rfloor_{\epsilon_i}) - f(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i})\| + 25\|f(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i} + \epsilon_i) - f(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i})\| + 25\|f(\lfloor i+1 \rfloor, \lfloor x \rfloor_{\epsilon_{i+1}} + \epsilon_{i+1}) - f(\lfloor i + 1 \rfloor, \lfloor x \rfloor_{\epsilon_{i+1}})\|$.*

Proof. Let $\zeta = \frac{3\pi}{2}$. Let $\omega : x \mapsto \zeta\theta_3(\sin(2\pi x))$. $\forall i, \int_i^{i+1} \omega = 1$ and ω is equal to 0 on $[i + \frac{1}{2}, i+1]$ for $i \in \mathbb{N}$. Let $\Omega = \int(\omega)$ its primitive, and $int : x \mapsto \Omega(x - \frac{1}{2})$. int is a function similar to the integer part: $\forall i$, $\forall x \in [i, i + \frac{1}{2}]$, $int(x) = i = \lfloor x \rfloor$. Figure 1 shows graphical representations of ω and int respectively.

Let $\Delta(i,x) = f(i, x + \epsilon(i)) - f(i,x)$. For all i,x, we have

$$\begin{aligned} \tfrac{\omega(x/\epsilon(i))}{\epsilon(i)}\Delta(i, \epsilon(i)\ int(x/\epsilon(i))) &= 0 \textit{ whenever } x - \lfloor x \rfloor_{\epsilon(i)} \geq \epsilon(i)/2 \\ &= \tfrac{\omega(x/\epsilon(i))}{\epsilon(i)}\Delta(i, \lfloor x \rfloor_{\epsilon(i)}) \textit{ otherwise.} \end{aligned}$$

Let G be the solution of the linear differential equation

$$\begin{cases} G(i,0) &= f(0) \\ \frac{\partial G}{\partial x}(i,x) &= \frac{\omega(x/\epsilon(i))}{\epsilon(i)}\Delta(i, \epsilon(i) int(x/\epsilon(i))) \end{cases}$$

An easy induction on j then shows that $G(i, j\epsilon(i)) = f(i, j\epsilon(i))$ for all $j \in \mathbb{N}$.

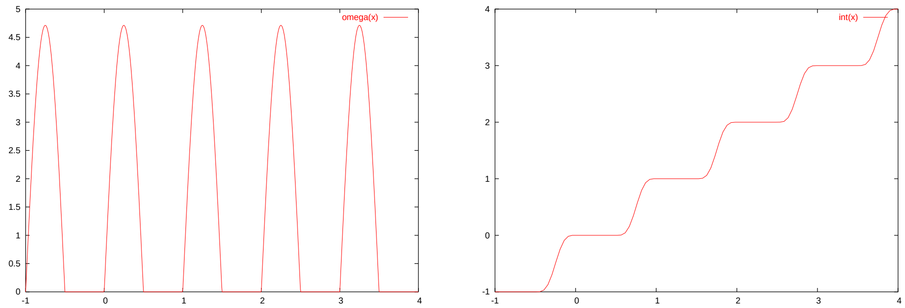

Figure 1. Graphical representations of ω and int.

On $[j\epsilon(i),(j+1)\epsilon(i))$,

$$G(i,x)-f(i,\lfloor x\rfloor_{\epsilon(i)})=\int_{j\epsilon(i)}^{x-j\epsilon(i)}\frac{\omega(t/\epsilon(i))}{\epsilon(i)}\Delta(i,\lfloor t\rfloor_{\epsilon(i)})dt,$$

hence, for all $i\in\mathbb{N}$,

$$\|G(i,x)-f(i,\lfloor x\rfloor_{\epsilon_i})\|\leq\|\Delta(i,\lfloor x\rfloor_{\epsilon_i})\|=\|f(i,\lfloor x\rfloor_{\epsilon_i}+\epsilon_i)-f(i,\lfloor x\rfloor_{\epsilon_i})\|.$$

Now, let $\Delta'(i,x)=G(i+1,x)-G(i,x)$. For all i,x we have

$$\begin{aligned}\omega(i)\Delta'(int(i),x)&=0 \ \textit{whenever}\ i-\lfloor i\rfloor\geq 1/2\\&=\omega(i)\Delta'(\lfloor i\rfloor,x)\ \textit{otherwise}\end{aligned}$$

Let F be the solution of linear differential equation

$$\begin{cases}F(0,x)=G(0,x)\\ \frac{\partial F}{\partial i}\quad\ =\omega(i)\Delta'(int(i),x)\end{cases}$$

An easy induction on i shows that $F(i,x)=G(i,x)$ for all integer i, and all $x\in\mathbb{R}$. Hence $F(i,x)=f(i,x)$ for all $i\in\mathbb{N}$, $x\in\mathbb{N}\epsilon_i$ and

$$\|F(i,x)-f(i,\lfloor x\rfloor_{\epsilon_i})\|\leq\|f(i,\lfloor x\rfloor_{\epsilon_i}+\epsilon_i)-f(i,\lfloor x\rfloor_{\epsilon_i})\|$$

for all $i\in\mathbb{N}$, $x\in\mathbb{R}$.

Now, $\frac{\partial F}{\partial i}$ is either 0 or $\omega(i)\Delta'(\lfloor i\rfloor,x)=\omega(i)(G(\lfloor i+1\rfloor,x)-G(\lfloor i\rfloor,x))$. In any case, it is derivable in x, and hence $\frac{\partial^2 F}{\partial x\partial i}$ is either 0 or $\omega(i)(\frac{\partial G}{\partial x}(\lfloor i+1\rfloor,x)-\frac{\partial G}{\partial x}(\lfloor i\rfloor,x))$.

When $x\in\mathbb{N}\epsilon_i$, bounding ω by 5 ($\zeta\leq 5$),

$$\|\frac{\partial F}{\partial i}\|\leq 5\|f(\lfloor i+1\rfloor,x)-f(\lfloor i\rfloor,x)\|.$$

When $x\in\mathbb{R}$,

$$\|\frac{\partial^2 F}{\partial x\partial i}\|\leq\|\frac{\partial G}{\partial x}(\lfloor i+1\rfloor,x\|+\|\frac{\partial G}{\partial x}(\lfloor i\rfloor,x)\|.$$

The term $\|\frac{\partial G}{\partial x}(\lfloor i \rfloor, x)\|$ can be either 0 or

$$\begin{aligned} 5\|\tfrac{\omega(x/\epsilon_i)}{\epsilon_i} \Delta(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i})\| &\leq \tfrac{25}{\epsilon_i}\|\Delta(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i})\| \\ &\leq \tfrac{25}{\epsilon_i}\|f(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i} + \epsilon_i) - f(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i})\|. \end{aligned}$$

A similar bound holds for the other term, replacing i by $i+1$.

Using mean value theorem,

$$\begin{aligned} \|\tfrac{\partial F}{\partial i}(i,x)\| &\leq \|\tfrac{\partial F}{\partial i}(i, \lfloor x \rfloor_{\epsilon_i})\| + \|\tfrac{\partial^2 F}{\partial x \partial i}(i,x)\|(x - \lfloor x \rfloor_{\epsilon_i}) \\ &\leq \|\tfrac{\partial F}{\partial i}(i, \lfloor x \rfloor_{\epsilon_i})\| + \epsilon(i)\|\tfrac{\partial^2 F}{\partial x \partial i}(i,x)\| \end{aligned},$$

which yields the expected bound.

Lemma 9. *If $f : \mathcal{C} \subset \mathbb{R} \to \mathbb{R}$ is defined over a closed interval containing $\overrightarrow{0}$, with bounds either rational or infinite, of class $\mathcal{C}^1$, and elementarily computable, then the primitive $\int(f)$ is in $\mathcal{L}^*$.*

Proof. Let $M_{\mathbb{N}} : \mathbb{N}^2 \to \mathbb{N}$ be the function given by Proposition 7 for function f: given some integer, K, $M_{\mathbb{N}}(K, \_)$ is a elementarily computable modulus of continuity of function f on $[-K, K]$.

For all $i \in \mathbb{N}$ and $j \in \mathbb{N}$, consider $x_j = j \exp(-M_{\mathbb{N}}(i+1, i))$, so that for all $x, y \in [x_j, x_{j+1}] \cap [-i-1, i+1]$, we have

$$|f(x) - f(y)| \leq exp(-i).$$

For all j, let p_j and q_j two integers such that $p_j \times \exp(-q_j)$ is at most $\exp(-i)$ far from $f(x_j)$. The functions $p_{\mathbb{N}} : \mathbb{N}^2 \to \mathbb{N}$, and $q_{\mathbb{N}} : \mathbb{N}^2 \to \mathbb{N}$ that map (i,j) to corresponding p_j and q_j are elementary.

By Proposition 9, these functions as well as function $M_{\mathbb{N}}$ can be extended to function $p : \mathbb{R}^2 \to \mathbb{R}$, $q : \mathbb{R}^2 \to \mathbb{R}$, $M : \mathbb{R}^2 \to \mathbb{R} \in \mathcal{L}$. Consider function $g : \mathbb{R} \times \mathcal{C} \to \mathbb{R}$ defined on all $(i,x) \in \mathbb{R} \times \mathcal{C}$ by $g(i,x) = p(i, \exp(M(i+1,i))x)e^{-q(i,\exp(M(i+1,i))x)}$. By construction, for i, j integer, we have

$$g(i, x_j) = p_j \exp(-q_j).$$

Consider the function F given by Lemma 8 for function g and $\epsilon : i \mapsto \exp(-M(i+1,i))$. We have

$$F(i, x_j) = g(i, x_j)$$

and

$$\|g(i, x_j) - f(x_j)\| \leq \exp(-i)$$

for all i, j.

For all $x \in \mathcal{C}$, and all integer $i \geq \|x\|$ we have (observe that if x_j denotes $\lfloor x \rfloor_\epsilon$, we have $x_j, x_{j+1} \in [-i-1, i+1]$)

$$\begin{aligned} \|F(i,x) - f(x)\| &\leq \|F(i,x) - F(i, \lfloor x \rfloor_\epsilon)\| + \|F(i, \lfloor x \rfloor_\epsilon) - g(i, \lfloor x \rfloor_\epsilon)\| \\ &\quad + \|g(i, \lfloor x \rfloor_\epsilon) - f(\lfloor x \rfloor_\epsilon)\| + \|f(\lfloor x \rfloor_\epsilon) - f(x)\| \\ &\leq \|F(i, \lfloor x \rfloor_\epsilon + \epsilon) - F(i, \lfloor x \rfloor_\epsilon)\| + 0 + \exp(-i) + \exp(-i) \\ &\leq \|g(i, x_{j+1}) - g(i, x_j)\| + 2\exp(-i) \\ &\leq \|g(i, x_{j+1}) - f(x_{j+1})\| + \|g(i, x_j) - f(x_j)\| \\ &\quad + \|f(x_{j+1}) - f(x_j)\| + 2\exp(-i) \\ &\leq 5 \times \exp(-i). \end{aligned}$$

Consider the function $G : \mathbb{R}^2 \to \mathbb{R}$ defined for all $i, x \in \mathbb{R}$ by the linear differential equation

$$\begin{cases} G(i,0) &= 0 \\ \frac{\partial G}{\partial x}(i,x) &= F(i,x) \end{cases}$$

Hence

$$G(i,x) = \int_0^x F(i,u)du.$$

We get

$$\|\frac{\partial G}{\partial x}(i,x) - f(x)\| = \|F(i,x) - f(x)\| \leq 5 \times \exp(-i)$$

and by mean value theorem on function $G(i,x) - f(x)$, we get

$$\|G(i,x) - \int_0^x (f)(x)\| \leq (5 \times \exp(-i))|x|,$$

when $i \geq \|x\|$.

Hence, $\int(f)(x)$ is the limit of $G(i,x)$ when i goes to $+\infty$ with integer values. We just need to check that schema LIM can be applied to function G of $\mathcal{L}^*$ to conclude: indeed, the limit of $G(i,x)$ when i goes to $+\infty$ will exist and coincide with this value, i.e. $\int(f)(x)$.

Since $\frac{\partial G}{\partial x} = F$, we have $\|\frac{\partial^2 G}{\partial i \partial x}\| = \|\frac{\partial F}{\partial i}\|$. Since $\frac{\partial G}{\partial i} = \int_0^x \frac{\partial F}{\partial i}(i,u)du$ implies

$$\|\frac{\partial G}{\partial i}\| \leq \int_0^x \|\frac{\partial F}{\partial i}\|du \leq |x| \times \|\frac{\partial F}{\partial i}\| \leq (x^2+1) \times \|\frac{\partial F}{\partial i}\|,$$

we only need to prove that we can bound $\|\frac{\partial F}{\partial i}\|$ by $K(x) \times \exp(-i)$ for some function $K \in \mathcal{L}^*$, and $i \geq \|x\|$.

But from Lemma 8, we know that for all i, x,

$$\begin{aligned} \|\tfrac{\partial F}{\partial i}(i,x)\| \leq\ & 5\|g(\lfloor i+1 \rfloor, \lfloor x \rfloor_{\epsilon_i}) - g(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i})\| \\ & +25\|g(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i} + \epsilon_i) - g(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i})\| \\ & +25\|g(\lfloor i+1 \rfloor, \lfloor x \rfloor_{\epsilon_{i+1}} + \epsilon_{i+1}) - g(\lfloor i+1 \rfloor, \lfloor x \rfloor_{\epsilon_{i+1}})\|. \end{aligned}$$

First term can be bounded by $5 \times \exp(-i) + 5 \times \exp(-i) = 10 \times \exp(-i)$.

Second term can be bounded by $25(\|g(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i} + \epsilon_i) - f(\lfloor x \rfloor_{\epsilon_i} + \epsilon_i)\| + \|f(\lfloor x \rfloor_{\epsilon_i} + \epsilon_i) - f(\lfloor x \rfloor_{\epsilon_i})\| + \|g(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i}) - f(\lfloor x \rfloor_{\epsilon_i})\|) \leq 25 \times \exp(-i) + 25 \times \exp(-i) + 25 \times \exp(-i) = 75 \times \exp(-i)$, for $i \geq \|x\|$.

Similarly for third term, replacing i by $i+1$.

Hence, when $i \geq \|x\|$,

$$\|\frac{\partial F}{\partial i}(i,x)\| \leq 160 \times \exp(-i),$$

and

$$\|\frac{\partial G}{\partial i}(i,x)\| \leq 160 \times (x^2+1) \times \exp(-i),$$

and so schema LIM can be applied on function G of $\mathcal{L}^*$ to get function $\int(f)$. This ends the proof.

Actually, the previous lemma can easily be extended a little bit to get any primitive:

Lemma 10. *Let h be elementarily computable and defined on* 0.

If $f : \mathcal{C} \subset \mathbb{R} \to \mathbb{R}$ is defined over a closed interval containing $\overrightarrow{0}$, with bounds either rational or infinite, of class $\mathcal{C}^1$, and elementarily computable, then the primitive of f equal to $h(0)$ in 0 is in $\mathcal{L}^*$.

Proof. Replace in previous proof the initial condition $G(i, 0) = 0$ of the differential equation defining function G, by $G(i, 0) = g(i)$ where $g : \mathbb{R} \to \mathbb{R}$ is a function converging to $h(0)$, obtained by extending a suitably chosen function $g : \mathbb{N} \to \mathbb{N}$.

We are now ready to prove the missing inclusion of Theorem 1.

Proposition 14. *Let $f : \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$ be some function over the reals of class $\mathcal{C}^2$, with $\mathcal{D}$ product of compact intervals with rational endpoints. If f is in $\mathcal{E}(\mathbb{R})$, then it belongs to $\mathcal{L}^*$.*

Proof. Putting together Lemma 3, Proposition 7 and Lemma 10 applied on f', we obtain this proposition when $k = l = 1$. The case $k > 1, l = 1$ can be obtained by adapting the previous arguments to functions of several variables. The case $l > 1$ is immediate since a function is in $\mathcal{L}^*$ if its projections are.

The missing inclusion of Theorem 2 can be proved similarly for all levels $n \geq 3$ of the Grzegorczyk hierarchy.

Proposition 15. *Let $f : \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$ be some function over the reals of class $\mathcal{C}^2$, with $\mathcal{D}$ product of compact intervals with rational endpoints. If f is in $\mathcal{E}_n(\mathbb{R})$, for $n \geq 3$, then it belongs to $\mathcal{L}^*_n$.*

9 Extensions

Observe now that, for non-compact domains we have:

Proposition 16. *Let $f : \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$ be some function over the reals of class $\mathcal{C}^2$, with $\mathcal{D}$ product of closed intervals with rational or infinite endpoints.*

If f and the derivatives of f are in $\mathcal{E}(\mathbb{R})$ then $f \in \mathcal{L}^*$.

Proof. This follows from Proposition 7 and more specifically Lemma 10.

Recall that we have, conversely $\mathcal{L}^* \subset \mathcal{E}(\mathbb{R})$ by Proposition 12.

Remark 8. If one suppresses the condition, in LIM schema, that the limit must be of class $\mathcal{C}^2$, then one does not need to assume in Lemma 10 that the function is of class $\mathcal{C}^1$. In that case, any function $f \in \mathcal{E}(\mathbb{R})$, differentiable, whose derivatives are in $\mathcal{E}(\mathbb{R})$, can be obtained as in Lemma 10, that is as a limit schema of functions of $\mathcal{L}^*$.

We have also the following corollary:

Corollary 1. *Let $f : \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$ be some function over the reals of class $\mathcal{C}^\infty$, with $\mathcal{D}$ product of compact intervals with rational endpoints. If f is $\mathcal{E}(\mathbb{R})$, then all its derivatives $f^{(n)}$, $n \geq 0$, belong to $\mathcal{L}^*$.*

Proof. From Lemma 3, for all n, $f^{(n+1)}$ is elementarily computable since it is of class $\mathcal{C}^2$ over a compact domain. Now, for all n, $f^{(n)}(x) \in \mathcal{L}^*$ from Lemma 10 applied on $f^{(n+1)}$.

We also have a kind of *normal form theorem*:

Proposition 17. *If constant function π is added to the base functions of $\mathcal{L}^*$, then every function of $\mathcal{L}^*$ can be defined using only* 1 *schema* LIM.

Proof. The previous proof shows that to represent a $\mathcal{C}^2$ function that belongs to $\mathcal{E}(\mathbb{R})$, using one LIM is sufficient, if π is considered as base function (in order to have the inclusion $\mathcal{L} \subset \mathcal{L}^*$. That means that all functions from $\mathcal{L}^*$ can be written with at most one LIM in that case.

A corollary of this proposition is that composing several LIM schemata is always equivalent to at most one for functions of our classes, if constant function π is considered as a base function. Otherwise, two limits are sufficient.

All previous results generalize to Grzegorczyk's hierarchy.

10 Variations on schemas

First, we can note that it is possible to change a bit our schemata in order to have a more natural LIM schema. The price to pay is a less natural LI schema, that we called CLI in [8].

Formally, we define CLI as follows:

Definition 7 (CLI **schema).** *From g,h and c, with h differentiable and first derivatives of h bounded by c,*

CLI(g, h, c) *is any solution defined on a product of closed intervals of the linear differential equation* $\frac{\partial f}{\partial y}(\overrightarrow{x}, y) = h(\overrightarrow{x}, y) f(\overrightarrow{x}, y)$ *with* $f(\overrightarrow{x}, 0) = g(\overrightarrow{x})$.

In this schema, if g goes to $\mathbb{R}^n$, $f = \text{CLI}(g, h, c)$ also goes to $\mathbb{R}^n$ and $h(\overrightarrow{x}, y)$ is a $n \times n$ matrix with elements in $\mathcal{L}$.

One first useful remark is to understand that replacing LI schema by CLI schema in the definition of class $\mathcal{L}$, does not change the statements of Propositions 8, 9 and 10.

Now, using this controlled linear integration schema, we do not need to impose a bound on the second derivatives in LIM schema, since the reason for this bound was to be able to state in Lemma 7 that partial derivatives of a function of the class have finite degree, and hence to be able to apply Euler's method in Proposition 12. Using CLI, we know that the first derivatives of the functions are bounded elementarily and hence that the second derivatives of the constructed function are also bounded elementarily. Observing the proof, this is sufficient.

So if we denote LIM_w the schema:

Definition 8 (LIM$_w$ **schema).** *Let $f : \mathbb{R} \times \mathcal{D} \subset \mathbb{R}^{k+1} \to \mathbb{R}^l$, $K : \mathcal{D} \to \mathbb{R}$ and $\beta : \mathcal{D} \to \mathbb{R}$ a polynomial with the following hypothesis: such that for all $\overrightarrow{x}$, $t \geq \|\overrightarrow{x}\||$, $\|\frac{\partial f}{\partial t}(t, \overrightarrow{x})\| \leq K(\overrightarrow{x}) \exp(-t\beta(\overrightarrow{x}))$,*

Then, on every product of closed intervals $I \subset \mathbb{R}^k$ on which $\beta(\overrightarrow{x}) > 0$, $F(\overrightarrow{x}) = \lim_{t \to +\infty} f(t, \overrightarrow{x})$ exists by Lemma 1. If F is of class $\mathcal{C}^2$, then we define $\mathrm{LIM}_w(f, K, \beta)$ as this function $F : I \to \mathbb{R}$.

We can then claim that if, in the definition of class $\mathcal{L}^*$ and $\mathcal{L}_n^*$, LIM$_w$ schema is substituted to LIM schema, and CLI schema is substituted to LI schema, then we still have Theorem 1 and Theorem 2, as well as all following lemmas and propositions (except last assertion of Lemma 7 as discussed above).

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