Deciding stability and mortality of piecewise affine dynamical systems*

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1 Introduction

This paper studies problems such as: given a discrete time dynamical system of the form $x(t+1) = f(x(t))$ where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a (possibly discontinuous) piecewise affine function, decide whether all trajectories converge to 0. We show in our main theorem (Theorem 2) that this Attractivity Problem is undecidable as soon as $n \geq 2$. The same is true of two related problems: Stability (is the dynamical system globally asymptotically stable?) and Mortality (decide whether all trajectories go through 0). In section 4 we show that Attractivity and Stability become decidable in dimension 1 for *continuous* functions, and these two notions become in fact equivalent. One can show with similar techniques that Mortality is also decidable for piecewise affine continuous functions of one variable.

It is well-known that Turing machines can be simulated by various types of dynamical systems, including hybrid systems and the piecewise affine dynamical systems studied in this paper. As an immediate corollary, one obtains the undecidability of problems such as the following: "given a particular initial state,

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does the resulting trajectory of the dynamical system ever reach (or converge to) the origin?" In a typical proof that such a simulation is possible, one usually associates a machine configuration to an element of the dynamical system's state space. The configurations of the Turing machine are mapped to a countable (and typically, nondense) subset of the state space. A correct simulation is obtained provided that the dynamics of the dynamical system are properly defined on this subset.

We now compare with the problems considered and the results obtained in this paper. We deal with *global* stability-like questions such as "do all trajectories converge to the origin?". This is similar in spirit to the question "does a Turing machine halt for every initial configuration?". The latter problem is known to be undecidable [8], and the proof is significantly more involved than the proof of undecidability of the halting problem. This suggests that establishing undecidability of stability problems is qualitatively different, and possibly much harder, than the usual simulation results. An additional complication is the following: unlike the problem of simulating a Turing machine with a dynamical system, it now becomes important to define the dynamics of the dynamical system on the entire state space, while ensuring certain desired properties. To this effect, we introduce an encoding that associates a legitimate machine configuration to *all* points in the state space (Lemma 1).

We finally note that while our main result could be established by using the undecidability [8] of a corresponding Turing machine problem, we take a parallel route, based on 2-counter machines. The advantages are that the paper becomes self-contained (the rather difficult proof in [8] is replaced by a much simpler argument, provided in Theorem 1, which establishes the undecidability of the corresponding problem for counter machines), and that the simulation is easier to describe.

This work was motivated by a question of Sontag [18]: is global asymptotic stability decidable for saturated linear systems? These are dynamical systems of the form $x(t+1) = \sigma(Ax(t) + b)$ where $x(t)$ lives in the state space $\mathbb{R}^n$ and σ denotes componentwise application of the saturated linear function $\sigma : \mathbb{R} \to [-1, 1]$ defined as follows: $\sigma(x) = x$ for $|x| \leq 1$, $\sigma(x) = 1$ for $x \geq 1$, $\sigma(x) = -1$ for $x \leq -1$. Saturated linear system therefore fall within the class of piecewise affine systems studied in this paper. They are however much more restricted. Note in particular that the corresponding transition function $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous since σ is continuous. We plan to publish undecidability results for a particular class of continuous piecewise affine systems in a future paper; see [3]. Note that discontinuous piecewise affine functions occur naturally as models of simple hybrid systems; see [19] and [4] for discrete time examples and [2] for an example in continuous time. Surveys of decidability and complexity results available for hybrid and nonlinear systems are given in [1], [7], [18] and [4].

2 Basic definitions

In the sequel X denotes a metric space and 0 some arbitrary point of X which is chosen as origin (when $X \subseteq \mathbb{R}^n$, we assume that 0 is the usual origin of $\mathbb{R}^n$).

Definition 1 *Let $f : X \to X$ be an arbitrary map on a metric space X.*

f is globally convergent if for every initial point $x_0 \in X$ the trajectory $x_{t+1} = f(x_t)$ converges to 0.

f is mortal if for every initial point $x_0 \in X$ there exists $t \geq 0$ such that $f^t(x_0) = 0$.

f is locally asymptotically stable if for any neighborhood U of 0 there is another neighborhood V of 0 such that for every initial point $x_0 \in V$ the trajectory $x_{t+1} = f(x_t)$ converges to 0 without leaving U (i.e., $x_t \in U$ for all $t \geq 0$ and $\lim_{t\to+\infty} x_t = 0$).

f is globally asymptotically stable if f is globally convergent and locally asymptotically stable.

A map $f : X \to X$ which is not mortal is called *immortal*. Asymptotic stability is discussed for instance in [17], where in particular dynamical systems with inputs ("control systems") are studied.

Next we define what we mean by a piecewise affine function. Define the *sign function* by

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{when} \quad x \geq 0 \\ 0 & \text{when} \quad x < 0 \end{cases}$$

and consider the natural extension of this function to $\mathbb{R}^m$ by applying the function componentwise. Let $n, m \geq 1$ and consider $\Omega \subseteq \mathbb{R}^n$ and $\{0,1\}^m = \{e_1, e_2, \ldots, e_{2^m}\}$. Let $C \in \mathbb{Q}^{m\times n}$ and $d \in \mathbb{Q}^m$. For any given e_i the set $H_i = \{x \in \Omega : \operatorname{sgn}(Cx + d) = e_i\}$ is a subset of Ω defined by an intersection of finitely many halfspaces. The sets H_i ($i = 1, \ldots, 2^m$) form a partition of Ω, i.e., $\Omega = \cup_{i=1}^{2^m} H_i$ and $H_i \cap H_j = \emptyset$ whenever $i \neq j$. A *piecewise affine function on* Ω is a function given by

$$f : \Omega \to \Omega : x \mapsto A_i x + b_i \text{ when } x \in H_i$$

for some $A_i \in \mathbb{Q}^n$ and $b_i \in \mathbb{Q}^n$.

Observe that the composition of two piecewise affine functions is still a piecewise affine function.

3 Stability and mortality for discontinuous piecewise affine functions

In this section we prove that mortality, attractivity and stability for discontinuous piecewise affine functions are undecidable. The proof consists in first showing that mortality for 2-counters machines is undecidable, then in proving that piecewise affine functions are able to simulate 2-counters machine in a sense strong enough to deduce the undecidability of all three properties for piecewise affine functions.

use the immortality problem

3.1 The mortality problem for 2-counter machines

d'immortalite parce

We consider counter machines: a n-counter machine is an abstract, synchronous, deterministic computing machine with a finite number of internal states $Q = \{0, 1, 2, \dots, m-1\}$. It operates on a finite number of nonnegative integer registers $R_1, \dots, R_n$. Depending upon its internal state and whether the registers are equal to 0 it can perform one of the following operations: leave the registers unchanged, increase some register R_j by 1, or decrease some register R_j by 1 (assuming $R_j \neq 0$).

increment

The instructions for the counter machines are tuples

$$[i, b_1, \dots, b_n, j, D, k]$$

where $i \in Q$ represents the present state, $b_j \in \{true, false\}$ represents whether register R_j is null, j the register which is modified by the instruction, $D \in \{Increment, Decrement, NoChange\}$ the operation, and $k \in Q$ the new internal state. For consistency, no two tuples begin with the same $n+1$ symbols. This definition of a counter machine is slightly different from that given in [9] but is easily seen equivalent in terms of computational power.

j'utilise dans

The value of the registers with the internal state of the machine constitutes a *configuration* of the machine. If a configuration has a corresponding instruction, the result of applying it is another configuration, a *successor* of the original. A configuration for which there is no tuple is said to be a *halting configuration*.

There is no loss of generality to assume that the only halting configuration is the one where the internal state is 0 and where the registers have value 0.

Extending the relation of successor to its transitive completion, each configuration with a halting successor can be termed *mortal*, the others that do not lead to halting configurations but rather run for ever are termed *immortal*.

The configuration space of n-counters machines P can be considered as $C = \mathbb{N}^n \times Q$. n-counters machines are special cases of dynamical systems over C: $P = (C, f_P)$ where $f_P : C \to C$ is the function that maps non-halting configurations to their successors, and the halting configuration $(0, 0)$ to itself.

We will use the following result, which is an analog of the result proved in [8] for Turing machines. Let us note that our result here is not a corollary of the result in [8]: the fact that counter machines can simulate Turing machines does not readily imply that the immortality problem for counter machines is as hard as for Turing machines.

Theorem 1 *The problem of determining if a given n-counters machine halts on all possible* configurations *(the machine is then said to be* mortal*) is undecidable. This assertion remains true when $n = 2$.*

Proof:

The proof is by reduction from the classical halting problem for counter machines; see [9]. Consider a counter machine M with m internal states labeled $q_1, q_2, \ldots, q_m$, n registers $R_1, \ldots, R_n$ and let $s = (r_1, r_2, \ldots, r_n, q_l)$ be a given configuration of M. Instructions of M have the form $[q_i, b_1, b_2, \ldots, b_n, j, D, q_k]$.

To establish the first part of the result we describe how to construct effectively a counter machine M' on $n+2$ registers $R_1, \ldots, R_n, V, W$ such that M' halts on all possible configurations if and only if M halts on s.

The machine M' has a special state denoted q_0. Each time that M' enters state q_0, it executes a sequence of instructions whose effect is to store r_i in R_i, $2\max(1, V)$ in W and 0 in V. After having done this, it moves into state q_l.

Then the machine starts a simulation of the machine M. The simulation is such that, before performing any of the instructions of M, the machine first increases the register's content of V by 1, decreases that of W by 1 and performs the instruction of the machine M only if W is not equal to 0. If $W = 0$ it returns to the special state q_0.

Thus, the instructions of the machine M

$$[q_i, b_1, b_2, \ldots, b_n, j, D, q_k]$$

are all changed into sixteen instructions for M';

$$\begin{gathered}
[q_i, b_1, b_2, \ldots, b_n, b^*_{n+1}, b^*_{n+2}, n+1, Increment, q'_i] \\
[q'_i, b_1, b_2, \ldots, b_n, b^*_{n+1}, b^*_{n+2}, n+2, Decrement, q''_i] \\
[q''_i, b_1, b_2, \ldots, b_n, b^*_{n+1}, True, n+2, NoChange, q_0] \\
[q''_i, b_1, b_2, \ldots, b_n, b^*_{n+1}, False, j, D, q_k]
\end{gathered}$$

where b^*_{n+1} and b^*_{n+2} range over all four possible combinations $b^*_{n+1}, b^*_{n+2} \in \{True, False\}$.

We claim that M' halts on all possible configurations if and only if M halts on s.

One of the implications is clear. If M' halts on all possible configurations, it must halt on the configuration $(r_1, \ldots, r_n, v, 0, q_0)$ for all possible $v \geq 0$. When started on $(r_1, \ldots, r_n, v, 0, q_0)$, the machine M' simulates $2\max(1, V)$ steps of M in starting state q_l before returning to state q_0. Thus, if M' halts on all possible configurations, M must halt on $(r_1, \ldots, r_n, q_l)$.

Assume now that M halts on $(r_1, r_2, \ldots, r_n, q_l)$ and let k be the number of steps after which it halts. We need to show that M' halts on all possible configurations. Let $s' = (r_1, \ldots, r_n, v, w, q_r)$ be an arbitrary configuration of M'.

The register W is regularly decremented when executing instructions of M'. It is therefore clear that, whatever w, the machine M' will halt on s' or W will reach 0 after finitely many steps. In the latter case, the machine will restart a simulation of M with an increased register content for W. After sufficiently many returns to q_0, the register W will eventually contain a value larger than $k+1$ and the machine M' will then halt since it will simulate k steps of M on $(r_1, r_2, \ldots, r_n, q_l)$.

It remains to show how to reduce the number or registers to two. Let M' be a counter machine on n registers $R_1, R_2, \ldots, R_n$. We construct a machine M''on two registers S and T such that M'' halts on all possible configurations if and only if M' does. The content of the registers R_i of M' are stored in the register S of M'' by the classical prime number encoding. The nonnegative integers $r_1, r_2, \ldots, r_n$ are encoded into the nonnegative integer s by $s = 2^{r_1}3^{r_2}5^{r_3}\ldots\pi(n)^{r_n}$ where $\pi(n)$ is the $(n+1)$th prime number. Incrementation (respectively, decrementation) of the register R_i can then be obtained by multiplying (respectively, dividing) s by $\pi(i)$. These incrementing and decrementing operations can be performed on M'' with the help of the register T. The register T can also be used to test divisibility of s by $\pi(i)$ and hence equality to zero of the registers R_i can be checked with the machine M''. Finally one can verify that this construction preserves mortality of counter machines and so mortality is undecidable for 2-counter machines.

register □

3.2 Simulating a n-counters machine by a piecewise affine function

In traditional simulations of counter machines or Turing machines by dynamical systems, a machine configuration is encoded by a single point of the dynamical system's state space [11, 16, 15, 14, 10, 6, 2]. Since we are interested in this section in the global behavior of dynamical system on $\mathbb{R}^2$, we will instead assign the same machine configuration to all points in a subbox of a certain box $\mathcal{N}^* \subseteq \mathbb{R}^2$.

Lemma 1 *Given a 2-counter m-state machine P with transition function $f_P : C \to C$, one can construct a piecewise affine function $g_P : \mathcal{N}^* \to \mathcal{N}^*$ and an encoding function $\nu' : \mathcal{N}^* \to C$ such that the following conditions hold.*

(i) $\mathcal{N}^* = [0, m[\times[0, 1[$ *and* $\nu'(\mathcal{N}^*) = C$.

(ii) $\nu'(x)$ *is equal to the halting configuration* $(0, 0, 0)$ *of P if and only if* $x \in [0, 1/2[^2$, *and in this case* $g_P(x) = 0$.

(iii) The following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{f_P} & C \\ {\scriptstyle \nu'}\uparrow & & \uparrow{\scriptstyle \nu'} \\ \mathcal{N}^* & \xrightarrow{g_P} & \mathcal{N}^* \end{array}$$

i.e., for all $x \in \mathcal{N}^*$, $f_P(\nu'(x)) = \nu'(g_P(x))$.

Proof: We first define ν'. This encoding maps a point $(x_1, x_2) \in \mathcal{N}^*$ to the unique configuration (w_1, w_2, q) such that $x_2 \in [1 - 1/2^{w_2}, 1 - 1/2^{w_2+1}[$ and $x_1 - q \in [1 - 1/2^{w_1}, 1 - 1/2^{w_1+1}[$. Note that $\nu'(\mathcal{N}^*) = C$ as required, and x_2 (respectively, x_1) encodes an empty counter if and only if $x_2 \in [0, 1/2[$ (respectively, $x_1 - q \in [0, 1/2[$).

The piecewise affine function g_P will be affine on each box B of the form $[q+\alpha, q+\alpha+1/2[\times[\beta, \beta+1/2[$ subbox. suffit pas. where $q \in \{0, \ldots, m-1\}$ and $\alpha, \beta \in \{0, 1/2\}$. By definition of ν' all points in this box encode a configuration in state q and the emptiness status of each counter is also uniquely defined (by the values of α and β). The next state q' and the operations to be applied to the counters are therefore the same for all configurations in $\nu'(B)$.

In the box $[0, 1/2[^2$ corresponding to the halting configuration $(0, 0, 0)$ of P we set $g_P(x_1, x_2) = (0, 0)$. In other boxes we proceed as follows. For $(x_1, x_2) \in B$, we take $g_P(x_1, x_2) = (x'_1, x'_2)$ where $1 - x'_2 = a(1 - x_2)$ and $1 - (x'_1 - q') = b(1 - (x_1 - q))$. Each constant a and b is set to 2 if the corresponding counter is decremented, to 1/2 if it is incremented, or to 1 if it is unchanged. It is clear that the map $g_P : \mathcal{N}^* \to \mathcal{N}^*$ thus defined makes the diagram commutative. □

3.3 Undecidability in two dimensions

Theorem 2 *The three problems below are all undecidable.*
Let a piecewise affine function $g : \mathbb{R}^2 \to \mathbb{R}^2$ be given.

1. *Mortality Problem: is g mortal?*
2. *Attractivity Problem: is g globally convergent?*
3. *Stability Problem: is g globally asymptotically stable?*

Proof: We first show that problem 1 is undecidable by a reduction from the immortality problem for 2-counter machines. Assume a 2-counter machine P is given. Let g'_P be the extension to $\mathbb{R}^2$ of map g_P of Lemma 1 obtained by setting $g'_P(x) = 0$ for $x \notin \mathcal{N}^*$. We shall prove that P has an immortal configuration iff g'_P has an immortal trajectory: i.e. iff there exists some sequence $x^{t+1} = g'_P(x^t)$ with $x^t \neq 0$ for all $t \geq 0$.

Assume first that such an immortal trajectory exists. Since g'_P is zero outside $\mathcal{N}^*$, $x^t \in \mathcal{N}^*$ for all $t \geq 0$. ¿From the commutative diagram of Lemma 1, we see that the sequence $c^t = \nu'(x^t)$ is a sequence of successive configurations of P.

¿From condition (ii) in the same lemma, $c^t \neq (0,0,0)$ for all $t \geq 0$. Configuration c^0 is therefore immortal.

Conversely, assume P to be immortal: there exists an infinite sequence of configurations c_t with $c_{t+1} = f_P(c_t)$, $c_t \neq (0,0,0)$. By condition (i) of Lemma 1, there exists $x^0 \in \mathcal{N}^*$ such that $\nu'(x^0) = c^0$. We claim that the trajectory $x^{t+1} = g_P(x^t)$ is immortal. Indeed, by the commutative diagram we have $\nu'(x^t) = c^t \neq 0$ for all $t \geq 0$, hence $x^t \neq 0$ by condition (ii) of Lemma 1.

The undecidability of problems 2 and 3 now follows from a simple observation. On the one hand, an immortal trajectory of g'_P does not converge to the origin since it remains in $\mathcal{N}^* \setminus [0, 1/2[^2$. On the other hand, any mortal trajectory of g'_P satisfies $x_t = 0$ for t large enough since 0 is a fixed point of g'_P. That is, for g'_P mortality is equivalent to global convergence and to global stability. $\square$

Remarks.

1. It is easily seen that these three problems remain undecidable for piecewise affine functions $g : \mathbb{R}^n \to \mathbb{R}^n$ whenever $n \geq 2$.

2. We do not know if these problems remain undecidable for a fixed number of partitions.

3. A related problem is the point-to-fixed-point problem, i.e., the problem of determining, for a given piecewise affine function $g : \mathbb{R}^n \to \mathbb{R}^n$ *and* initial point $x_0 \in \mathbb{R}^n$, if the iterates $x_{t+1} = g(x_t)$ eventually reach a fixed point. This problem is known to be undecidable for $n = 2$ and for less than 800 partitions; see [11]. The decidability of the case $n = 1$ was proposed as an open problem in [11], and it seems to be open to this date. In fact, we are not aware of a decision algorithm for the case $n = 1$ even when there are only *two* partitions.

4 Decidability in one dimension

Theorem 3 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map from such that $f(0) = 0$. Then, the following properties are equivalent:*

(a) f is globally convergent.

(b) For every $x > 0$ we have $f(x) < x$ and $f^2(x) < x$, and for every $x < 0$ we have $x < f(x)$ and $x < f^2(x)$.

(c) f is globally asymptotically stable.

Proof: We first prove that (a) implies (b). Suppose that f is globally convergent. Furthermore, suppose, in order to derive a contradiction, that there exists some $x > 0$ such that $f(x) \geq x$. If we have $f(y) \geq y$ for all $y > 0$, then the sequence $f^k(x)$ is nondecreasing, which contradicts global convergence. Therefore, there exists some $y > 0$ such that $f(y) < y$. Using continuity, there exists some $z > 0$ such that $f(z) = z$, which again contradicts global convergence. This shows that $f(x) < x$ for all $x > 0$. Since f is globally convergent, it is clear that f^2 is also globally convergent, and the preceding argument also establishes that $f^2(x) < x$ for all $x > 0$. The conditions for the case where $x < 0$ are established by a symmetrical argument.

We now assume that the conditions in (b) hold, and proceed to establish property (c). For $x > 0$, we define $F_-(x) = \min_{0 \le z \le x} f(z)$. Since $f(0) = 0$, it follows that $F_-(x) \le 0$ for any $x > 0$. We claim that f maps the interval $I = [F_-(x), x]$ into $[F_-(x), x)$. Indeed, for any positive $z \in I$, we have $F_-(x) \le f(z) < z \le x$. If $z \in I$ is negative, then $F_-(x) \le z < f(z)$. Also, using the continuity of f and the definition of $F_-(x)$, a negative $z \in I$ must be the image $f(y)$ of some $y \in [0, x]$. Therefore, $f(z) = f^2(y) < y \le x$, which completes the proof of the claim.

The property established in the preceding paragraph implies that if $f^k(x) > 0$, then $f^{k+l}(x) < f^k(x)$, for all $l \ge 1$. Thus, the subsequence of $\{f^k(x)\}$ obtained by restricting to k for which $f^k(x)$ is positive, is monotonically decreasing. It must therefore converge, and the only possible limit is zero, due to the continuity of f. By an entirely symmetrical argument, we also conclude that the subsequence obtained by restricting to k for which $f^k(x)$ is negative is monotonically increasing. Hence, $f^k(x)$ must converge to zero. Furthermore, since the positive and negative subsequences of $\{f^k(x)\}$ are monotonic, for every initial x, it is easily seen that there exist arbitrarily small invariant neighborhoods of 0. This establishes global asymptotic stability as well.

The fact that (c) implies (a) is an immediate consequence of the definitions. □

A decision algorithm follows immediately from Theorem 3. For this algorithmic application we assume that our piecewise affine function f is defined by equations with rational coefficients (i.e. the endpoints of intervals where f is affine and the corresponding slopes are rational numbers). A generalization to a larger class of "finitely representable" coefficients (e.g. algebraic numbers) is straightforward (and arbitrary real coefficients can be allowed if we work with an algebraic model of computation). Generalizing to a larger class than piecewise affine functions (e.g. to piecewise polynomial functions) is also straightforward.

Corollary 1 *Let $f : E \to E$ be a piecewise affine continuous function, where E is either $\mathbb{R}$ or a compact interval in $\mathbb{R}$ that contains 0. There is an algorithm for deciding the global asymptotic stability of f.*

Proof: For the case where $E = \mathbb{R}$, it suffices to test the conditions (b) in Theorem 3, which is straightforward. For the case where E is an interval of the form $[a, b]$, we note that Theorem 3 remains valid, and the same decision procedure applies. Alternatively, we could extend the function f to outside $[a, b]$ (e.g. by $f(x) = f(b)$ for $x > b$ and $f(x) = a$ for $x < a$), and note that f and its extension share the same stability and convergence properties. □

Without a continuity assumption the situation is quite different. For instance, the map $f : [0, 1] \to [0, 1]$ defined by: $f(x) = 2x$ for $0 \le x \le 1/2$,

$f(x) = 0$ for $1/2 < x \leq 1$ is globally convergent but it is not globally asymptotically stable. We leave it as an open problem whether there is a decision algorithm for discontinuous piecewise affine functions.

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