Continuous Models. Computations. Distributed Algorithms.

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Habilitation à Diriger les Recherches

Nancy,
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Some Topics

- Tools for verification
- Frontier between Tractability/Non-tractability
- Complexity in Blum Shub Smale Model
- Programming with Rules and Strategies
- Exotic (ex Probabilistic) Rewriting
- Continuous Time Models

http://www.loria.fr/~bournez/load/HDR/cv-commente.pdf

Objectives

Main objective

Understand computation theories for continuous systems.

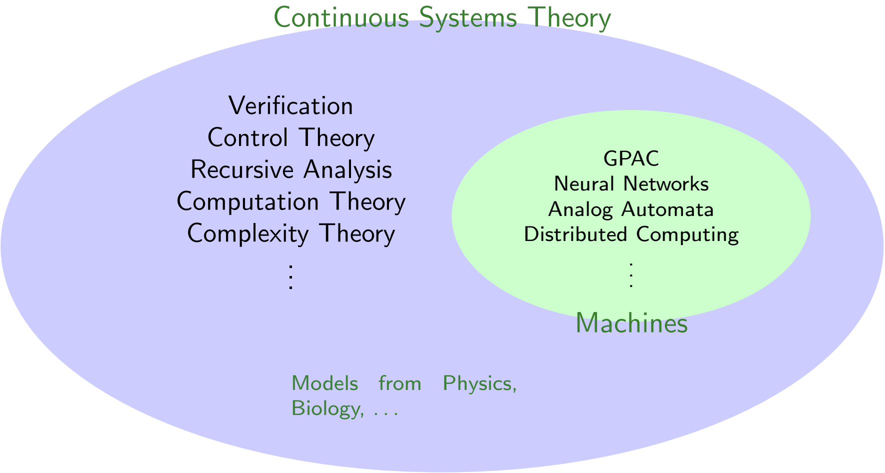
Dynamical Systems
over a continuous space
$\mathcal{H} = (\mathbb{R}^n, f)$

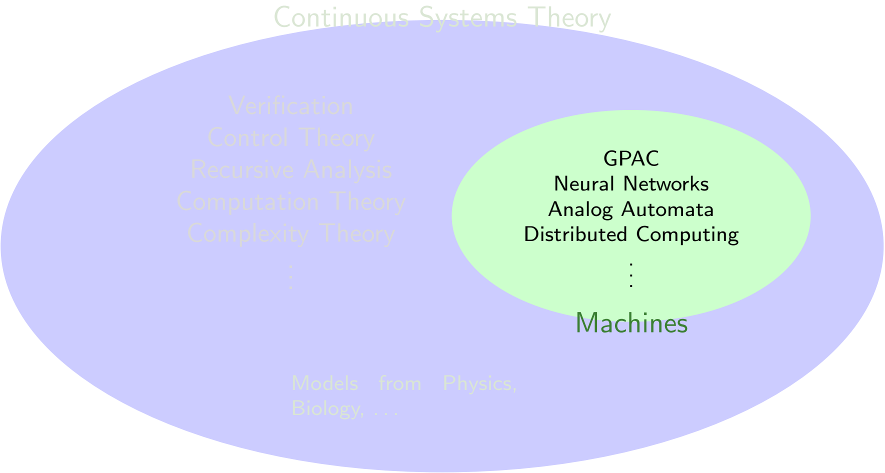
Discrete-Time

$$\begin{cases} y(t+1) & = & f(y(t)) \\ y(0) & = & x \end{cases}$$

Continuous-Time

$$\begin{cases} y' & = & f(y(t)) \\ y(0) & = & x \end{cases}$$


Continuous Systems Theory
Verification
Control Theory
Recursive Analysis
Computation Theory
Complexity Theory
⋮
GPAC
Neural Networks
Analog Automata
Distributed Computing
⋮
Machines
Models from Physics, Biology, . . .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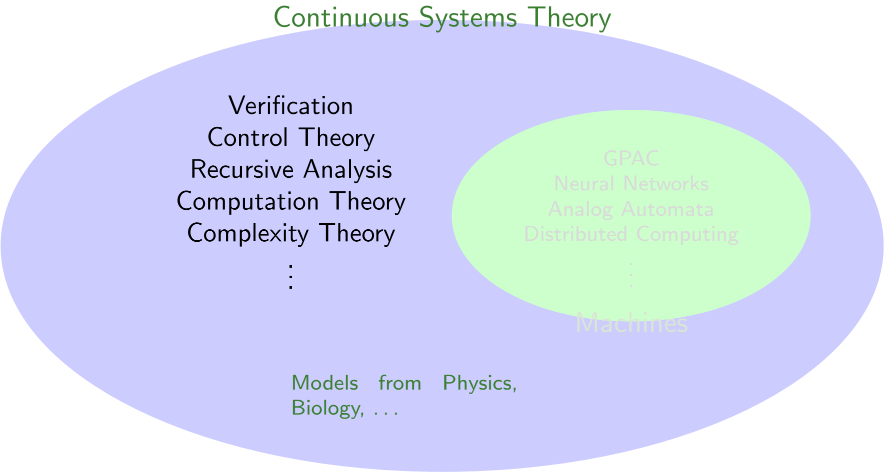
A (discrete time) Picture

Church Thesis	“What is effectively calculable is computable”
Thesis M	“What can be calculated by a machine is computable”
Thesis?	“What can be calculated by a model is computable”

(following [Copeland2002])

Understanding computational power of models helps to understand

- limits of mechanical reasoning.
- limits of machines.
- limits of models.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Main Focus

Properties

- **Reachability**. Given $\mathcal{H}$, x_0, $X \subset \mathbb{R}^n$, decide if there is a trajectory going from x_0 to X.
- **Stability**. Given $\mathcal{H}$, decide if all trajectories go to the origin.

Proofs and constructions from recursive analysis lead limited insights on true difficulty of considered problems

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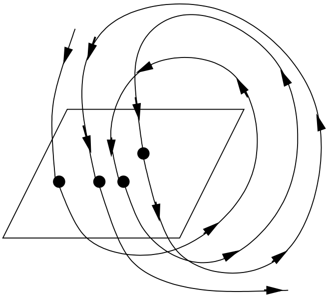
Dynamic Undecidability

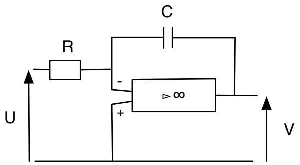


Dynamic Undecidability Results:

- [Moore90]
- [Ruohonen93]
- [Siegelmann-Sontag94]
- [Asarin-Maler-Pnueli95]
- [Branicky95]
- [Graça-Campagnolo-Buescu2005]

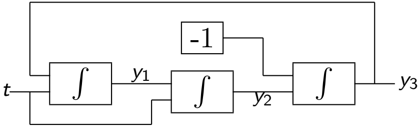
Example

A Modern Electronic Integrator



$$V(t) = -1/RC \int_0^t U(t)dt$$

Generating cos(t)



$$\begin{cases} y_1 &= \cos(t) \\ y_2 &= \sin(t) \\ y_3 &= -\sin(t) \end{cases}$$

The GPAC

[Shannon41]'s GPAC

A mathematical abstraction of the (mechanical) Vannevar Bush MIT Differential Analyzer (1931).

- Basic blocks: constant, adder, integrator, multiplier.
- Shannon's 41 characterization is incomplete. Corrections by [PourEl-Richards74], [Lipshitz-Rubel87], [Graça-Costa03].

Proposition (Graça-Costa03)

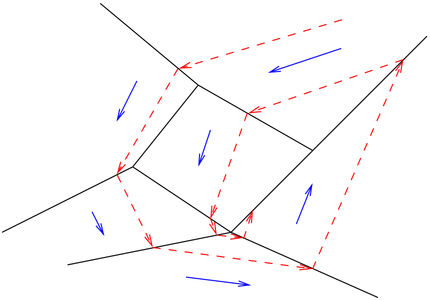
A scalar function $f : \mathbb{R} \to \mathbb{R}$ is generated by a GPAC iff it is a component of the solution of a system

$$y' = p(t, y), \tag{1}$$

where p is a vector of polynomials.

Piecewise Constant Derivative systems

[Asarin-Maler-Pnueli94]'s PCD Systems



$$x' = f(x)$$

$f : \mathbb{R}^d \to \mathbb{Q}^d$ piecewise constant

Theorem (Asarin-Maler-Pnueli94, Asarin-Maler95)

Reachability properties are

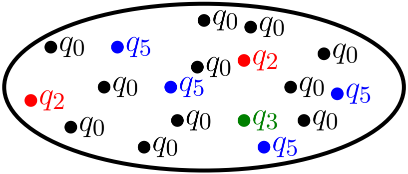
- Σ_1*-complete for the discrete time model.*
- Σ_k*-hard, for all* k*, for the continuous time model.*

Population Protocols

[Angluin-Aspnes-Diamadi-Fisher-Peralta2004]'s sensor networks model

A passively-mobile population of finite-state agents interacts with pairwise interactions $\delta : Q \times Q \rightarrow Q \times Q$.

Example: "Count to 5" protocol.
$Q = \{q_0, q_1, \ldots, q_5\}$
$\delta(q_i, q_j) = (q_5, q_5)$ if $i + j \geq 5$
$\delta(q_i, q_j) = (q_{i+j}, q_0)$ otherwise



Characterization (Angluin-Aspnes-Eisenstat2006)

Population protocols compute precisely relations definable in Presburger's Arithmetic.

Asynchronous Version

Reasonable Hypothesis

Interactions happen following an homogeneous Poisson process.

Microscopic Dynamic

$$Q = \{q_1, q_2, q_3, q_4\}$$

$$\begin{array}{ll} \delta(q_1, q_2) = q_2 & [\beta] \\ \delta(q_2, q_1) = q_2 & [\beta] \\ \delta(q_4, q_2) = q_3 : 1/2, q_4 : 1/2 & [\nu] \\ \delta(q_2, q_4) = q_3 : 1/2, q_4 : 1/2 & [\nu] \end{array}$$

Macroscopic Dynamic

Kermack-McKendrick SIR model.

$$\begin{cases} S' = -\beta S I \\ I' = \beta S I - \nu I \\ R' = \nu I. \end{cases}$$

Epidemic Rate

$$R = \beta S_0 / \nu.$$

Question

Power of such models?

Relating Models

1. Prove equivalence of models:

   ex: $f : \mathbb{R} \rightarrow \mathbb{R}$ is $\mathcal{A}$-computable iff it is $\mathcal{B}$-computable.

2. Discuss discretizations of computable functions:

   ex: If $f : \mathbb{R} \rightarrow \mathbb{R}$ is $\mathcal{A}$-computable, $f(\mathbb{N}) \subset \mathbb{N}$, then $DP(f) = f_{|\mathbb{N}}$ is $\mathcal{B}$-computable.

3. Generalize classical discrete results to the continuous case:

   ex: Class P_Σ can be characterized à la [Bellantoni Cook'92] over any-arbitrary structure Σ.

4. Discuss Hardness of Associated Problems:

   ex: Completeness results for (polynomialy bounded time) reachability problem.

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In Appendix A

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In relations with Paulin de Naurois's PhD

Bellantoni and Cook's Idea

Theorem (Bellantoni-Cook92)

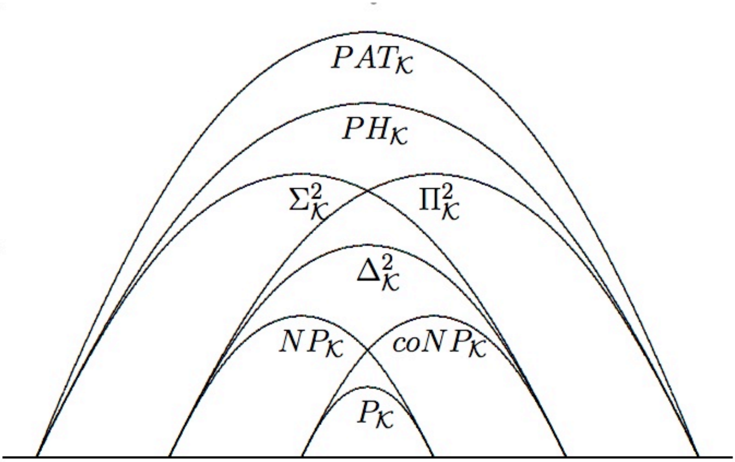
Distinguishing **"safe"** *arguments from* **"normal"** *arguments captures polynomial time.*

$$\mathrm{Add}(\underline{s(;x)}\ ;\ \underline{y}\ ) = s(\ ;\ \mathrm{Add}(x;y)\ )$$

Normal Parameter   Safe Parameter

1. A normal argument can become safe, but not vice versa.
2. Recurrence parameters must be normal
3. Recurrence value must be safe

Inclusion Relations



Results (with Cucker, de Naurois, Marion)

- $\mathbb{Z}_2 = (\{\mathbf{0}, \mathbf{1}\}, =, \mathbf{0}, \mathbf{1})$

PSPACE	Leivant-Marion95
PH	Bellantoni94
NP	Bellantoni94
P	Bellantoni-Cook92, Leivant94
NC	Leivant98
NC^1	Bloch94, Leivant-Marion2000

- $\mathcal{K} = (\mathbb{K}, \{op_i\}_{i \in I}, rel_1, \ldots, rel_l, \mathbf{0}, \mathbf{1})$

$P_{\mathcal{K}}$	Safe Recursion (S.R)
$\Delta^i_{\mathcal{K}}$	S.R with Predicative Minimisations
$\mathrm{D}\Delta^i_{\mathcal{K}}$	S.R. with Digital Predicative Minimisations
$\mathrm{PAR}_{\mathcal{K}}$	S.R with Substitutions
$\mathrm{PAT}_{\mathcal{K}}$	S.R. with Predicative Substitutions
$\mathrm{DPAT}_{\mathcal{K}}$	S.R. with Digital Predicative Substitutions

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In relations with Emmanuel Hainry's PhD

$\mathbb{R}$-Recursive Functions: $\mathbb{R} \to \mathbb{R}$

[Moore96]'s Idea

$$REC: \begin{cases} f(\mathbf{x},0) &= g(\mathbf{x}) \\ f(\mathbf{x},y+1) &= h(\mathbf{x},y,f(\mathbf{x},y)) \end{cases} \qquad \mathrm{INT}: \begin{cases} f(\mathbf{x},0) &= g(\mathbf{x}) \\ \frac{\partial f}{\partial y}(\mathbf{x},y) &= h(\mathbf{x},y,f(\mathbf{x},y)) \end{cases}$$

Classical Settings: $\mathbb{N}^k \to \mathbb{N}^l$

$\mathcal{R}ec = [0, S, U; COMP, REC, MU]$.

Continuous Settings: $\mathbb{R}^k \to \mathbb{R}^l$

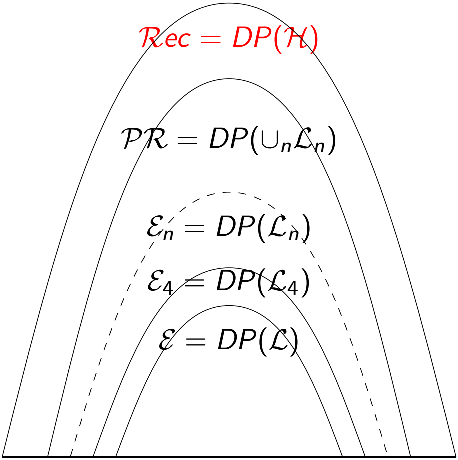
$\mathcal{G} = [0, 1, U; COMP, INT, MU]$

- Several problems in [Moore96] about *MU* schema.
- Corrections & Developments:
  - [Campagnolo-Moore-Costa2000]
  - [Mycka2003]
  - [Mycka-Costa2004]

Continuous Settings: $\mathbb{R}^k \to \mathbb{R}^l$

$\mathcal{L} = [0, 1, -1, \pi, U, \theta_m; COMP, LI]$

Some results (with Dr. Hainry): Discretizations



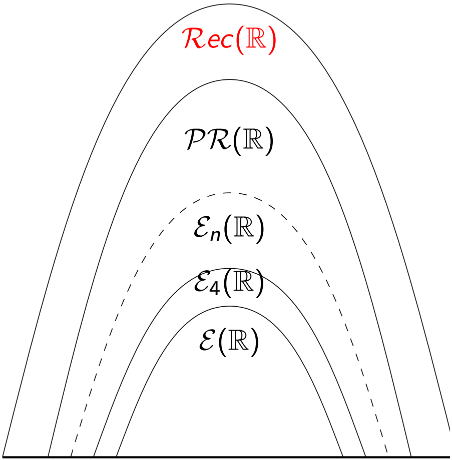
Theorem

There is a minimization operator UMU *with* $DP(\mathcal{H}) = \mathcal{R}ec$ *where* $\mathcal{H} = \mathcal{L} + \mathrm{UMU}$.

$\mathcal{H} = [0, 1, U, \theta_3; \mathrm{COMP}, \mathrm{CLI}, \mathrm{UMU}]$

(all other relations from Campagnolo, Costa, Moore) $DP(f) = f_{|\mathbb{N}}$ for $f(\mathbb{N}) \subset \mathbb{N}$.

Some results (with Dr. Hainry): Recursive Analysis



For $\mathcal{C} = [\mathcal{F}; \mathcal{O}]$, write $\mathcal{C}^*$ for $\mathcal{C}^* = [\mathcal{F}; \mathcal{O}, \mathrm{LIM}]$.

Theorem

For functions of class $\mathcal{C}^2$ defined on a compact domain,

- $\mathcal{L}^* = \mathcal{E}(\mathbb{R})$.
- $\mathcal{L}_n^* = \mathcal{E}_n(\mathbb{R})$.
- $\mathcal{H}^* = \mathcal{R}ec(\mathbb{R})$

Theorem

Computable functions over the reals can be characterized algebraically in a machine independent way.

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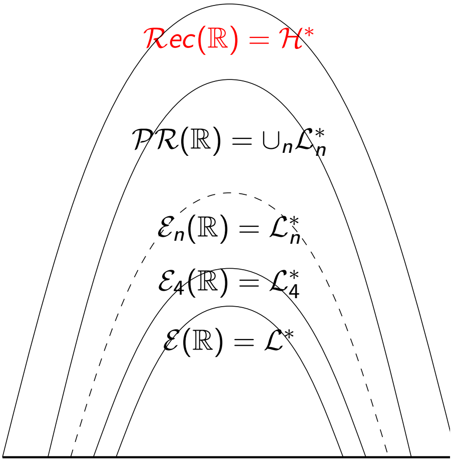

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In relations with Emmanuel Hainry's PhD + Campagnolo, Graça (PAI)

A result (with Graça, Hainry, Campagnolo)

Theorem

Let a and b be computable reals. A function $f : [a, b] \to \mathbb{R}$ is computable iff it is GPAC-computable.

General Picture

- GPAC generated functions are differentially algebraic, hence analytic, and computable.
- computable functions include non-analytic functions: e.g. $min(0, x)$.

...

Reasonable models?

...

- Smooth systems can simulate Turing machines in a finite time.
- Discontinuous/PCD systems can recognize arithmetical sets

Objective A

Understand whether there might be an unifying concept for continuous systems similar to Church thesis.

- A candidate: polynomial differential equations.

Some Arguments I: Modeling Strength

- All examples considered in [Hirsh-Smale74], [Murray93] are of this type.
  - Examples: Lorenz's system, Lotka-Volterra, Kermack-McKendrick SIR model, ...
- Strong stability properties [Graça2007]
  - E.g.: Any system $x' = f(t, x)$, where each component of f is a composition of polynomial and GPAC-generated functions is equivalent to a higher dimensional system $y' = p(t, y)$, where p is a vector of polynomials.

Some Arguments II: Computability

- This corresponds to a notion of machine
  - Differential Analyser
  - Analog Electronic
  - ...
- Classical Recursion can be related to GPAC-computability
  - A function is GPAC-computable iff its is computable, over compact domains.

Objective B

Understand if there is a well-founded complexity theory for continuous time systems.

Obstacles:

- Time, Space contraction phenomena.
- Lack of model relations.

A Discussion I: General Systems

Theorem (Vergis et al86)

The error of Euler's method for $y' = f(y)$, $y(0) = x$ *is*

$$||y(T) - y_N^*|| \leq \frac{h}{\lambda}\left[\frac{R}{2} + \frac{\sigma}{h^2}\right](e^T\lambda - 1),$$

where

- y_N^* *is approximation after* N *steps.*
- h *is the step.*
- λ *is Lipschitz constant for* f *on* $[0, T]$
- $R = \max ||y''(t)||$

N is polynomial in R and $1/\epsilon$, but not in T!

- Same phenomena for all numerical methods.
- [Smith2006] Under some adhoc conditions (e.g. assumptions on solutions), one can eliminate exponential dependence in T.
- Are nicer statements possible, or is this inherent to numerical methods?

A Discussion II: General Systems

1. Can we characterize $P(\mathbb{R})$ algebraically?
2. Can Bellantoni-Cook's idea be used for distinguishing two types of arguments in involved schemas?
3. Can $P(\mathbb{R})$ be related to a notion of GPAC-computability where error is given as a function of a polynomial of t?

A Discussion III: Other Approaches

- Dissipative systems
  - [Gori-Meer2002]: An abstract settings to discuss minimizers of a Lyapunov function E.
  - [BenHur-Siegelmann-Fishman2002]: A settings for studying exponentially converging flows (eg. [Faybusovich91]'s flow to solve linear programming problems).
- Classical Problems Seen with the Toolbox of Analysis
  - E.g. [Costa-Mycka2005]: Two classes of $\mathbb{R}$-recursive functions can be separated iff $P \neq NP$.

Objective C

Better understand the effects of noise and imprecisions on computations.

Obstacles:

- Models of noise and imprecision.
- Contradicting results.

A Discussion I: Some (discrete-time) approaches

- Probabilistic Noise: $P(x_{i+1} \in B) = \int_q z(f(x_i, a_i), q) d\mu$
  - Bounded space implies regular for a wide class of systems [Maass-Orponen98].
  - Gaussian noise forbid recognition of arbitrary regular languages [Maass-Sontag99].

- Non-deterministic Noise: $||x_{i+1} - f(x_i)|| \leq \epsilon$
  - [Fränzle99]: (ad-hoc) Robustness implies decidability over compact domains.
  - [Asarin-Bouajjani02]: ($Reach = Reach_\omega$) Robustness implies decidability

    . . .
  - [Asarin-Collins05]: Stochastic Turing machines compute precisely Π_2
  - [Henzinger-Raskin99]: Open relations still yield to undecidability
  - [Gupta-Henzinger-Jagadeesan97]: Perturbing trajectories still yield to undecidability.

A Discussion II: Some directions

1. Continuous-time systems?

2. Frontier between decidability/undecidability according to models of noise:
   - Do undecidability results still hold for robust systems?

3. How complexity (i.e. not only computability) increase with noise?

- Models of noise, and imprecisions, and the relevance of formal statements about them.

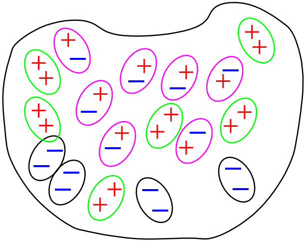
Objective D

Understand new models (e.g. sensor networks, or telecommunication networks) using continuous systems.

Difficulties:

- Justification of the microscopic/macroscopic transformation.
- Legitimacy of models.
- Power of models.

A Discussion I: Second example



$$\begin{cases} ++ \rightarrow \frac{1}{2}+, \frac{1}{2}- & [\lambda] \\ +- \rightarrow + & [\lambda] \\ -+ \rightarrow + & [\lambda] \\ -- \rightarrow \frac{1}{2}+, \frac{1}{2}- & [\lambda] \end{cases}$$

This corresponds to a description of a polynomial ordinary differential equation.
This population protocol computes $\sqrt{2}/2$.

1. Can all algebraic numbers be computed?
2. Can we characterize the input/output relation?

A Discussion II: Statements

Statements:

- The idea of going to thermodynamic limit is not new.
- But classical models of distributed algorithmic forbid macroscopic approximation (non spatial homegeneity).
- Previously mentioned models legitimate a macroscopic approximation.

A Discussion III: Directions

1. Investigate microscopic/macroscopic approximation (ex: variants of population protocols).
2. Investigate their computational power:
   - equilibria, stability.
   - input / output relation.
3. Investigate suitable models
   - for systems (e.g. population protocols)
   - for dynamic of systems (e.g. evolutionary game theory)

   and their relations.

All these models are particular continuous time systems (polynomial ordinary differential equations).

PhDs

Liliana Ibanescu 2004, (PSA, ENSIC)	"Programmation par règles et stratégies pour la génération automatique de mécanismes de combustion d'hydrocarbures polycycliques"
Paulin de Naurois 2004, (Cotutelle de Thèse)	"Completeness Results and Syntactic Characterizations of Complexity Classes over Arbitrary Structures"
Emmanuel Hainry Today, (PAI, ARA SOGEA)	"Modèles de Calculs sur les Réels. Résultats de Comparaison"
Florent Garnier 2007 (FTR&D, RNTL AVERROES)	"Terminaison en temps moyen fini de systèmes de règles probabilistes."