Proposal for M2 Research Internshipfor ENS Lyon for 2025

NP (non-deterministic polynomial time) & Analog computations.

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General Presentation

Today's theoretical computer science, and in particular classical computability and complexity consider mostly computations over a discrete space with a discrete space. This aims at modeling today's computers, which are *digital* computers working over bits. This covers today's machines and today's classical models such as Turing machines working over words over a finite alphabet with a discrete time.

But machines where time is continuous can be considered and can indeed be built. Such machines are analog and are working over continuous quantities like voltage. Notice that first ever built programmable computers were actually analog machines. This includes for example the differential analysers that were first mechanical machines working over quantities encoded by angles of shafts, and later on electronic machines working over quantities like continuous voltages. Such machines were typically used to solve ordinary differential equations. Building today and analog computers does not yield to major difficulties: see example [5] for a book about the history of computing, not forgetting analog computing history as in most of the books.

It turns out that the corresponding computability and complexity theory has not received so much attention: even if models of computation where space could be continuous and time remains discrete have been considered (see e.g. [3], or [6]), these models are still discrete time.

Notice that there is no hope to get a unification of the type of Church-Turing thesis for all such models. However, there is indeed something similar in many aspects for **continuous time and space** models.

The purpose of this internship is to focus on these latter models: i.e. truly analog (continuous time and space) models of computation such as differential analyzers or electronic analog computers. In this context, for many reasons, the equivalent of a Turing machine can be considered as a *polynomial ordinary differential equation (pODE)*. Indeed, a (projection of a) solution of such a pODE, that could be considered as the analog of *computable functions* enjoys many stability properties similar to the stability properties of computable functions. All common analytic functions are in this class, an observation similar to the fact that all common functions in mathematics are computable. Such functions are stable by most of the operations (addition, multiplication, subtraction, division, inverse, composition, ODE solving, etc. . .). Some analytic computable functions are known not to be in this class. However, if a modern definition of computability is considered for pODEs, then computable functions for Turing machines and by pODEs coincide. Etc . . .

Context of the work

We recently proved that there is a very natural and well-founded way of measuring time of computation for such functions: time corresponds to the length of the trajectory.

This surprising result, was awarded the ICALP'2016 best paper award (Track B), and led to the (European PhD) Ackermann Award 2017 to Amaury Pouly. Amaury Pouly was cosupervised by Daniel Graç in Portugal and myself in Palaiseau, France. We used this result in bioinformatics, and we proved that kinetic mechanisms can simulate Turing machines. We received the CMSB'2017 best paper award for this result and the "prix du journal La Recherche" 2019 (for computer science) in 2019 for this result.

We recently understood how to measure space: space corresponds to the involved precision.

This has been obtained by Manon Blanc, currently doing her PhD, and was presented in [2]. We got MFCS 2023 best paper award for a preliminary result last year characterizing PSPACE with discrete ODEs [1].

On one hand, this demonstrates that analog models of computation (in particular old and first ever considered models of machines) are equivalent to modern models both at the computability (we established this fact about 10 years ago [4]) and complexity level. This latter fact is really more surprising.

More importantly, on the other hand, this opens new lights on classical computability and complexity: Indeed, this proves that polynomial time (class P or FP for respectively decision problems and functions, class $PSPACE$ or $FPSPACE$) can be defined very easily using only very simple concepts from analysis: polynomial ordinary differential equations and length of curves, and precision. Indeed, this states for example the following rather unexpected fact: a function over the real is computable in polynomial time iff it can be uniformly approximated by solutions of a polynomial ordinary differential equation of polynomial length.

A natural next question is how to deal with non-determinism. Can we characterize NP with such models? To what corresponds non-determinism when talking about analog models of computations, that is to say ordinary differential equations?

Description of the work

The purpose of this internship is to understand if classes such as NP can also be characterized. One of the purposes if for example to formulate the question $P = NP$? in the framework of (mathematical) analysis.

For examples, replacing an ordinary differential equation $y' = f(y)$, by an differential inclusion $y' \in f(y)$ seems to be a natural way to obtain a concept close to non-determinism. Can we prove that this works well? That NP corresponds exactly to such a concept in a natural way?

An alternative is to consider wider classes of ordinary differential equations, such as discontinuous ODEs.

We have some preliminary statements in that direction, but the purpose is to reach a nice(r) and elegant characterization of NP.

Comments

The actual topic of the work is related to computability and complexity theory. This requires only common and basic knowledge in ordinary differential equations. Most of the intuitions of our today's constructions come from classical computability and complexity.

There is no specific prerequisite for this internship. This subject can be extended to a PhD. Possibilities of funding according to the administrative situation of candidates.

The supervisor is also open on variations on these questions according to the preferences of the candidate (focusing on algorithmic aspects " how to derive an efficient algorithm using continuous methods", logical aspects "what is the associated proof theory", etc.).

References

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