Medvedev degrees of effective subshifts on groups

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Phd student under the supervision of Cristóbal Rojas and Sebastián Barbieri, Pontificia Universidad Católica de Chile, joint work

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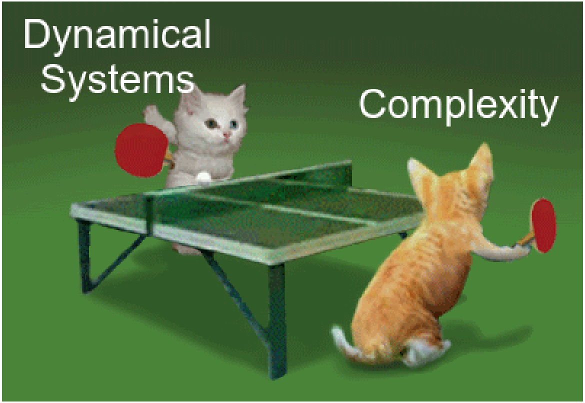
Context



Source: 3dgifanimation.blogspot.com

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Entropy (Topological, G amenable)

- Measures uncertainty / information / more interpretations
- Nonincreasing by factors
- Conjugacy invariant
- Takes values in $[0, \infty)$
- Sends $\sqcup$ and $\times$ to max and sum

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We want to classify

Question

What are the possible entropies of this class of subshifts?

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What is known for SFT's in $\mathbb{Z}^d$

1. On $\mathbb{Z}^2$, there are nonempty SFT's with no computable points (Myers 1974; Hanf 1974).
2. A set has computable points if and only if it has Medvedev degree 0.
3. On $\mathbb{Z}^2$, there are nonempty SFT's with $>_{\mathfrak{M}} 0$ (Myers 1974; Hanf 1974).
4. On $\mathbb{Z}^d$, $d \geq 2$ there are nonemtpy SFT's of all Π^0_1 degrees (Simpson 2014)
5. On $\mathbb{Z}$, all SFT's are $\equiv_{\mathfrak{M}} 0$ (Folklore)

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Effective subshifts

Remark

On $\mathbb{Z}^d$, effective subshifts = subshifts + effectively closed sets.

Theorem (Miller 2012)

On $\mathbb{Z}$, there are effective subshifts of all Π^0_1 degrees.

Main theorem (N.C. 2023)

The same holds on a finitely generated infinite group with decidable word problem.

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Countable setting

- Let $E, F \subset \mathbb{N}$
- $E \leq_T F$
- $E \leq_m F$
- $E \leq_? F$
- $X \leq_{\mathfrak{M}} Y$ we compare subsets of the Cantor space $A^{\mathbb{N}}$ (A finite).

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The definition of $(\mathfrak{M}, \leq_{\mathfrak{M}})$

Definition

Let P, Q be subsets of $A^{\mathbb{N}}$. We write

$$P \leq_{\mathfrak{M}} Q$$

if there is a computable function Φ such that $\Phi(Q) \subset P$.
We define $\equiv_{\mathfrak{M}}$ by $\leq_{\mathfrak{M}}$ and $\geq_{\mathfrak{M}}$.
Medvedev degrees = equivalence classes under $\equiv_{\mathfrak{M}}$.

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Examples of problems

1. Graph theory: infinite paths, matchings, colorings,
2. Invariant measures associated to dynamical systems
3. Solutions to some equation
4. Subshifts!

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Some properties of $\mathfrak{M}$

For the relation $\leq_{\mathfrak{M}}$:

- The empty set is maximal
- If P has a computable element, it is minimal for $\leq_{\mathfrak{M}}$. We write

$$P \equiv_{\mathfrak{M}} 0.$$

- $(\mathfrak{M}, \leq_{\mathfrak{M}})$ is a lattice!
- $X \vee Y = X \times Y$
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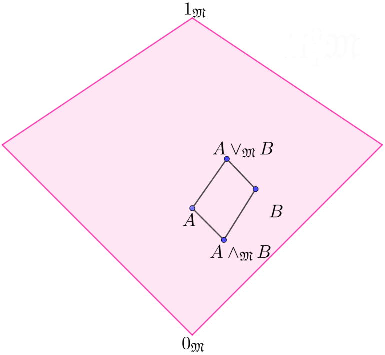
The lattice of Π^0_1 degrees

Π^0_1 degrees form a sublattice of $(\mathfrak{M}, \leq_{\mathfrak{M}})$



From $A^{\mathbb{N}}$ to A^G

Problem

Define Medvedev degrees on A^G.

If G is f.g. infinite with decidable word problem, it admits a computable bijection $\nu : \mathbb{N} \to G$.
We obtain a homeomorphism

$$A^{\mathbb{N}} \to A^G.$$

This transfers all computability notions from $A^{\mathbb{N}}$ to A^G, in particular Medvedev degrees.

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If G is as above, then effective subshift = subshift + effectively closed set. This coincides with notions already present.

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Proof sketch for Main Theorem:

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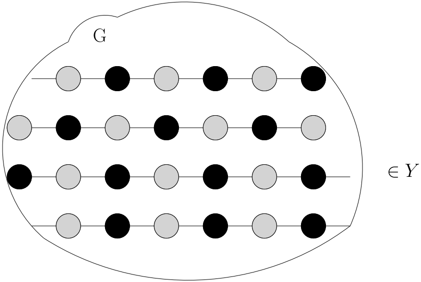
Suppose first that G contains $\mathbb{Z}$, and try to deduce the classification from that for $\mathbb{Z}$.

Special case: G contains $\mathbb{Z}$.

In this case, we can write G as union of copies of $\mathbb{Z}$.

Given $X \subset A^{\mathbb{Z}}$, we can define $Y = \{x \in A^G \mid \forall g \in G, gx|_{\mathbb{Z}} \in X\}$.



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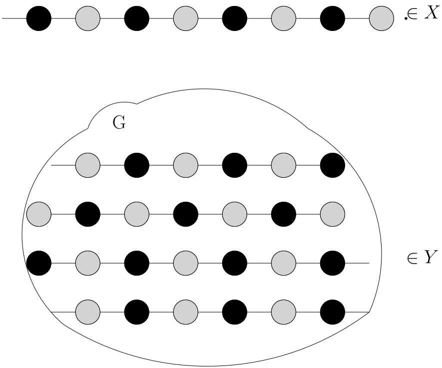

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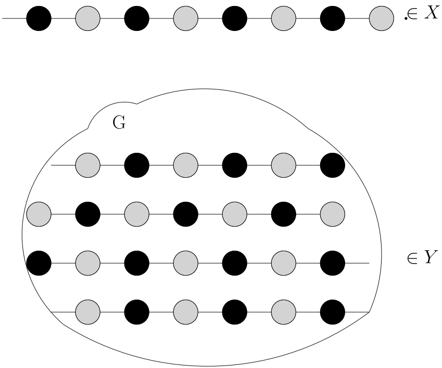

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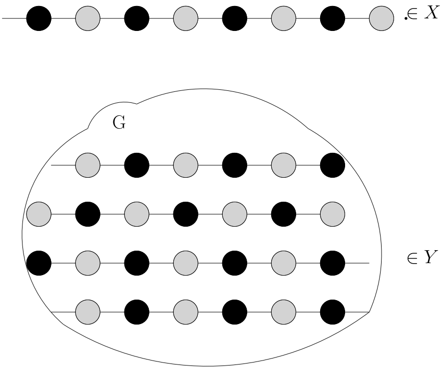
$Y \geq_{\mathfrak{M}} X$ is easy



The function $A^G \to A^{\mathbb{Z}}, y \mapsto y|_{\mathbb{Z}}$ is computable. This proves that $Y \geq_{\mathfrak{M}} X$.

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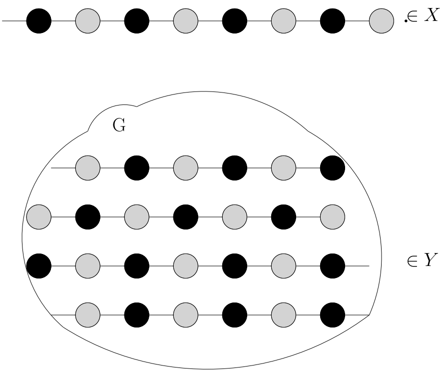

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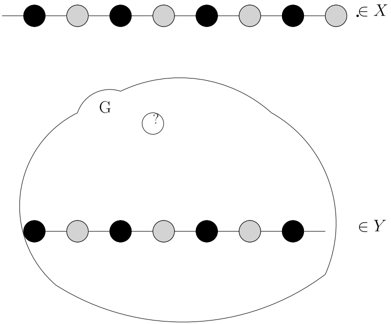

- To compute an Y configuration from an X configuration, we need $\mathbb{Z} \leqslant G$ to have decidable **subgroup membership problem**.
- Under the extra hypothesis, we have $X \equiv_{\mathfrak{M}} Y$.

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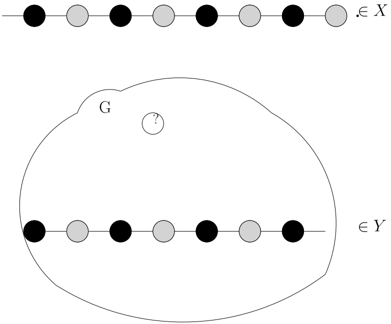

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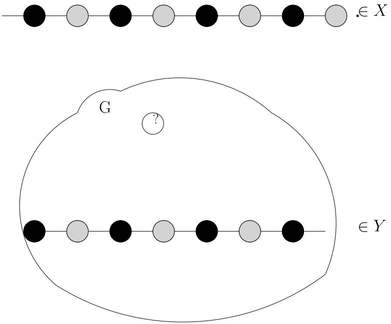

- To compute an Y configuration from an X configuration, we need $\mathbb{Z} \leqslant G$ to have decidable **subgroup membership problem**.
- Under the extra hypothesis, we have $X \equiv_{\mathfrak{M}} Y$.

Problem

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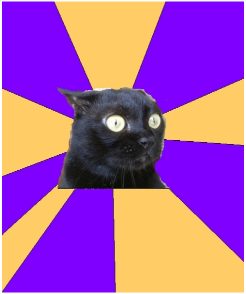
Some groups do not contain $\mathbb{Z}$.



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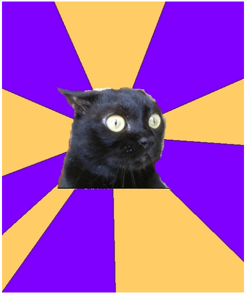

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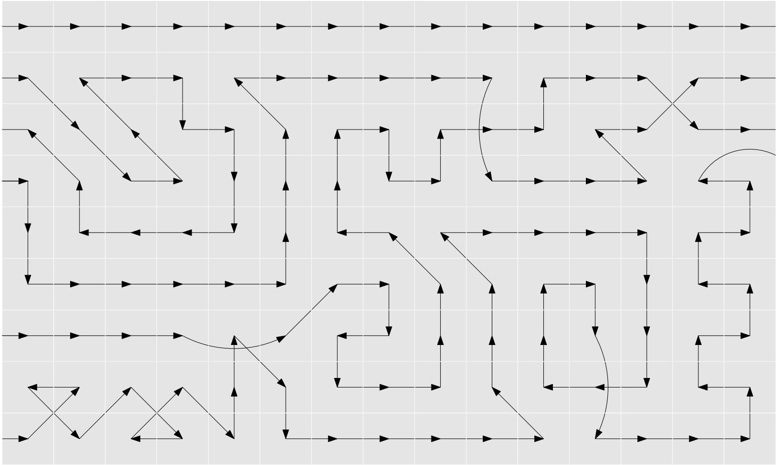
An example on $\mathbb{Z}^2$



Translation-like actions

Definition

Let G be a f.g. group, and d be a word length metric on G. A group action $\mathbb{Z} \curvearrowright G$ is called translation-like if

1. All orbits are infinite
2. There is $J \in \mathbb{N}$ such that $d(g, g * 1) \leq J$

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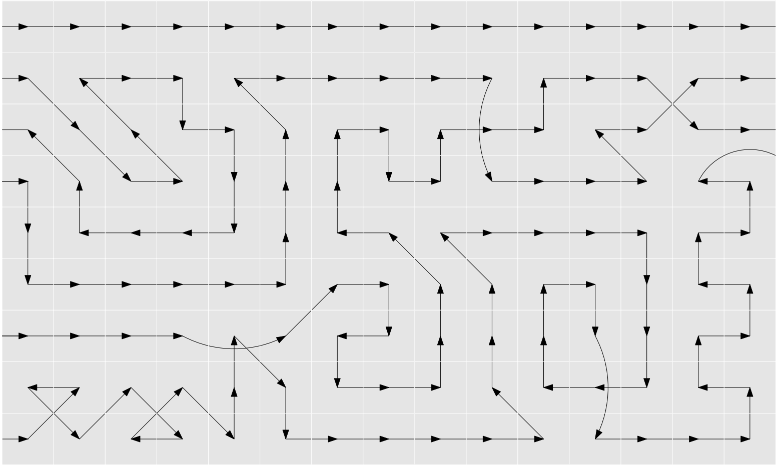

Translation-like actions can be coded as subshifts!

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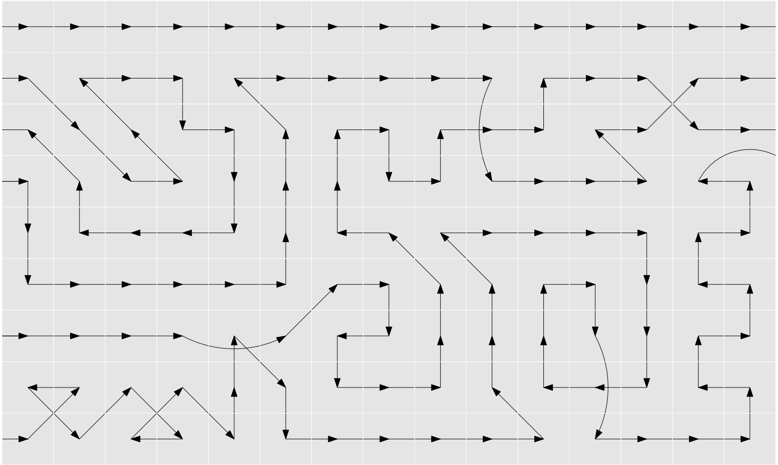

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Seward's result

Theorem (Seward 2014)

Every finitely generated group admits a translation-like action by $\mathbb{Z}$.

We need a computable version:

Main Lemma (N.C. 2023)

With the adittional hypothesis of decidable word problem, there is a computable translation-like action, and with decidable orbit membership problem.

This means that it is decidable whether a pair of groups elements are in the same orbit.

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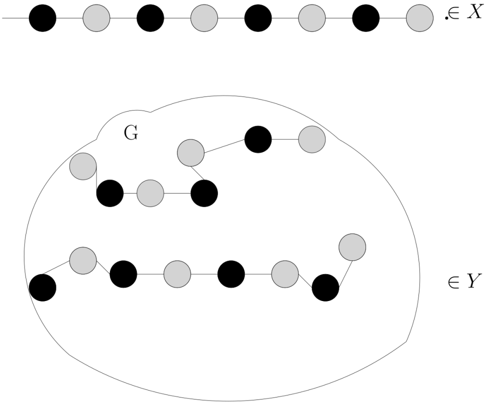
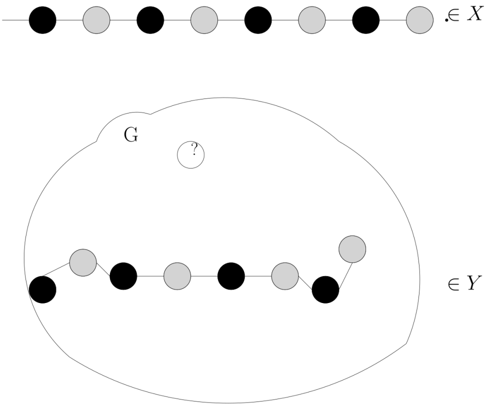


Given $X \subset A^{\mathbb{Z}}$, we can construct $Y \subset (A \times B)^G$ which describes translation-like actions, and elements of X.

The proof of Main Theorem



Main Lemma allows to construct a computable function $X \to Y$.

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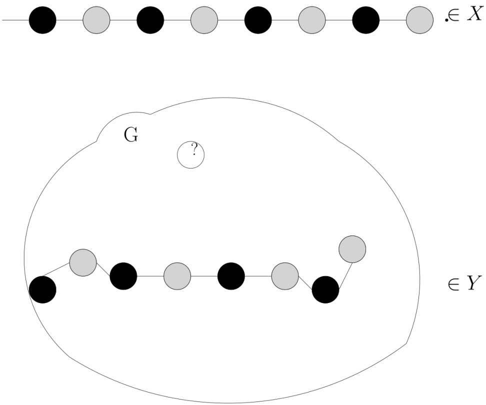

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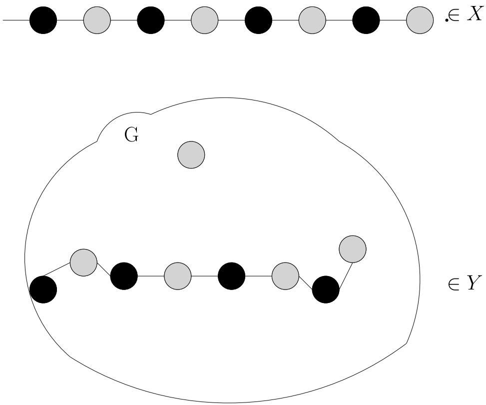

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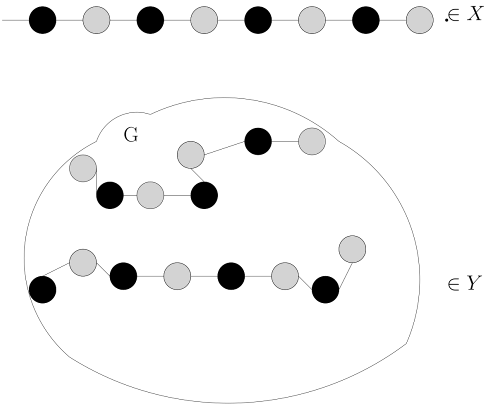

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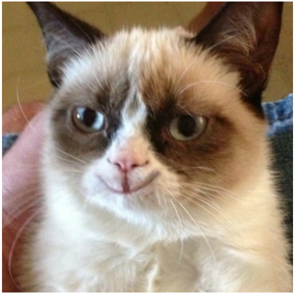
Theorem

Let G be finitely generated, infinite, and with decidable word problem. There are effective subshifts of all Π_1^0 degrees.



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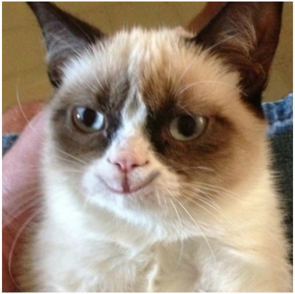

Things to solve in the Main Lemma

1. Best case possible: G admits a transitive translation-like actions by $\mathbb{Z}$
2. Seward 2014: G admits a transitive translation-like action by $\mathbb{Z}$ if and only if G has one or two ends.
3. Ends = geometric property of a graph (or Cayley graph)
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2. We define transitive translation-like actions locally, i.e. by finite pieces.
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The end

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