Totalization of ODEs

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Ordinary differential equations

Let $y : [a, b] \to E \subseteq \mathbb{R}^n$ be the unique solution of:

$$\begin{cases} y' = f(y(t)) \\ y(a) = y_0 \end{cases}$$

- Obtain y: if f is continuous, limit of sequence of continuous functions
- Compute y: if f is continuous, Ten thousand monkeys [CG09]

Question 1:

Relaxing continuity for f, when can we obtain y from f ?

Question 2:

What is the set theoretical complexity of y relative to f? Borel hierarchy, arithmetical hierarchy etc

Antidifferentiation

Antidifferentiation is a particular type of ODE solving when the derivative is known explicitely

- Let $F : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be a function differentiable on $[a, b]$
- Let $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be such that $F'(x) = f(x)$ for all $x \in [a, b]$

Goal 1:
Obtain F from f

Goal 2:
Describe the set theoretical complexity of F relative to f

Conditions on the derivative

If f satisfies (A) then by (B) we get:

$$F(x) = F(a) + \int_a^x f(t)dt \text{ for } x \in [a, b]$$

- (A) f continuous
  (B) Fundamental theorem of calculus
  $F \in C^1([a, b])$
- (A) f bounded, continuous almost everywhere ($\mu_L(D_f) = 0$)
  (B) Lebesgue-Vitali theorem
  $F \in C^1([a, b])$ almost everywhere
- (A) f Lebesgue integrable
  (B) Lebesgue differentiation theorem
  $F \in BV$, Bounded variation

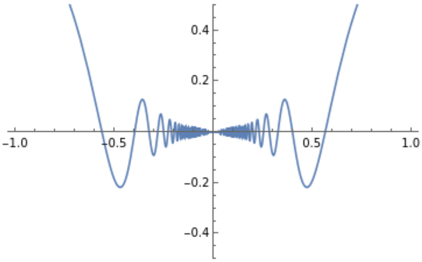
Lebesgue integral is not enough

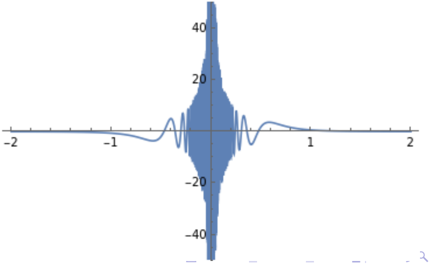
Goal: generalize integration

Investigate antidifferentiation for non-Lebesgue integrable derivatives

Function Ω

$$\Omega(x) = \begin{cases} x^2 \sin(\frac{1}{x^2}) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases} \qquad \Omega'(x) = \begin{cases} 2x \sin(\frac{1}{x^2}) - \frac{2}{x}\cos(\frac{1}{x^2}) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

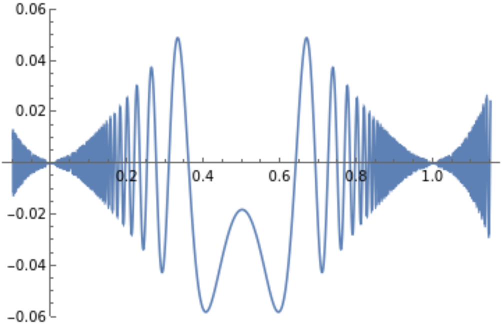




Two problematic discontinuities

- Function with derivative with unbounded discontinuities in two points:

$$\Omega_2(x) = \begin{cases} x^2(1-x)^2 \sin\left(\frac{1}{x^2(1-x)^2}\right) & \text{if } \; 0 < x < 1 \\ 0 \; \text{ if } \; x = 0, 1 \end{cases}$$



Extension of Lebesgue integral

Question:

Is there a way to obtain such antiderivatives?

- Denjoy, 1912, [Den12], iterative process, transfinite induction
- Perron, 1914, [Per14], equivalent to Denjoy
- Luzin, 1915, variation absolute continuity
- Kurzveil, 1957, [Kur57], gauge integral, similar to Riemann
- Henstock, 1957, [Hen57], equivalent to Kurzveil

Denjoy totalization

- Condition on f: f is a derivative

Definition 1 (Nonsummable points of f)

Let E be a closed set $E \subseteq [a, b]$ and f a Lebesgue measurable function. A point $x \in E$ is a *nonsummable point of f on E* if f is not Lebesgue integrable in every $I \in E$, I an open interval containing x.

Theorem 2

If F is a differentiable function on $[a, b]$ and $E \subseteq [a, b]$ is closed, then the nonsummable points of f on E form a closed nowhere dense set[1] *in E*

[1] A subset A of a topological space X is nowhere dense in X if the closure of A has empty interior

Takeaway message

- Bad behaved points are few in the domain
- Outside of them we can simply integrate

Procedure:
We can obtain $F(d) - F(c)$ for all $[c, d] \subseteq [a, b]$, $c < d$ disjoint with this set of bad behaved points,take the limits and then describe a new set of bad behaved points within the previous one

Intuition:

1. Lebesgue integration
2. Limits
3. Transfinite induction

Transfinite process

- Let $E_1 = \{x \in [a, b]$ such that x is a nonsummable point of f on $[a, b]\}$, and let $\{(a_i, b_i)\}_i$ be its contiguous intervals
- Obtain $F(d) - F(c)$ for all $[c, d] \subseteq [a, b]$ such that $[c, d] \cap E_1 = \emptyset$
- Since F is continuous, take limits to obtain $F(b_i) - F(a_i)$ for all i.

Inductive step

Let $E_2 = \{x \in [a, b]$ such that x is a nonsummable point of f on $E_1\}$

- We know by the same theorem that E_2 is nowhere dense in E_1
- Repeat the above for E_2
- Proceed by transfinite induction, taking intersections at limit ordinals

Theorem 3 (Cantor-Baire Stationary Principle, [EV10])

Let $\{E_\alpha\}_{\alpha<\Omega}$ be a family of closed subsets of $\mathbb{R}^n$, indexed by the countable ordinal numbers. Suppose $\{E_\alpha\}_{\alpha<\Omega}$ is decreasing; i.e., $E_\alpha \subseteq E_\beta$ if $\alpha \geq \beta$. Then there exists $\alpha^* < \Omega$ such that $E_\alpha = E_{\alpha^*}$ for $\alpha \geq \alpha^*$.

- If $E_{\alpha+1} \subseteq E_\alpha$ with $E_{\alpha+1}$ nowhere dense in E_α, $\Rightarrow E_{\alpha+1} \subset E_\alpha \Rightarrow E_{\alpha^*} = \emptyset$
- The process converges, $E_{\alpha^*} = \emptyset \Rightarrow F(d) - F(c)$ for all $[c, d] \subseteq [a, b]$; Totalization

Theorem 4 ([DK91])

The operation of antidifferentiation is not Borel. [2]

Theorem 5 ([Wes20])

The operation of antidifferentiation is $\mathbf{\Pi}^1_1$-complete.

[2] More precisely, there is no Borel set $B \subseteq C[a, b]^{\mathbb{N}}$ such that for f derivative $f \in B \iff F(b) - F(a) > 0$.

Back to ordinary differential equations

Let $y : [a, b] \to E \subseteq \mathbb{R}^n$ be the unique solution of:

$$\begin{cases} y' = f(y(t)) \\ y(0) = y_0 \end{cases} \tag{1}$$

Question 1:

When can we obtain y from f with a totalization?

Question 2:

What is the complexity of such procedure?

Lebesgue integration $\longrightarrow$ Ten Thousand Monkeys

Nonsummability points $\longrightarrow$ Discontinuity points for f

Definition 6 (sequence of f-removed sets on E)

Let $\{E_\alpha\}_{\alpha<\omega_1}$ be a transfinite sequence of sets and $\{f_\alpha\}_{\alpha<\omega_1}$ a transfinite sequence of functions $f_\alpha = f\restriction_{E_\alpha}: E_\alpha \to \mathbb{R}^r$ defined as following:

- Let $E_0 = E$
- For every α successor ordinal, let E_α be:
  $E_\alpha = \{x \in E_{\alpha-1} : f_{\alpha-1} \text{ is discontinuous in } x\} = D_{f_{\alpha-1}}$
- For every α limit ordinal, let E_α be $E_\alpha = \cap_\beta E_\beta$ with $\beta < \alpha$

Conditions on right-hand term f

- Every derivative is a Baire one function, i.e. it is the limit of a sequence of continuous functions. [3]

Hypothesis on f

Let f be Baire one and such that for every closed set $K \subseteq E$ the set of discontinuity points of function $f \upharpoonright_K$ is a closed set

Theorem 7 (Bournez, Gozzi)

If f satisifies the hypothesis then there exists an ordinal $\alpha < \omega_1$ such that $E_\beta = \emptyset$ for all $\beta \geq \alpha$.

[3] Let X, Y be two separable, complete metric spaces. A function $f : X \to Y$ is Baire one if there exists a sequence of continuous functions from X to Y, $\{f_m\}_m$, such that $\lim_{m\to\infty} f_m(x) = f(x)$ for all $x \in X$.

Main claim

Theorem 8 (Bournez, Gozzi)

If f satisifies the hypothesis then we can obtain the solution from f and the initial condition via transfinite induction up to an ordinal number α such that $\alpha < \omega_1$.

- The induction works for each E_α using a Ten thousand monkeys approach where $f_\alpha = f \restriction_{E_\alpha}$ is continuous and taking limits
- The method is bound to terminate for some countable ordinal due to previous theorem
- The transfinite number of steps corresponds to the first ordinal α such that $E_\alpha = \emptyset$ and represents the complexity of ODEs solving for f on E

Example

Let $E = [-5,5] \times [-5,5]$ and let $f : ([-5,5] \times [-5,5]) \to \mathbb{R}^2$ be $f(x,y) = (1, 2x \sin\frac{1}{x} - \cos\frac{1}{x})$ if $x \neq 0$ and $f(x,y) = (1,0)$ otherwise. Consider the IVP on $[-2,2]$:

$$z'(t) = f(z(t)) = \begin{cases} z_1'(t) = 1 \\ z_2'(t) = \begin{cases} 2z_1(t)\sin\frac{1}{z_1(t)} - \cos\frac{1}{z_1(t)} & \text{if } z_1(t) \neq 0 \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

$$\begin{cases} z_1(-2) = -3 \\ z_2(-2) = 9\sin(-\frac{1}{3}) \end{cases}$$

- Unique solution $z_1(t) = t - 1$ and $z_2(t) = (t-1)^2 \sin(\frac{1}{t-1})$ for $t \neq 1$ and $z_2(1) = 0$
- $D_f = E_1 = \{(0,y) : y \in [-5,5]\}$ nowhere dense in E
- f Baire one, $f(E_1) = (1,0)$ and $D_{f\restriction_K}$ closed for K closed
- Since $f(E_1) = (1,0)$ then $D_{f\restriction_{E_1}} = E_2 = \emptyset$

Conclusions and open problems

- We can construct more complex cases based on the previous example with $E_\alpha \neq \emptyset$ for all $\alpha < \omega_1$
- In parallel with results from [DK91] and [Wes20] for antidifferentiation we expect to obtain similar complexity results for ODEs solving
- Connection with ITTMs [HL00]

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Thank you!