Measuring the robustness of dynamical systems

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Introduction

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- Motivated by the field of verification.
- Informal conjecture: undecidability in verification does not happen for robust systems.

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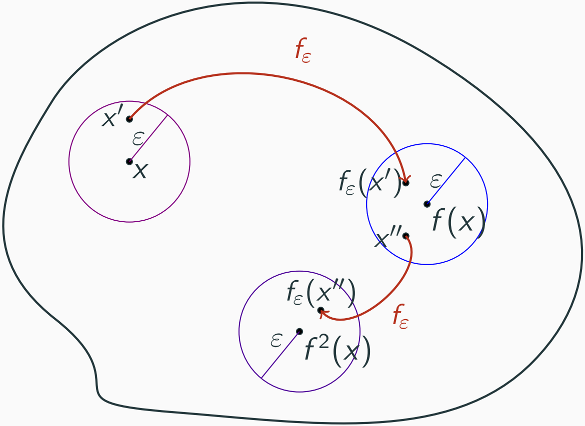
- Approach: Asarin and Bouajjani (LICS '01).
- Our goal:
  - go from computability to complexity,
  - quantifying the robustness to characterise FPSPACE.


f_ε
x'
ε
x
$f_\varepsilon(x')$
ε
$f(x)$
x''
$f_\varepsilon(x'')$
f_ε
ε
$f^2(x)$

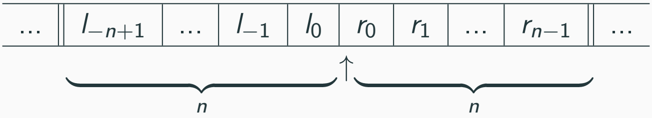
Frame : We want to talk about discrete- and continuous-time dynamical systems. We first use the frame of Turing machines

(seen as a particular dynamical system).

Space-perturbation of a Dynamical System

n-perturbation of a TM

Let $\mathcal{M}_n$, the n-space-perturbed version of TM $\mathcal{M}$: the idea is that the n-perturbed version of the machine $\mathcal{M}$ is unable to remain correct at a distance more than n from the head of the machine.



Some Properties (from Asarin and Bouajjani '01)

Let $L_n(\mathcal{M})$ be the n-perturbed language of $\mathcal{M}$.

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- $L(\mathcal{M}) \subseteq L_\omega(\mathcal{M}) \subseteq \cdots \subseteq L_2(\mathcal{M}) \subseteq L_1(\mathcal{M})$.

Perturbated reachability co-ce

Theorem (Perturbed reachability is co-c.e., from Asarin-Bouajjani 01)

$$L_\omega(\mathcal{M}) \in \Pi^0_1.$$

Say L is robust if $L = L_\omega$:

$$L \text{ robust} \Rightarrow L \text{ decidable}.$$

- Definition of the notion of perturbation of a TM
- Some properties

A Characterisation of PSPACE

Definition
For $f : \mathbb{N} \to \mathbb{N}$, we write $L_{\{f\}}(\mathcal{M})$ the set of words accepted by $\mathcal{M}$ with a "space-perturbation" f:

$$L_{\{f\}}(\mathcal{M}) = \{w | \ w \in L_{f(\ell(w))}(\mathcal{M})\}.$$

A Characterisation of PSPACE

Theorem (Polynomial precision robust $\Leftrightarrow$ PSPACE)

$L \in \mathrm{PSPACE}$ *iff for some* $\mathcal{M}$ *and some polynomial* p,

$$L = L(\mathcal{M}) = L_{\{p\}}(\mathcal{M}).$$

- Characterisation of PSPACE with space-perturbated TMs

Reachability Relations

Reachability Relation for Dynamical Systems

To each rational discrete time dynamical system $\mathcal{P}$ is associated its reachability relation $R^{\mathcal{P}}(\cdot,\cdot)$ on $\mathbb{Q}^{d} \times \mathbb{Q}^{d}$.

$\rightarrow$ two rational points $\mathbf{x}$ and $\mathbf{y}$, $R^{\mathcal{P}}(\mathbf{x},\mathbf{y})$ holds iff there exists a trajectory of $\mathcal{P}$ from $\mathbf{x}$ to $\mathbf{y}$.

Reachability Relation With Small Perturbations

- Consider now a discrete-time dynamical system $\mathcal{P}$ with function $\mathbf{f}$ Lipschitz on a compact domain:
  For $\varepsilon > 0$ we consider the ε-perturbed system $\mathcal{P}_\varepsilon$.

- Its trajectories are defined as sequences $\mathbf{x}_t$ satisfying $d(\mathbf{x}_{t+1}, \mathbf{f}(\mathbf{x}_t)) < \varepsilon$ for all t.

Reachability Relation With Small Perturbations

- We denote reachability in the system $\mathcal{P}_\varepsilon$ by $R_\varepsilon^{\mathcal{P}}(\cdot,\cdot)$.

- $R_\omega^{\mathcal{P}} = \bigcap_\epsilon R_\varepsilon^{\mathcal{P}}(\cdot,\cdot)$ is co-c.e..
- Say $R^{\mathcal{P}}$ is robust when $R_\omega^{\mathcal{P}} = R^{\mathcal{P}}$:

$$R^{\mathcal{P}} \text{ robust} \Rightarrow R^{\mathcal{P}} \text{ computable.}$$

Link with δ-reachability

Proposition (Robust $\Leftrightarrow$ reachability true or ϵ-far from being true)

We have $R_\omega^{\mathcal{P}} = R^{\mathcal{P}}$ iff for all $\mathbf{x}, \mathbf{y} \in \mathbb{Q}^d$, either $R^{\mathcal{P}}(\mathbf{x}, \mathbf{y})$ is true or $R^{\mathcal{P}}(\mathbf{x}, \mathbf{y})$ is false and there exists $\epsilon > 0$ such that it is ϵ-far from being true.

$\Rightarrow$ relation with δ-reachability (Gao, Kong, Chen, Clarke '06).

Ball decision problem

Input:

- A point $\mathbf{x}$
- A set B
- A dynamic given by the function $\mathbf{f}$
- The promise that the dynamics starting from $\mathbf{x}$ never ends up on the border of B.

Ball decision problem

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Output: Does the dynamics starting from $\mathbf{x}$ reach B ?

Theorem (Polynomially robust to precision $\Rightarrow$ PSPACE)

Take a Lipschitz system on a compact, with $\mathbf{f}$ *poly. time computable, whose domain* X *is a closed rational box, and that for all rational* $\mathbf{x}$, $R^{\mathcal{P}}(\mathbf{x})$ *is closed and* $R^{\mathcal{P}}(\mathbf{x}) = R^{\mathcal{P}}{}_{\{p\}}(\mathbf{x})$ *for a polynomial* p*. Then the ball decision problem is in* PSPACE.

Complexity of reachability relations

Theorem (Polynomially robust to precision $\Leftarrow$ PSPACE)

Any PSPACE *language is reducible to PAM's reachability relation:*
$R^{\mathcal{P}} = R^{\mathcal{P}}{}_{\{p\}}$*, for some polynomial* p*.*

- Perturbated reachability relation for rational dynamical systems
- Characterisation of PSPACE in that framework (with additionnal properties)

Alternative view: Drawability

Drawability: geometric properties

But we could also see it as a relation over the reals and use the framework of CA, regarding subsets of $\mathbb{R}^d \times \mathbb{R}^d$.

Definition

A closed subset of $\mathbb{R}^d$ is said *c.e. closed* if we can effectively enumerate the rational open balls intersecting it.

Drawability: geometric properties

From the statements of CA, the following holds:

Theorem

Consider a computable discrete time system $\mathcal{P}$ whose domain is a computable compact. For all $\mathbf{x}$, *$cls(R^{\mathcal{P}}(x)) \subseteq \mathbb{R}^d$ is a c.e. closed subset.*

And With Reachability Properties

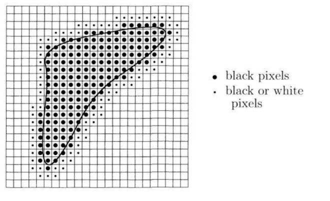
Theorem (Perturbed reachability is co-c.e.)

Consider a dynamical system, with $\mathbf{f}$ *locally Lipschitz, computable, whose domain is a computable compact, then, for all* $\mathbf{x}$*,* $cls(R_\omega^{\mathcal{P}}(\mathbf{x})) \subseteq \mathbb{R}^d$ *is a co-c.e. closed subset.*

Say $R^{\mathcal{P}}$ is robust when $R_\omega^{\mathcal{P}} = R^{\mathcal{P}}$: $R^{\mathcal{P}}$ robust $\Rightarrow$ $R^{\mathcal{P}}$ computable.

Plot twist

From CA, Computable $\Leftrightarrow$ can be plotted.



Theorem
Assume $R^{\mathcal{P}}$ is closed and can be plotted effectively in a name of $\mathbf{f}$. Then the system is robust, i.e. $R^{\mathcal{P}}_{\omega} = R^{\mathcal{P}}$.

- We have a nice extension to geometric properties

Time-perturbation

Another type of perturbation

Definition

Given $f : \mathbb{N} \to \mathbb{N}$, we write $L^{\{f\}}(\mathcal{M})$ for the set of words accepted by $\mathcal{M}$ with time perturbation f:

$$L^{\{f\}}(\mathcal{M}) = \{w \mid w \in L^{f(\ell(w))}(\mathcal{M})\}.$$

Another Characterisation of PTIME with length

Theorem (Polynomially robust to time $\Leftrightarrow$ PTIME)

A language $L \in \mathsf{PTIME}$ iff for some $\mathcal{M}$ and some polynomial p, $L = L(\mathcal{M}) = L^{\{p\}}(\mathcal{M})$. Any PTIME language is reducible to PAM's reachability: $R^{\mathcal{P}} = R^{\mathcal{P},(p)}$ for some polynomial p.

A Characterisation of PTIME

Theorem (Polynomially length robust $\Rightarrow$ PTIME)

Assume distance d is time metric and $R^{\mathcal{P}} = R^{\mathcal{P},(p)}$ for some polynomial p. Then $R^{\mathcal{P}} \in \mathrm{PTIME}$.

- Definition of another type of perturbation
- Characterisation of PTIME

Conclusion

- Characterisation of FPSPACE with the suitable notion of perturbation.
- Characterisation ot PTIME with the suitable notion of perturbation.
- Extension to drawability.